Approximation of the Effective Hamiltonian Through a Degenerate Elliptic Problem

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The Effective Hamiltonian

Definition

Let $H \in \mathrm{C}^0(\mathbb{T}^d \times \mathbb{R}^d)$ be a convex Hamiltonian. Its effective Hamiltonian $\bar{H}$ maps each $P \in \mathbb{R}^d$ to the unique constant $\bar{H}(P)$ such that the *cell problem*

$$H(x, P + \nabla u) = \bar{H}(P)$$

has a viscosity solution $u \in \mathrm{C}^0(\mathbb{T}^d)$.

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Difficulties for solving the cell problem numerically:

- two unknowns: u and $\bar{H}(P)$,
- u is not unique (not even up to a constant),
- no boundary values.

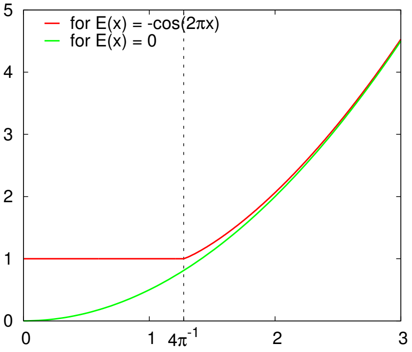
Example for the Effective Hamiltonian

Consider the Hamiltonian

$$H(x,p) = |p|^2 + E(x),$$

where E is the potential for d uncoupled penduli:

$$E(x) = -\sum_{i=1}^{d} \cos(2\pi x_i).$$



The Inf-Max Formula

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Theorem (Inf-Max Formula, Contreras et al., 1998)

For a convex Hamiltonian $H \in C^0(\mathbb{T}^d \times \mathbb{R}^d)$ *satisfying [...] and the coercivity condition*

$$\inf_{x \in \mathbb{R}^d} H(x,p) \overset{|p|\to\infty}{\longrightarrow} \infty,$$

the effective Hamiltonian is given by

$$\bar{H}(P) = \inf_{u \in C^\infty(\mathbb{T}^d)} \max_{x \in \mathbb{T}^d} H\big(x, P + \nabla u(x)\big).$$

An Elliptic Approximation of $\bar{H}(P)$

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Theorem (Evans, 2003)

Let $H \in C^2(\mathbb{T}^d \times \mathbb{R}^d)$ be strictly convex and [...], let u_k be a solution of

$$\nabla \cdot \left(e^{kH(x,P+\nabla u_k)} H_p(x, P + \nabla u_k)\right) = 0,$$

and define

$$\bar{H}_k(P) := \frac{1}{k} \ln \int_{\mathbb{T}^d} e^{kH(x,P+\nabla u_k)} \, \mathrm{d}x.$$

Then $\bar{H}_k(P) \to \bar{H}(P)$ as $k \to \infty$.

Solving the Elliptic PDE

Two possibilities to solve

$$\nabla \cdot \left(e^{kH(x,P+\nabla u_k)} H_p(x, P+\nabla u_k)\right) = 0$$

have been tested:

- standard finite elements,
- finite differences for

$$k\left(H_x(x,\nabla u_k) + \nabla^2 u_k H_p(x,\nabla u_k)\right) H_p(x,\nabla u_k) + \operatorname{tr} H_{xp}(x,\nabla u_k) + H_{pp}(x,\nabla u_k) : \nabla^2 u_k = 0.$$

Using standard finite elements

Ansatz space: $X_{\mathscr{G}} := \{u \in \mathscr{L}_q(\mathscr{G}) \,|\, \langle u \rangle = 0\}$

Find $u \in X_{\mathscr{G}}$ subject to periodic boundary conditions such that

$$a(u,\varphi) := \int_{\mathbb{T}^d} e^{kH(x,P+\nabla u)} H_p(x,P+\nabla u) \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \forall \varphi \in X_{\mathscr{G}}.$$

Linearization of a at some point $u \in X_{\mathscr{G}}$:

$$\delta a(u,v,\varphi) = \int_{\mathbb{T}^d} e^{kH} \nabla v^{\mathsf{T}} \left(k\, H_p H_p^{\mathsf{T}} + H_{pp}\right) \nabla\varphi \,\mathrm{d}x$$

Problem: Newton's method breaks down for $k > 10$.

A Finite Difference Approximation

We use the finite difference approximations

$$\partial_x^{\Delta x} = \frac{1}{2\Delta x}(u_{i+1,j} - u_{i-1,j}),$$
$$\partial_{xx}^{\Delta x} = \frac{1}{\Delta x^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}),$$
$$\partial_{xy}^{\Delta x} = \frac{1}{4\Delta x^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})$$

and define

$$L_{i,j} = H_p(x_{i,j}, \nabla_{\Delta x} u_{i,j}) H_p^{\mathsf{T}}(x_{i,j}, \nabla_{\Delta x} u_{i,j}) + \frac{1}{k} H_{pp}(x_{i,j}, \nabla_{\Delta x} u_{i,j})$$
$$R_{i,j} = H_x^{\mathsf{T}}(x_{i,j}, \nabla_{\Delta x} u_{i,j}) H_p(x_{i,j}, \nabla_{\Delta x} u_{i,j}) + \frac{1}{k} \operatorname{tr} H_{xp}(x_{i,j}, \nabla_{\Delta x} u_{i,j}),$$

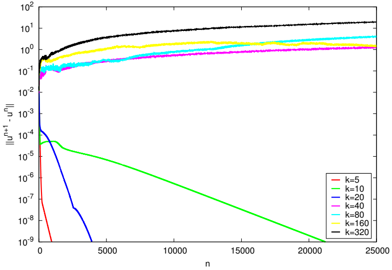
to obtain the discrete equation $L_{i,j} : \nabla^2_{\Delta x} u_{i,j} + R_{i,j} = 0$.

Iterative Solver

Iterative solution of $L_{i,j} : \nabla^2_{\Delta x} u_{i,j} + R_{i,j} = 0$:

$$u^{n+1}_{i,j} = \frac{\Delta x^2}{2\,\mathrm{tr}\,L^n_{i,j}} \left[L^n_{i,j} : \left(\nabla^2_{\Delta x} u^n_{i,j} + \frac{2u^n_{i,j}}{\Delta x^2}\mathbb{I} \right) + R^n_{i,j} \right].$$

Bad News:
Stability only for $k < C(\Delta x)$.



$\Delta x = \frac{1}{32}$

Stabilizing the Solver

Idea: Add a grid-dependent viscosity term:

$$(L_{i,j} + \Delta x\, \beta_{i,j}\, \mathbb{I}) : \nabla^2_{\Delta x} u_{i,j} + R_{i,j} = 0.$$

We choose

$$\beta_{i,j} = \max\Big\{ \beta_{min}, \tfrac{1}{2} \big| \nabla_p \big(H_x^{\mathsf{T}}(x_{i,j}, \nabla_{\Delta x} u_{i,j}) H_p(x_{i,j}, \nabla_{\Delta x} u_{i,j}) \big) \big| \Big\}.$$

Good News: In numerical tests, the solver remains stable even for $k = \infty$.

But: Can we justify this approach?

The Limit $k \to \infty$

Theorem (Evans, 2003)

Let $H \in \mathrm{C}^2(\mathbb{T}^d \times \mathbb{R}^d)$ *be strictly convex and [...] and let* $u \in \mathrm{C}^0(\mathbb{T}^d)$ *be a limit point of the sequence* $(u_k)_{k>0}$ *of solutions to*

$$\nabla \cdot \left(e^{kH(x,P+\nabla u)} H_p(x, P + \nabla u)\right) = 0.$$

Then, u is a viscosity solution of the Aronsson equation

$$-H_p(x, \nabla u) H_p^\mathsf{T}(x, \nabla u) : \nabla^2 u = H_x(x, \nabla u) \cdot H_p(x, \nabla u).$$

Evaluation of $\bar{H}$

Question: How do we obtain $\bar{H}$ for $k = \infty$?

For $k < \infty$, we have the formula

$$\bar{H}_k(P) := \frac{1}{k} \ln \int_{\mathbb{T}^d} e^{kH(x, P + \nabla u_k)} \, \mathrm{d}x.$$

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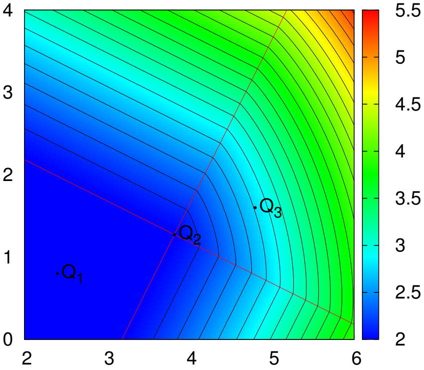
Assuming everything is smooth and passing to the limit we find

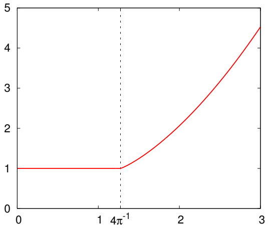
$$\bar{H}(P) = \max_{x \in \mathbb{T}^d} H(x, P + \nabla u_\infty).$$

A Test Problem

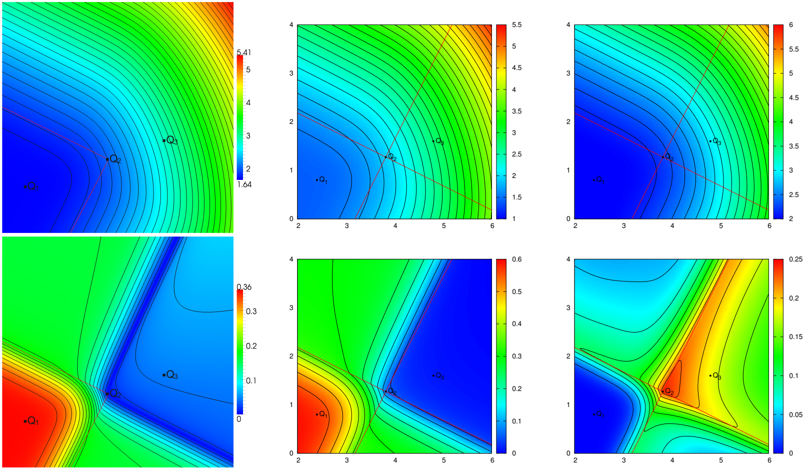
$$H(x,p) = \frac{1}{10}|p|^2 - \cos\big(2\pi(2x_1 + x_2)\big) - \cos\big(2\pi(2x_2 - x_1)\big)$$



eff. Hamiltonian $\bar{H}$



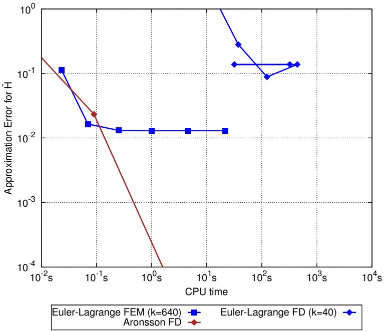
eff. Hamiltonian for 1d pendulum



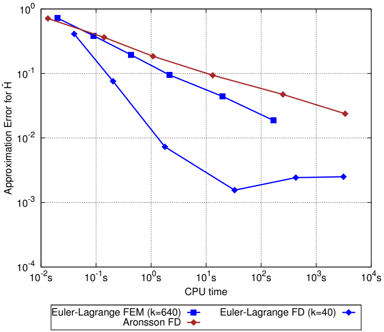
FE, $k = 12$ FD, $\beta_{i,j} = 0$, $k = 6$ FD, $\beta_{i,j} > 0$, $k = \infty$

Numerical Results

$$H(x,p) = \frac{1}{10}|p|^2 - \cos\big(2\pi(2x_1 + x_2)\big) - \cos\big(2\pi(2x_2 - x_1)\big)$$



in $Q_1 = (\frac{12}{5}, \frac{4}{5})$



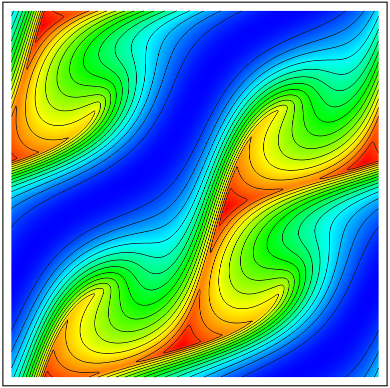
in $Q_3 = (\frac{24}{5}, \frac{8}{5})$

Questions

Can we ...

- ... prove the method to approximate $\bar{H}$?
- ... find a better (e.g., monotone) discretization?
- ... justify (or even minimize) the viscosity $\beta_{i,j}$?
- ... use a more efficient solver (e.g., a Newton method)?

The End



Thank you for your attention!