Adaptive two-three layer modelling of stratified flows

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joint with

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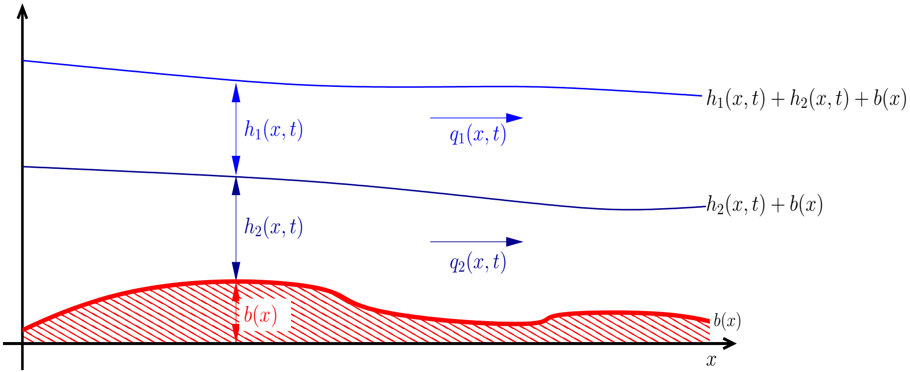
Padua, Hyp2012

Outline

Outline

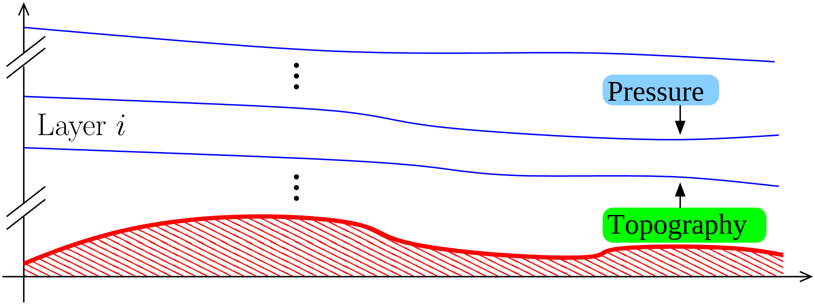
2-layer flows



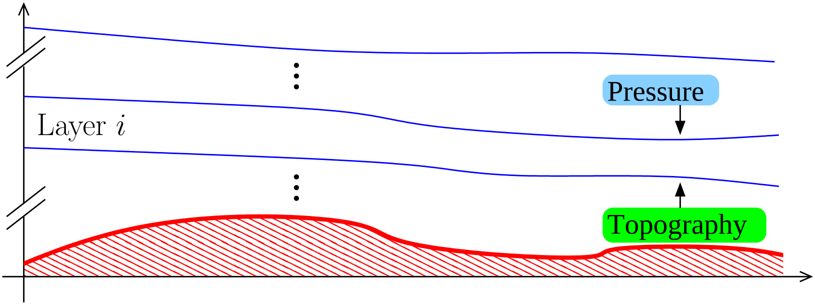
Variables:

- heights h_i, discharges q_i
- densities ρ_i
- bottom topography b

System of equations



System of equations

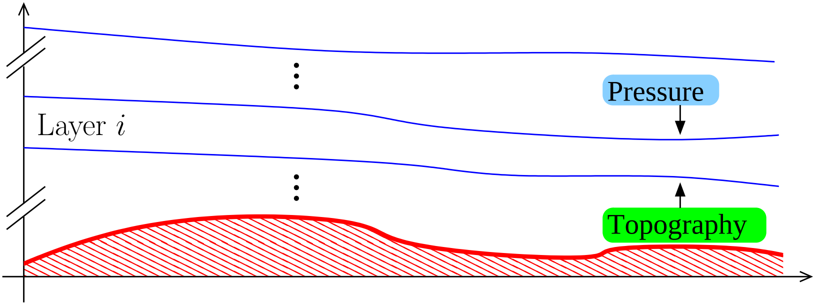


For each layer i, one has the equations

$$
\begin{aligned}
\partial_t h_i + \partial_x q_i &= 0, \\
\partial_t q_i + \partial_x \left(\frac{q_i^2}{h_i} + \frac{g}{2} h_i^2 \right) &= -g h_i \left(\sum_{k<i} r_{ki} \partial_x h_k + \sum_{k>i} \partial_x h_k + \partial_x b \right),
\end{aligned}
$$

with $r_{ki} = \frac{\rho_k}{\rho_i}$.

System of equations



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Outline

Approximations (Schijf/Schönfeld)

External and *internal* eigenvalues:

$$
\begin{aligned}
\lambda_{ext}^{\pm} &\approx U_{avg} \pm (g(h_1 + h_2))^{\frac{1}{2}} \\
\lambda_{int}^{\pm} &\approx U_{con} \pm \left(g' \frac{h_1 h_2}{h_1 + h_2} \left[1 - \frac{(u_1 - u_2)^2}{g'(h_1 + h_2)} \right] \right)^{\frac{1}{2}},
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$$

where $g' = (1 - r)g$, U_{con} and U_{avg} weighted averages of velocities.
Accuracy of approximation depends on r and $|u_1 - u_2|$.

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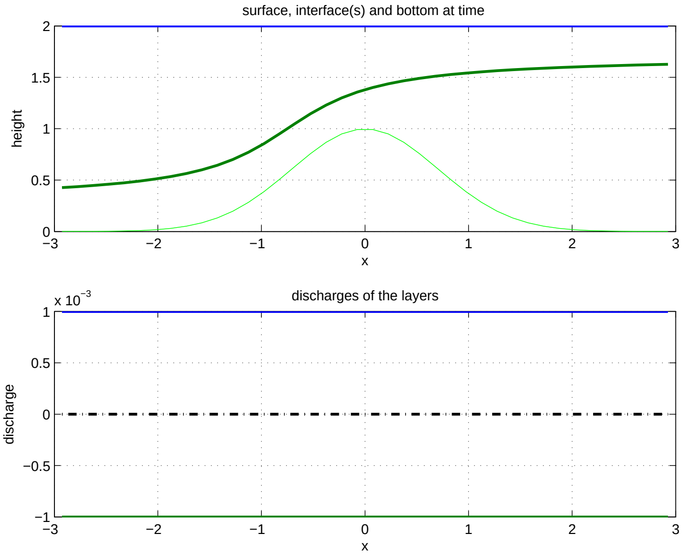
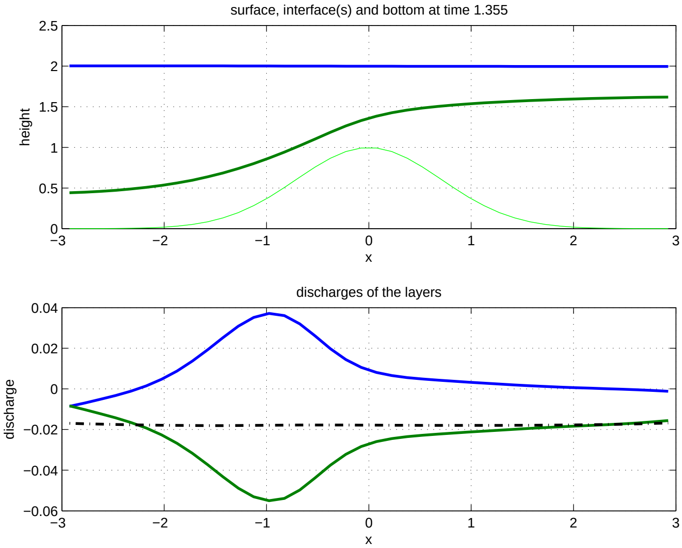
Approximative criterion for loss of hyperbolicity:

$$\kappa := \frac{(u_1-u_2)^2}{g'(h_1+h_2)} > 1 \, .$$

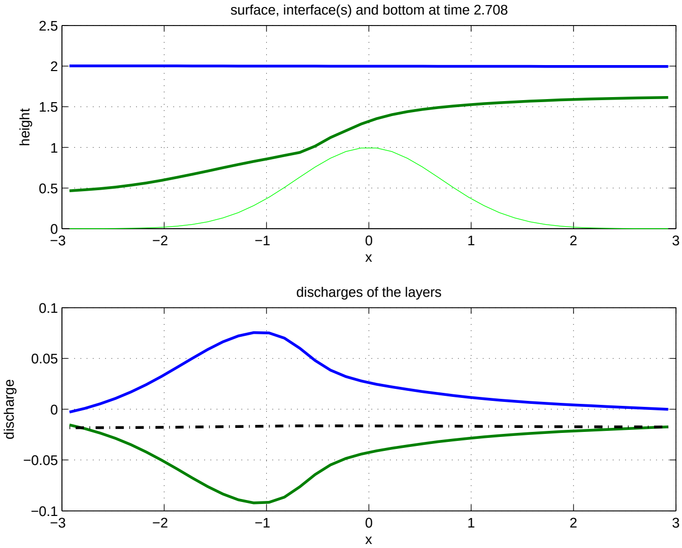
Breakdown of hyperbolicity



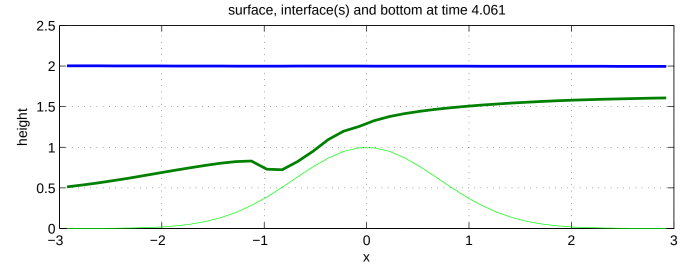
Breakdown of hyperbolicity

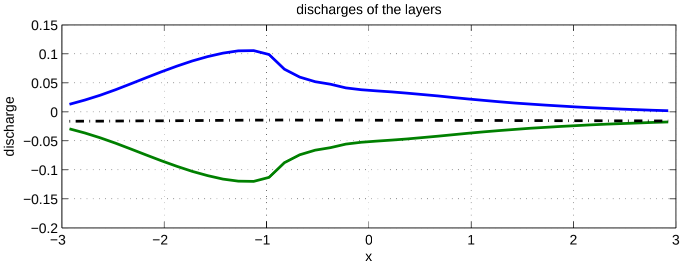


Breakdown of hyperbolicity

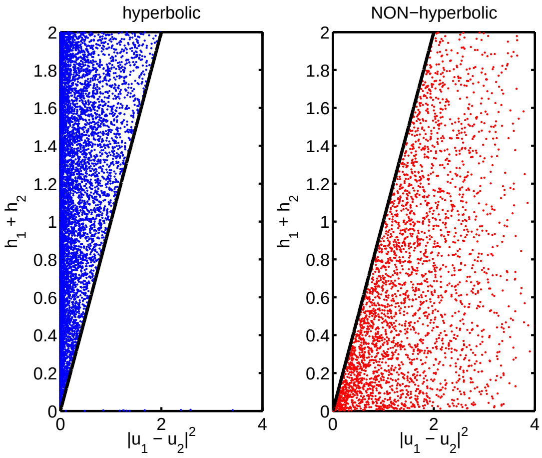


Breakdown of hyperbolicity





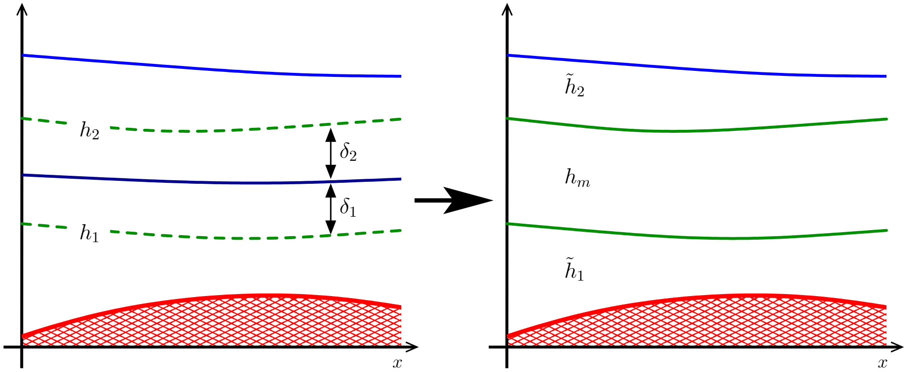
Hyperbolic region for original system



$$\kappa := \frac{(u_1 - u_2)^2}{g'(h_1 + h_2)} > 1 \, .$$

Outline

Introducing an intermediate layer



6 values (h_i, q_i, ρ_i) $\rightsquigarrow$ 9 values

Transformation formulas

Assumptions:

- mass (M), momentum (Q), total height (H) preserved
- velocities: $u_1 = \tilde{u}_1$, $u_2 = \tilde{u}_2$
- densities: $\rho_1 = \tilde{\rho}_1$, $\rho_2 = \tilde{\rho}_2$

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Formula (e.g.):

$$\rho_m = \frac{\rho_1 + \rho_2}{2} \Rightarrow \qquad \begin{aligned} \tilde{h}_1 &= h_1 - \frac{1}{2} h_m, \\ \tilde{h}_2 &= h_2 - \frac{1}{2} h_m, \\ u_m &= \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 + \rho_2}. \end{aligned}$$

Analyzing the transformation

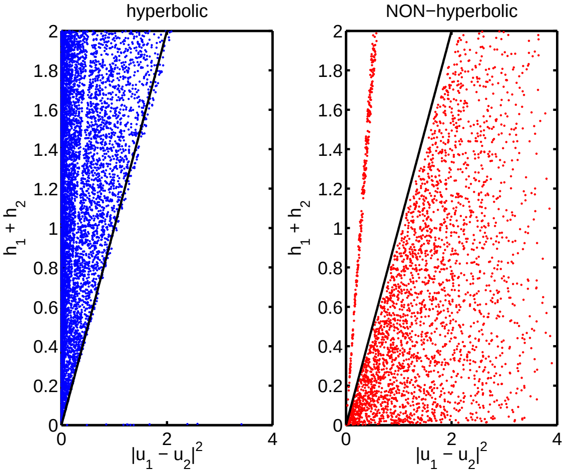
Take $h_m = h_m^{max} \rightsquigarrow$ 3-layers degenerate into 2 (or 1) layers:

$$\begin{aligned} \tilde{\kappa} &= \frac{(\tilde{u}_1 - u_m)^2}{\tilde{g}'(\tilde{h}_1 + h_m)} \\ &= \frac{\rho_2}{\rho_1 + \rho_2} \cdot \frac{(u_1 - u_2)^2}{g'(h_1 + h_2)} \approx \frac{1}{2}\kappa, \end{aligned}$$

since $\rho_1 \approx \rho_2$.

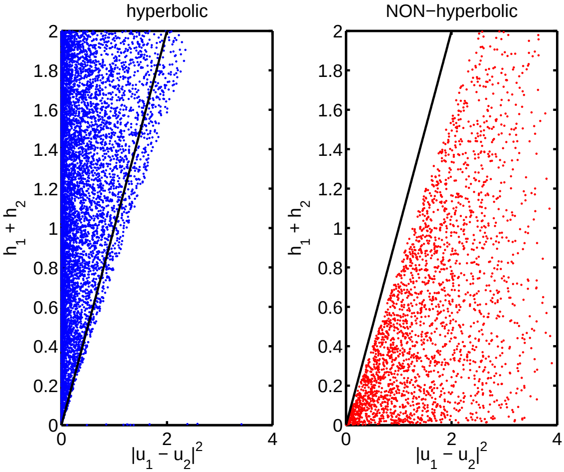
Hyperbolic region for 3-layer system

$$\text{ratio } \theta := \frac{h_2}{h_1+h_2} \approx 0.11$$



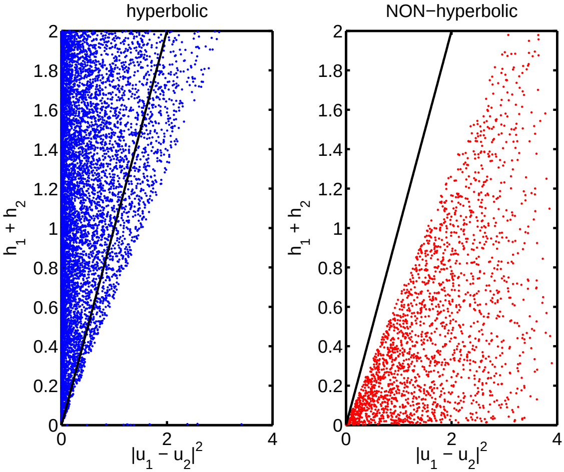
Hyperbolic region for 3-layer system

ratio $\theta := \frac{h_2}{h_1+h_2} \approx 0.22$



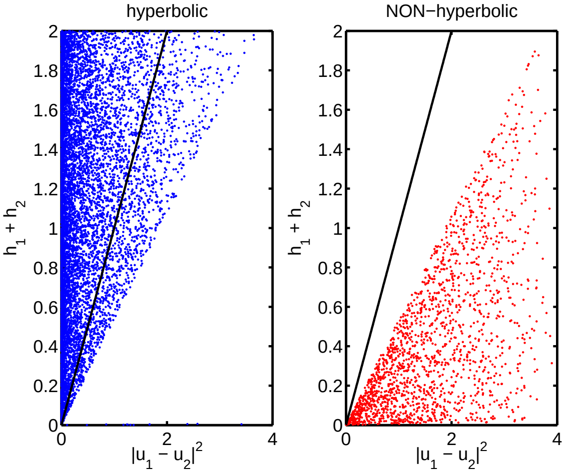
Hyperbolic region for 3-layer system

$$\text{ratio } \theta := \frac{h_2}{h_1 + h_2} \approx 0.33$$

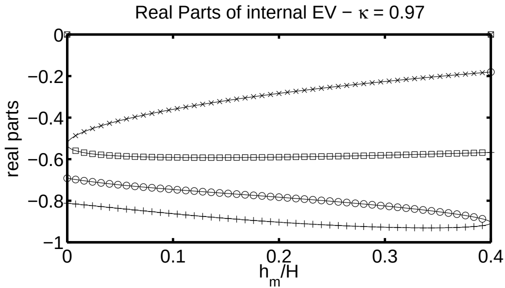


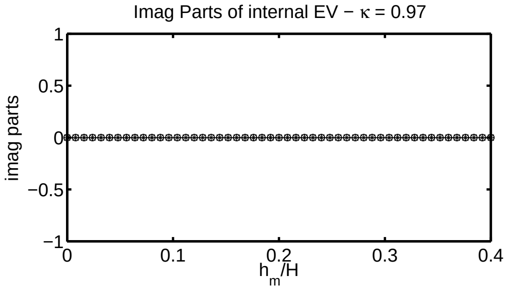
Hyperbolic region for 3-layer system

$$\text{ratio } \theta := \frac{h_2}{h_1+h_2} \approx 0.44$$



Internal eigenvalues





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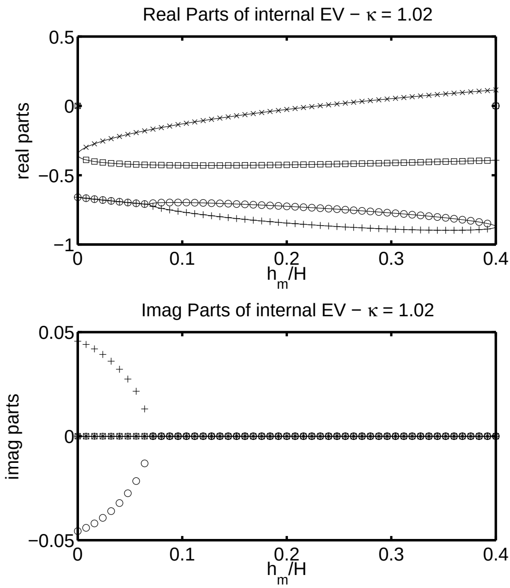
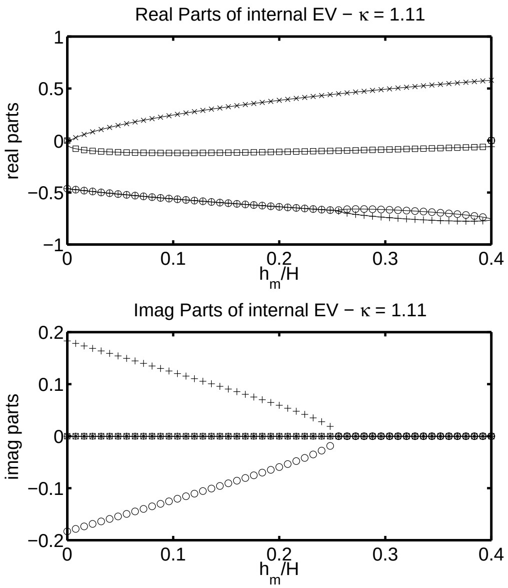


Internal eigenvalues



Optimization of h_m

- Two-layer indicator:

$$\delta(U) := |\Re(\lambda_{int}^{+}) - \Re(\lambda_{int}^{-})| - |\Im(\lambda_{int}^{+}) - \Im(\lambda_{int}^{-})|,$$

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Optimization of h_m

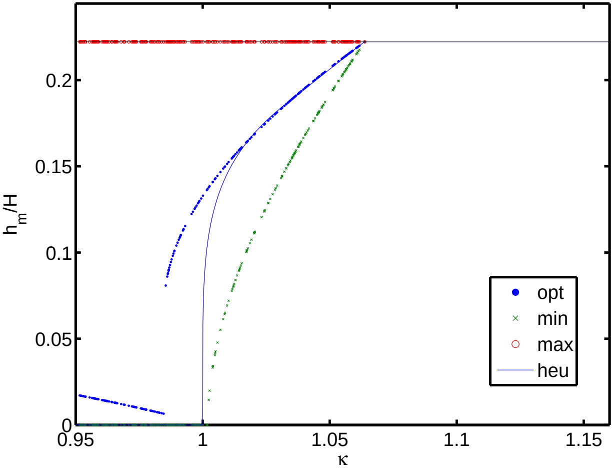
- Two-layer indicator:

$$\delta(U) := |\Re(\lambda_{int}^{+}) - \Re(\lambda_{int}^{-})| - |\Im(\lambda_{int}^{+}) - \Im(\lambda_{int}^{-})|,$$

- Multilayer indicator: minimum over all pairs of eigenvalues
- Optimization in the range $\mathcal{I}_{h_m} = (0, h_m^{\max}) = (0, 2\,\min(h_1, h_2))$.

Optimal layer heights for 3-layer system

h_m/H vs. κ for ratio $\theta := \frac{h_2}{h_1+h_2} \approx 0.11$



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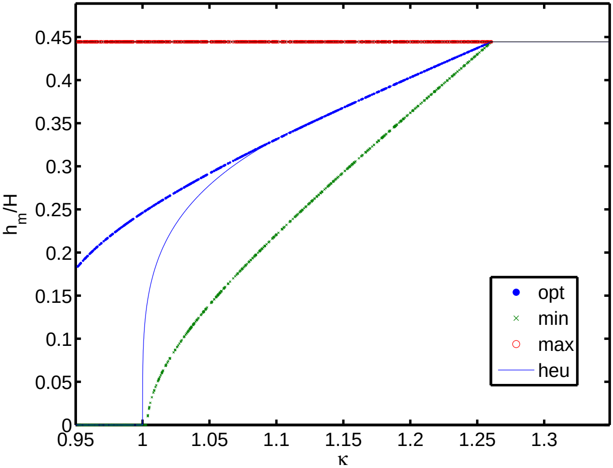

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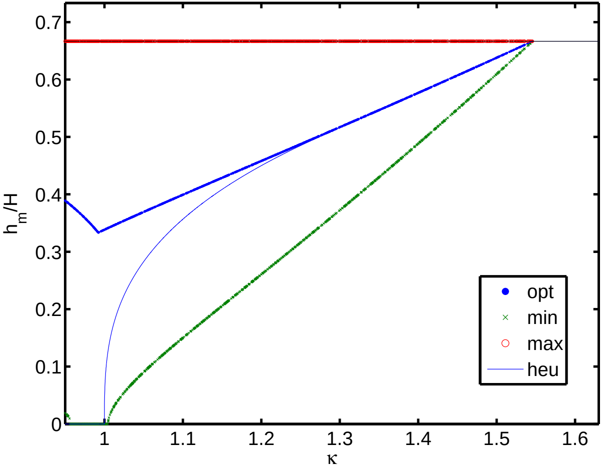

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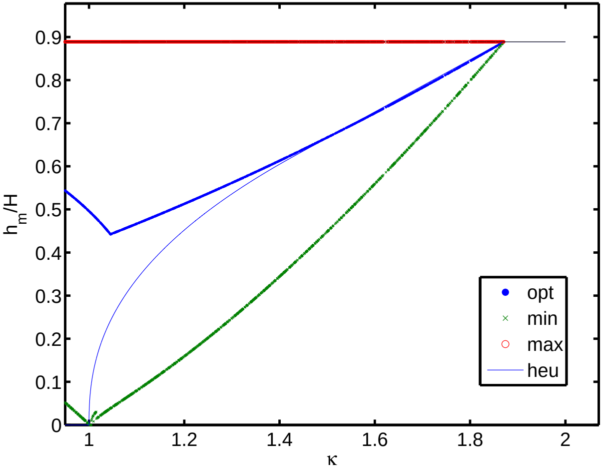
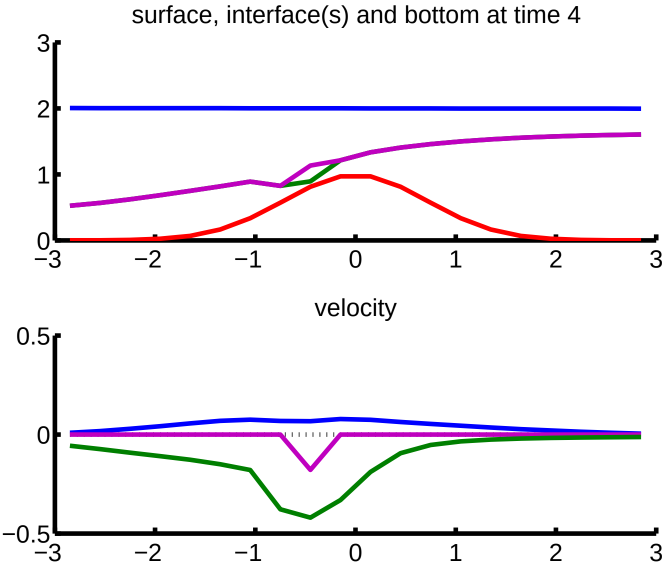


Outline

Numerical results



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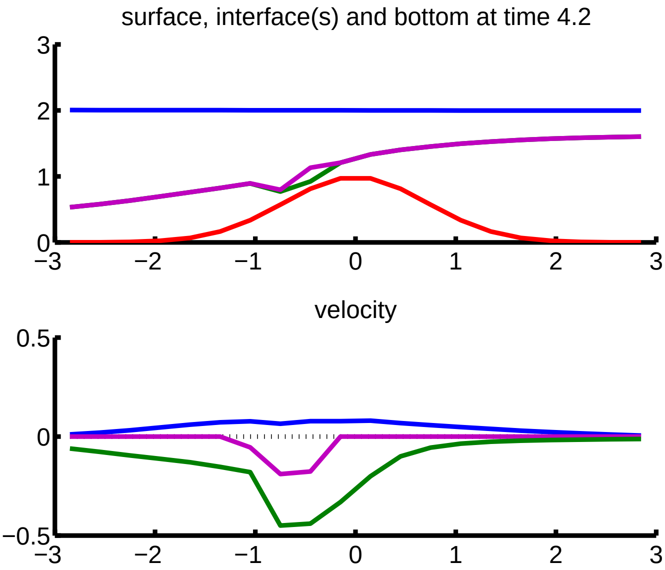

Outline

Equations

For the *i*th layer:

$$\begin{aligned} \partial_t h_i + \partial_x q_i &= 0, \\ \partial_t q_i + \partial_x \left(\frac{q_i^2}{h_i} + \frac{g}{2} h_i^2 \right) &= -g h_i \left(\sum_{k<i} r_{ki} \partial_x h_k + \sum_{k>i} \partial_x h_k + \partial_x b \right) \\ &\quad + \mu_i \left(\partial_z u(\eta_{i-1}, t) - \partial_z u(\eta_i, t) \right) \end{aligned}$$

with μ_i kinematic viscosity coefficients (+ turbulent viscosity),
$\eta_i = \sum_{j=1+i}^{N} h_j + b$ (cf. Audusse, 2005).

Sublayers and profiles

Split ith layer into $N_{sub,i}$ sublayers:

$$
\begin{aligned}
z_{i,j} &= \eta_{i-1} - j \cdot \frac{h_i}{N_{sub,i}} \\
z_{i,j-\frac{1}{2}} &= \eta_{i-1} - (j + \frac{1}{2}) \cdot \frac{h_i}{N_{sub,i}}, \qquad j = 1, \dots, N_{sub,i}
\end{aligned}
$$

Attach velocities $u_{i,j}$ to midpoints, discretize z-derivatives:

$$\partial_z u(z_{i,j}, t) \approx \frac{u_{i,j-1} - u_{i,j}}{z_{i,j-1-\frac{1}{2}} - z_{i,j-\frac{1}{2}}}$$

Viscosity matrix

Within *i*th layer:

$$\frac{u_{i,j}^{n+1} \cdot (h_i^* / N_{sub,i}) - q_{i,j}^*}{\Delta t}$$
$$= \mu_i \left(\frac{u_{i,j-1}^{n+1} - u_{i,j}^{n+1}}{z_{i,j-1-\frac{1}{2}} - z_{i,j-\frac{1}{2}}} - \frac{u_{i,j}^{n+1} - u_{i,j+1}^{n+1}}{z_{i,j-\frac{1}{2}} - z_{i,j+1-\frac{1}{2}}} \right)$$

Viscosity matrix

Within ith layer:

$$\frac{u_{i,j}^{n+1} \cdot (h_i^* / N_{sub,i}) - q_{i,j}^*}{\Delta t} = \mu_i \left(\frac{u_{i,j-1}^{n+1} - u_{i,j}^{n+1}}{z_{i,j-1-\frac{1}{2}} - z_{i,j-\frac{1}{2}}} - \frac{u_{i,j}^{n+1} - u_{i,j+1}^{n+1}}{z_{i,j-\frac{1}{2}} - z_{i,j+1-\frac{1}{2}}} \right)$$

$$\Leftrightarrow \quad u_{i,j}^{n+1} \cdot (h_i^* / N_{sub,i}) - \Delta t \mu_i \left(\frac{u_{i,j-1}^{n+1} - u_{i,j}^{n+1}}{z_{i,j-1-\frac{1}{2}} - z_{i,j-\frac{1}{2}}} - \frac{u_{i,j}^{n+1} - u_{i,j+1}^{n+1}}{z_{i,j-\frac{1}{2}} - z_{i,j+1-\frac{1}{2}}} \right) = q_{i,j}^*$$

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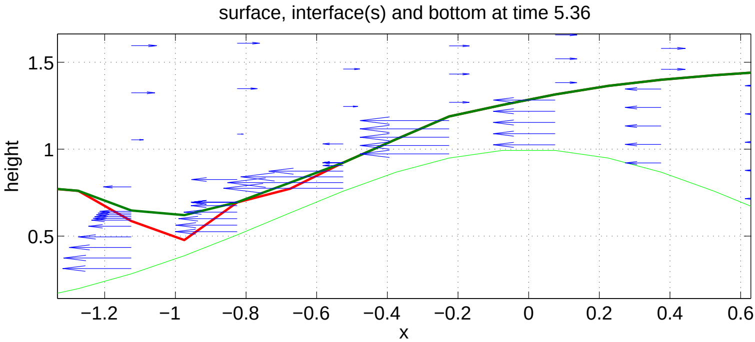
$$\Leftrightarrow \quad u_{i,j}^{n+1} \cdot (h_i^* / N_{sub,i}) - \Delta t \mu_i \left(\frac{u_{i,j-1}^{n+1} - u_{i,j}^{n+1}}{z_{i,j-1-\frac{1}{2}} - z_{i,j-\frac{1}{2}}} - \frac{u_{i,j}^{n+1} - u_{i,j+1}^{n+1}}{z_{i,j-\frac{1}{2}} - z_{i,j+1-\frac{1}{2}}} \right) = q_{i,j}^*$$

$$\rightsquigarrow \quad A(h_\cdot^*, \Delta t) u_{\cdot,\cdot}^{n+1} = q_{\cdot,\cdot}^*.$$

Profiles


surface, interface(s) and bottom at time 5.36
height
1.5
1
0.5
−1.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
x

Algorithm

Timestep splitting:

- Solve coupled hyperbolic balance laws:

$$U^* = U^n - \frac{\Delta t}{\Delta x}\left(G^-_{\frac{1}{2}}(U^n, b) - G^+_{-\frac{1}{2}}(U^n, b)\right)$$

  - Introduce or enlarge layers if neccessary

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- Solve equation systems given by viscous terms:

$$A(h^*_\cdot, \Delta t)u^{n+1}_{\cdot,\cdot} = q^*_\cdot$$

$$\rightsquigarrow \quad \begin{pmatrix} h_i^{n+1} \\ q_i^{n+1} \end{pmatrix} = \frac{h_i^*}{N_{sub,i}} \sum_j \begin{pmatrix} 1 \\ u_{i,j}^{n+1} \end{pmatrix} \rightsquigarrow U^{n+1}.$$

  - Store velocity profile

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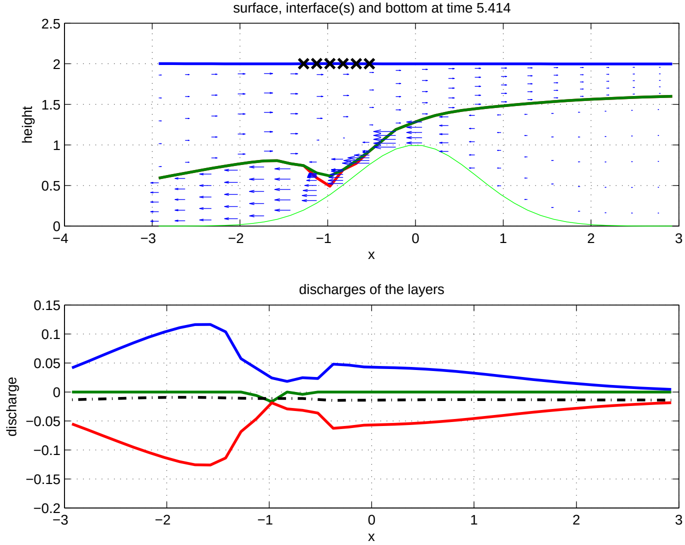
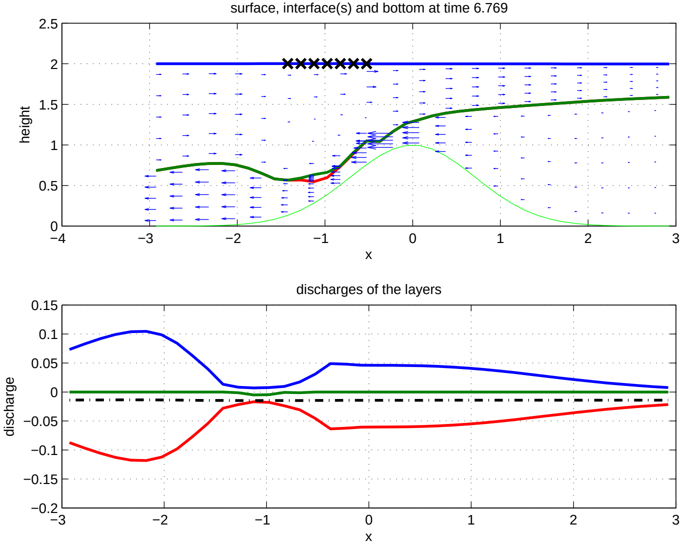
  - Store velocity profile
- Dissolve layers if possible

Outline

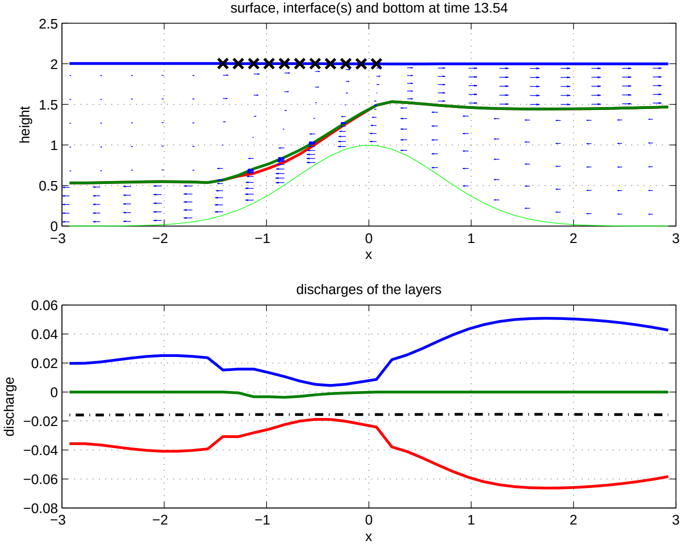
Test case 1


surface, interface(s) and bottom at time 5.414
height
x
discharges of the layers
discharge
x

Test case 1

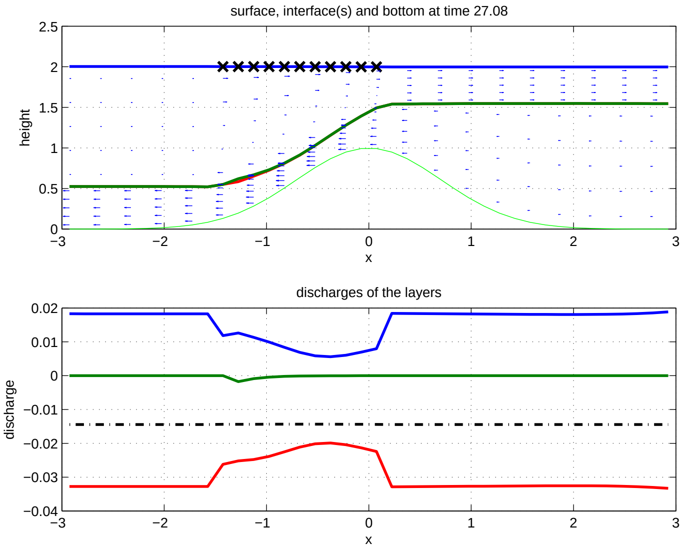

surface, interface(s) and bottom at time 6.769
height
x
2.5
2
1.5
1
0.5
0
−4
−3
−2
−1
0
1
2
3
discharges of the layers
discharge
x
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−3
−2
−1
0
1
2
3

Test case 1

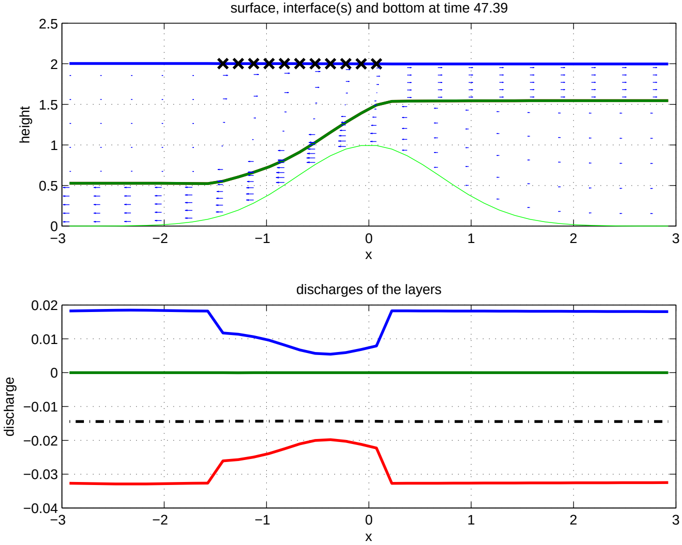


surface, interface(s) and bottom at time 13.54
height
x
discharges of the layers
discharge
x

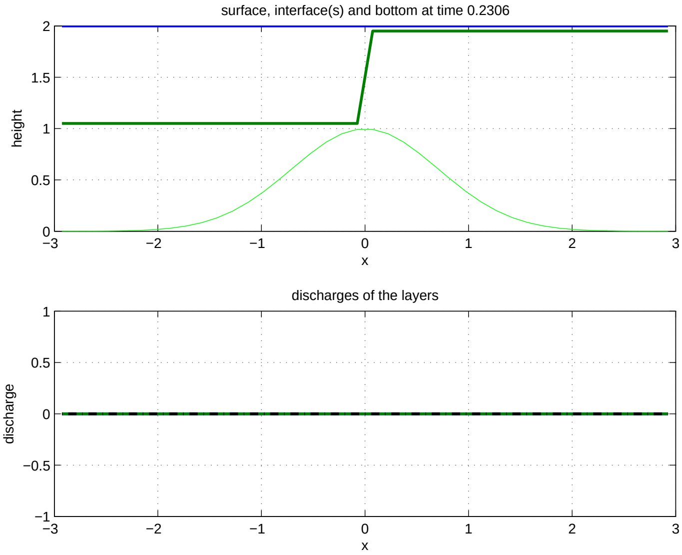
Test case 1


surface, interface(s) and bottom at time 27.08
height
x
discharges of the layers
discharge
x

Test case 1

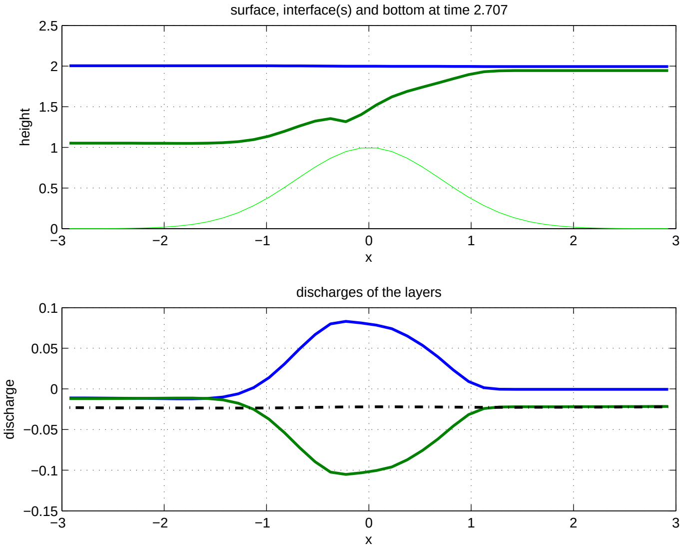

surface, interface(s) and bottom at time 47.39
height
x
discharges of the layers
discharge
x

Test case 2



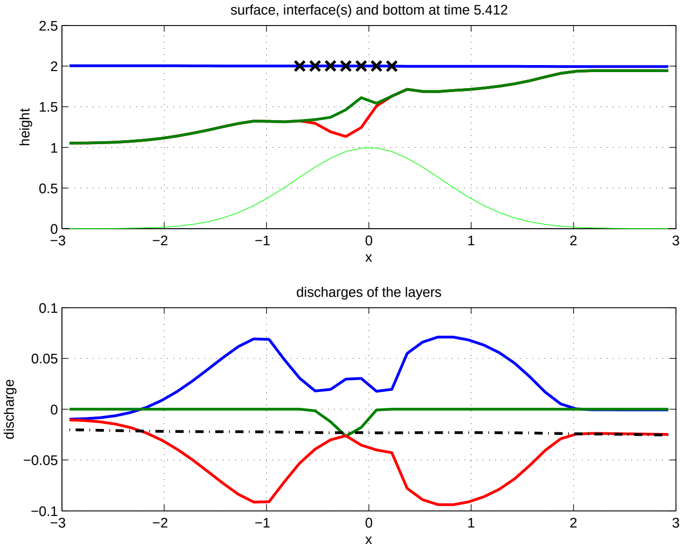
surface, interface(s) and bottom at time 0.2306
height
x
discharges of the layers
discharge
x

Test case 2

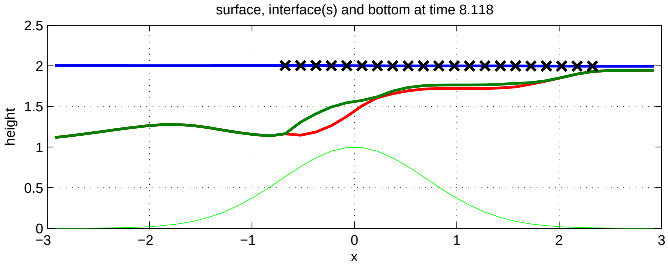


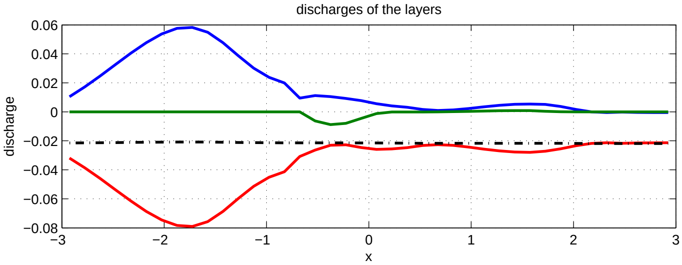
surface, interface(s) and bottom at time 2.707
height
2.5
2
1.5
1
0.5
0
−3
−2
−1
0
1
2
3
x
discharges of the layers
discharge
0.1
0.05
0
−0.05
−0.1
−0.15
−3
−2
−1
0
1
2
3
x

Test case 2

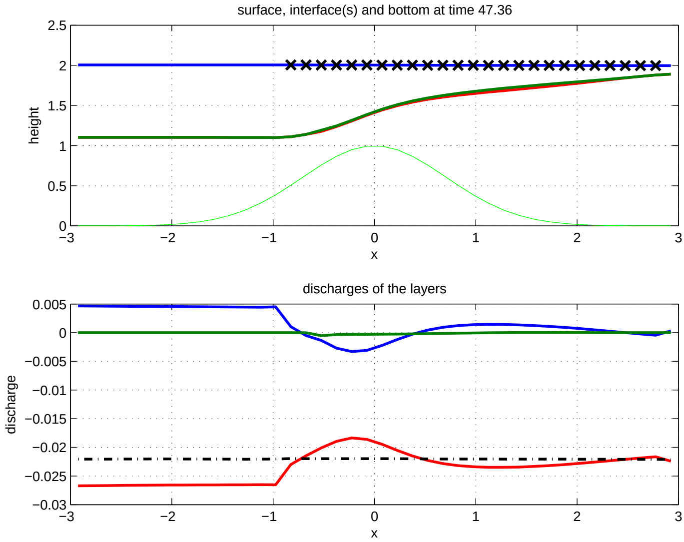

surface, interface(s) and bottom at time 5.412
height
2.5
2
1.5
1
0.5
0
−3
−2
−1
0
1
2
3
x
discharges of the layers
discharge
0.1
0.05
0
−0.05
−0.1
−3
−2
−1
0
1
2
3
x

Test case 2


surface, interface(s) and bottom at time 8.118
height
2.5
2
1.5
1
0.5
0
−3
−2
−1
0
1
2
3
x


discharges of the layers
discharge
0.06
0.04
0.02
0
−0.02
−0.04
−0.06
−0.08
−3
−2
−1
0
1
2
3
x

Test case 2



surface, interface(s) and bottom at time 47.36
height
x
discharges of the layers
discharge
x

Conclusion

Refined modeling:

- Introduction of intermediate layer
- velocity profiles (sublayers, Audusse et Al.)
- viscous and friction effects (Audusse et Al., Gerbeau/Perthame)

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- Coarsening (layer-wise)
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Thank you!