Compactness Estimates for Hyperbolic Conservation Laws

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Joint work with **Fabio Ancona** and **Olivier Glass**

Scalar conservation laws

A scalar conservation law

$$u_t + f(u)_x = 0 \tag{1}$$

where

- $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is the state variable,
- $f : \mathbb{R} \to \mathbb{R}$ is (uniformly) strictly convex

$$f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}.$$

Entropy solutions

Distributional weak solution of (1)

$$\int\int [u\varphi_t + f(u)\varphi_x] = 0 \quad \forall \varphi \in C^1_{\mathrm{loc}}. \tag{2}$$

Lax stability condition for admissibility

$$u(t,x-) \geq u(t,x+) \quad \text{for a.e } t > 0, \quad \forall x \in \mathbb{R}. \tag{3}$$

u is an entropy admissible weak solution of (1) if u satisfies (2) and (3).

Semigroup $(S_t)_{t\geq 0}$

Let

$$S_t : L^1(\mathbb{R}) \to L^1(\mathbb{R})$$

be the semigroup such that for every initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$u(t,\cdot) = S_t(u_0)$$

is the unique entropy admissible weak solution of (1).

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Kolmogorov's entropy

Lax's question: is it possible to give a quantitive estimate of the compactness of S_t ?

Kolmogorov's ε-entropy concept

Let (X, d) be a metric space and K a totally bounded subset of X. For $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε. Then the ε-entropy of K is definied as

$$H_\varepsilon(K \mid X) \;:=\; \log_2 N_\varepsilon(K).$$

Applications: rely on Kolmogorov's ε-entropy to:

- provide estimates on the accuracy and resolution of numerical methods.
- analyze computational complexity of conservation laws (derive number of needed operations to compute solutions with an error $< \varepsilon$).

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Problem

Given any $L, m, M > 0$, define

$$\mathcal{C}_{[L,m,M]} := \left\{u_0 \in \mathbf{L}^1(\mathbb{R}) \mid Supp(u_0) \subset [-L, L], \|u_0\|_{L_1} \leq m, \|u_0\|_{L^\infty} \leq M\right\}.$$

Our goal: Giving an estimate on

$$H_\varepsilon(S_T(C_{[L,m,M]})|\mathbf{L}^1(\mathbb{R})).$$

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For $\varepsilon > 0$ and $T > 0$, one has

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid \mathbf{L}^1(\mathbb{R})) \leq C \cdot \frac{1}{\varepsilon}.$$

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Sketch of proof

Key role: $f'' \geq c > 0 \implies$ Oleinik estimate

$$u(T,y) - u(T,x) \leq \frac{y-x}{c\,T}, \qquad x < y \qquad (u(T,x) \doteq S_T(u_0)(x)).$$

Equivalently, the function $x \mapsto \frac{x}{cT} - u(T,x)$ is nondecreasing.
Thus, $\frac{\cdot}{cT} - S_T(\mathcal{C}_{[L,m,M]})$ is a set of bounded and nondecreasing functions.

Lemma

For $L > 0$ and $V > 0$ set

$$\mathcal{I}_{L,V} \;=\; \{w : [0,L] \to [0,V] \mid w \text{ is nondecreasing}\}.$$

Then, for $0 < \varepsilon \leq \frac{LV}{6}$, the following holds:

$$H_{\varepsilon}(\mathcal{I}_{L,V} \mid L^1([0,L])) \leq \frac{1}{\varepsilon} \cdot 4LV.$$

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Lower estimates for $H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]})|L^1(\mathbb{R}))$.

Main steps towards lower estimates on $H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}))$

1. Controllability type result.
   Introduce a suitable parametrized class $\mathcal{F}$ of piecewise affine functions such that
   $$\mathcal{F} \subset S_T(C_{[L,m,M]}).$$

2. Combinatorial computation.
   Provide a lower estimate for $H_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R}))$. More precisely,
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Consider the set

$$\mathcal{A}_{[L,m,M,b]} = \Big\{ u_T \in BV(\mathbb{R}) \mid \mathrm{Supp}(u_T) \subset [-L, L], \|u_T\|_{L^1} \le m, \\ \|u_T\|_{L^\infty} \le M, Du_T \le b \Big\}.$$

($Du_T(J) \le b \cdot |J|$ for every Borel set $J \subset \mathbb{R}$)

Proposition 1.

Given any $L, M, m, T > 0$, for

$$h \le \min\left(M, \frac{m}{L}, \frac{L}{8T|f''(0)|}\right)$$

sufficiently small, one has

$$\mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]} \subset S_T(\mathcal{C}_{[L,m,M]}),$$

with $L_T = L - 2T|f''(0)|h$.

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Backward construction

GOAL: given $u_T \in \mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]}$, find $u_0 \in \mathcal{C}_{[L,m,M]}$ s.t. $S_T(u_0) = u_T$.

Reversing the direction of time

$$w_0(x) := u_T(-x), \quad w(t,x) := S_t(w_0)(x).$$

Set

$$u(t,x) \;:=\; w(T-t,-x), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Observe that

$$u(T,\cdot) = u_T(\cdot).$$

Moreover

- u is a weak distributional solution of our conservation law.
- estimates along generalized characteristics of w yield

$$u(0,\cdot) = w(T,-\cdot) \in \mathcal{C}_{[L,m,M]}.$$

$\Longrightarrow$ To prove $u_T \in S_T(\mathcal{C}_{[L,m,M]})$ it remains to show that $u(t,x)$ is entropy admissible.

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Lax admissibility cond'ns

AIM: show that $w(t,\cdot) \in \text{Lip}(\mathbb{R})$ ($\Longrightarrow$ $u(t,\cdot)$ satisfies Lax admissibility cond'ns $u(t,x-) \geq u(t,x+)$)

Notice: Oleinik's estimate:

$$w(t,y) - w(t,x) \leq \frac{1}{ct}(y-x), \quad x < y.$$

provide an upper bound on w_x.

$\Longrightarrow$ Look for a lower bound on w_x.

Observe that $u_T \in \mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]} \Longrightarrow Dw_0(]x,y[) \geq -\frac{1}{2T f''(0)}(y-x)$.

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Lower semi-Lipschitz estimate

The initial bound of the slope of $w_0(\cdot)$

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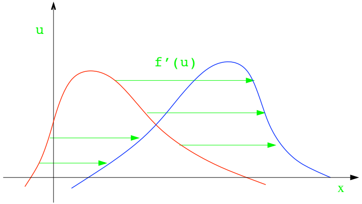
propagates over time interval $[0, T]$



$\Longrightarrow$ Lower semi-Lipschiz estimate of $w(t, \cdot)$ for all $t \in [0, T]$

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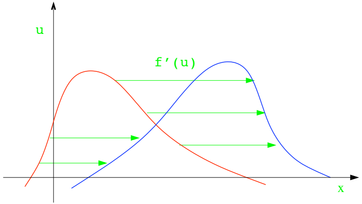

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Piecewise affine functions

Given $n \in \mathbb{Z}^+$ and $h > 0$, for every n-tuple $\iota = (\iota_i)_{i=0,1,\ldots,n-1} \in \{-1, 1\}^n$, we construct $\mathcal{F}_\iota$ as follows

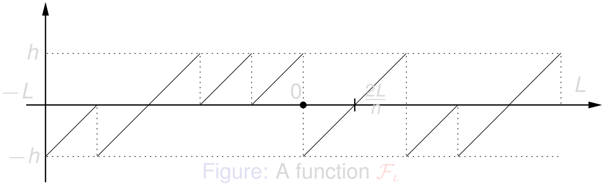

Figure: A function $\mathcal{F}_\iota$

$$h \leq M, \quad h \leq \frac{m}{L}, \quad \frac{nh}{L} \leq b$$

Observe that $\mathcal{F}_\iota \in \mathcal{A}_{[L,m,M,b]}$. Thus,

$$\mathcal{F} = \{\mathcal{F}_\iota \mid \iota \in \{-1,1\}^n\} \subset \mathcal{A}_{[L,m,M,b]}.$$

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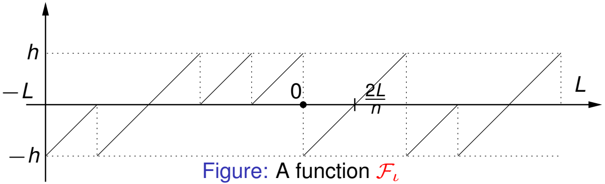

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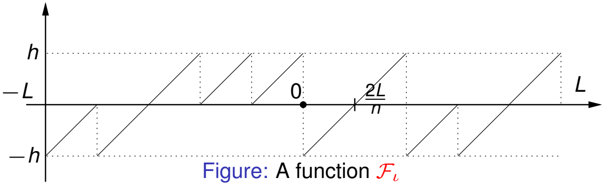

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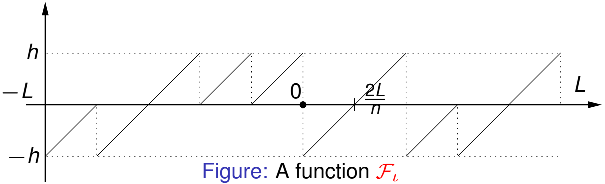

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Aim: Provide a lower estimate for $H_\epsilon(\mathcal{F} \mid L^1(\mathbb{R}))$.

Lower estimate for $H_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R}))$

For any $\iota, \bar\iota \in \{-1,1\}^n$, one has

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar\iota}\|_{L^1} \leq \frac{2hL}{n} d(\iota, \bar\iota).$$

where $d(\iota,\bar\iota) \doteq \text{Card}\,\{k \in \{1,\dots,n\} \mid \iota_k \neq \bar\iota_k\}$. It follows that

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar\iota}\|_{L^1} \leq \varepsilon \quad \Longleftrightarrow \quad d(\iota,\bar\iota) \leq \frac{n\varepsilon}{2hL}.$$

Therefore, for any fixed $\bar\iota \in \{-1,1\}^n$, let

$$C(\varepsilon) = \text{Card}\,\{\iota \in \{-1,1\}^n \mid \|\mathcal{F}_\iota - \mathcal{F}_{\bar\iota}\| \leq \varepsilon\}$$

We have

$$C(\varepsilon) = \sum_{\ell=0}^{\lfloor \frac{n\varepsilon}{2hL} \rfloor} \binom{n}{\ell} = \frac{1}{2^n}\mathbb{P}\left(X_1 + \dots + X_n \leq \lfloor \frac{n\varepsilon}{2hL} \rfloor\right)$$

where $X_1, \dots, X_n$ are indep. random variables with Bernoulli distribution $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$.

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Lower estimate for $H_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R}))$ ... continued

By Hoeffding's inequality, we obtain

$$C(\varepsilon) \leq 2^n \exp(-\frac{1}{\varepsilon})\frac{4bL^2}{27}.$$

Notice that: each element of an 2ε-cover of $\mathcal{F}$ contains at most $C(2\varepsilon)$ functions of $\mathcal{F}$.

Therefore, since the cardinality of $\mathcal{F}$ is 2^n, it follows that the number of sets in an ε-cover of $\mathcal{F}$ is at least

$$N_\varepsilon(\mathcal{F}) \geq \frac{2^n}{C(2\varepsilon)} \geq \exp\left(\frac{1}{\varepsilon} \cdot \frac{2bL^2}{27}\right).$$

$$\Longrightarrow \qquad H_\varepsilon(\mathcal{F} \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \cdot \frac{2bL^2}{27\ln(2)}.$$

Hence,

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Lower compactness estimates

Since $\mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]} \subset S_T(\mathcal{C}_{[L,m,M]})$, one obtains

F. Ancona, O. Glass and N–, CPAM (to appear)

Consider $f \in C^2(\mathbb{R}, \mathbb{R})$ such that $f''(u) \geq c > 0$ and $f'(0) = 0$. Given any L, m, M, define

$$\mathcal{C}_{[L,m,M]} := \left\{ u_0 \in L^1(\mathbb{R}) \mid Supp(u_0) \subset [-L, L], \|u_0\|_{L_1} \leq m, \|u_0\|_{L^\infty} \leq M \right\}.$$

Then, for any $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]} \mid L^1(\mathbb{R}))) \geq \frac{1}{\varepsilon} \frac{L^2}{48 \cdot \ln(2) \cdot |f''(0)| T}.$$

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$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]} \mid L^1(\mathbb{R}))) \approx \varepsilon^{-1}.$$

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Thank you for your attention.