Generalized Buckley-Leverett system

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Introduction

- Let us consider a pisto-like displacement of fluids in porous media.
- A displacement of oil by water.
- Piston-like means: The process of motion in porous media, always has two parts. One of oil (only) and other of water (only), i.e. immiscible.
- The liquids are also assumed incompressible.

Non-linear porous-media theory

The mathematical model is described by:

- $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d -$ points in the time-space domain.
- $m(\mathbf{x}) -$ porosity.
- $\mathbf{v}_i(t, \mathbf{x}) \in \mathbb{R}^d, (i = 1, 2) -$ seepage velocity field.
- $s_i(t, \mathbf{x}) \in \mathbb{R}, (i = 1, 2) -$ saturation of each i^{th} component,

$$0 \leq s_1, s_2 \leq 1, \quad s_1 + s_2 = 1. \tag{1}$$

Conservation of Mass (Continuity equation):

$$m\, \partial_t(\rho_i\, s_i) + \mathrm{div}_{\mathrm{x}}(\rho_i\, \mathbf{v}_i) = 0, \qquad (i = 1, 2) \tag{2}$$

where ρ_i is mass density of the i^{th}-phase of liquids.

From the incompressibility assumption, we have

$$\partial_t(s_i) + \mathrm{div}_x(\mathbf{v}_i) = 0, \qquad (i = 1, 2) \tag{3}$$

where we took $m \equiv 1$ without loss of generality.

Conservation of Linear Momenta (Darcy's law equation):

$$\frac{\mu_i}{k_0\, k_{ri}(s_1)}\mathbf{v}_i = -\nabla_x p_i + \rho_i\, g\, h, \quad (i = 1, 2) \tag{4}$$

where for each component $i = 1, 2$, $p_i(t, \mathbf{x})$ is the pressure, μ_i is the dynamic viscosity and k_{ri} is the relative permeability.
Moreover, $k_0(\mathbf{x})$ is the absolute permeability of the porous medium and $\rho\, g\, h$ is the external gravitational force, which is used dropped.

From the above considerations and, denoting $\lambda_i = \mu_i/(k_0\, k_{ri}(s_1))$ $(i = 1, 2)$, we have

$$\lambda_i\, \mathbf{v}_i = -\nabla_x p_i. \tag{5}$$

Then, from equations (3) and (5), we are in condition to formulate the Muskat problem.

One remarks that, in each part of the medium, where just exist one component, the saturation s_i is obviously constant and equals one.

Muskat problem (original formulation):

$$\begin{aligned} \lambda_o\, \mathbf{v}_o &= -\nabla_x p_o, \quad \mathrm{div_x}(\mathbf{v}_o) = 0, \quad \text{in } Q_o, \\ \lambda_i\, \mathbf{v}_w &= -\nabla_x p_i, \quad \mathrm{div_x}(\mathbf{v}_w) = 0, \quad \text{in } Q_w, \\ p_o &= p_w, \qquad \mathbf{v}_w \cdot n_t = \mathbf{v}_o \cdot n_t, \quad \text{on } \Gamma_t, \end{aligned} \tag{6}$$

where the under-scrip o, w stand respectively for oil and water. Further, we have assumed that, $p_c = 0$, i.e. the capillarity pressure is zero on Γ_t.

- Q_o, Q_w– regions of oil and water respectively.
- Γ_t– free boundary between Q_w and Q_o.
- n_t– unitary normal field to Γ_t.

Therefore, Muskat problem (original formulation) is a time-dependent elliptic-diffraction problem with a free-boundary.

Muskat problem (weak formulation):

First, we define on $Q_w \cup \Gamma_t \cup Q_o$:

- $u(t,x) := s_w(t,x)$, hence $s_o(t,x) = 1 - u(t,x)$.
- $\mathbf{v}(t,x) := \mathbf{v}_w(t,x) + \mathbf{v}_o(t,x)$ – total velocity.
- Moreover, we define the pressure p on $Q_w \cup \Gamma_t \cup Q_o$ as

$$p(t,x) := \begin{cases} p_w & \text{in } Q_w \cup \Gamma_t, \\ p_o & \text{in } Q_o. \end{cases} \tag{7}$$

After some algebraic computation, we have the following system, (also) called **Buckley-Leverett system**:

$$\begin{gathered} \partial_t u + \mathrm{div}_x(\mathbf{v}\, g(u)) = 0, \\ \mathrm{div}_x(\mathbf{v}) = 0, \quad h(u)\,\mathbf{v} = -\nabla_x p. \end{gathered} \tag{8}$$

Remarks:

- The Muskat problem (original formulation) was introduced in 1934 by Morris Muskat; Two fluid systems in porous media. The encroachment of water into oil sand, Physics Vol. 5.
- The Muskat problem (weak formulation) appears for the first time into the paper: Weak formulation of a multidimensional Muskat problem, written by L. Jiang and Z. Chen, 1990. On that paper are proved without functional formalism that, both formulations are equivalent.
- The Muskat is an open problem! Let us look closer to the weak formalism.

From equation (8), it follows that, we have to deal with a scalar non-homogeneous conservation law

$$\partial_t u + \mathrm{div}_{\mathrm{x}} \varphi(t, x, u) = 0,$$

with

$$\varphi(t, x, p) = \mathbf{v}(t, x)\, g(p), \tag{9}$$

and $\mathbf{v}$ is expected to be just in L^2! Do not mind w.r.t. g and h regularities, used to be good enough...

The best existence (and pre-compactness) result for this type of equation is given be E. Panov, Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux, Arch. Rational Mech. Anal., 195 (2010), 643–673.

One of the hypothesis on that important paper is $\max_{\lambda \in [a,b]} |\varphi(t,x,\lambda)| \in L^p$ $(p > 2)$, which is not our case. (The regularity assumption is related to Murat's Interpolation Lemma).

Moreover, even if we have enough regularity, the paper uses an extension of the H-measure introduced by Luc Tartar, and also, the same idea of the localization property. In this way, φ should be non-degenerated, that is

$$\Big(\forall a, b \in \mathbb{R}, a < b\Big) \Big(\forall (\tau, \xi) \in \mathbb{R}^{d+1} \setminus \{0\}\Big)$$
$$\Big(\lambda \in (a,b) \Longrightarrow (\tau,\xi) \cdot \big(1, \partial_\lambda \varphi(t,x,\lambda)\big) \neq 0 \text{ for a.a. } (t,x)\Big).$$

The most simple and one of the most important examples of non-homogeneous flux functions is given by (9) and, it does not satisfy the above condition! Indeed, let us give a simple example.

Example: We consider the 2-dimensional case and denote $\mathbf{v} = (V_1, V_2)$. Therefore, we must have for $\mathcal{L}^3$-a.e. $(t,x) \in \mathbb{R} \times \mathbb{R}^2$

$$(1, \mathbf{v}(t,x)\, g'(\lambda)) \cdot (\tau, \xi) = \tau + g'(\lambda)\Big(V_1(t,x)\, \xi_1 + V_2(t,x)\, \xi_2 \Big) \neq 0,$$

for all $(\tau, \xi_1, \xi_2) \in \mathbb{R}^3, (\tau^2 + \xi_1^2 + \xi_2^2 \neq 0)$, and $\lambda \in (a,b) \subset \mathbb{R}, a < b$, which is false once one takes $\tau = 0$, $\xi_1 = \pm V_2$, $\xi_2 = \mp V_1$.

Therefore, we could not attack the existence of solution to Muskat problem on that way!

- Some ideas have been proposed to by-pass this difficult, but most of them focused in the saturation equation.
- N. Chemetov and WN considered a new idea. They proposed a generalized Darcy's law equation, in fact a regularization of the standard one, i.e.

$$\tau\,\partial_t \mathbf{v} - \nu\,\Delta \mathbf{v} + h(u)\,\mathbf{v} = -\nabla p, \tag{10}$$

  where τ, ν are small positive parameters. They introduced a Generalized Buckley-Leverett system.
- The new formulation proposed brings enough regularity of the seepage velocity field. They showed solvability of the generalized system using the idea of Kinetic Theory.

Functional notation

Let $T > 0$ be a real number, $\Omega \subset \mathbb{R}^d$ (with $d = 1, 2$ or 3) an open and bounded domain having a C^2- smooth boundary Γ. We define

$$\Omega_T := (0, T) \times \Omega, \qquad \Gamma_T := (0, T) \times \Gamma.$$

We use standard notations for the Lebesgue function space $L^p(\Omega)$, the Sobolev spaces $W^{s,p}(\Omega)$ and $H^s(\Omega) \equiv W^{s,2}(\Omega)$.

The vector counterparts of these spaces are denoted by $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$ and $\mathbf{H}^s(\Omega) := (H^s(\Omega))^d$.

For any $\mathbf{u} \in \mathbf{L}^2(\Omega)$, satisfying $\operatorname{div}(\mathbf{u}) = 0$ in $\mathcal{D}'(\Omega)$, the normal component of $\mathbf{u}$, i.e. $\mathbf{u_n} := \mathbf{u} \cdot \mathbf{n}$, exists and belongs to $H^{-1/2}(\Gamma)$.

We will also use the following divergence free spaces

$$\begin{aligned}
\mathbf{V}^s(\Omega) \quad &: \quad = \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \mathrm{div}\,(\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\; d\mathbf{x} = 0\}, \\
\mathbf{V}^s(\Gamma) \quad &: \quad = \{\mathbf{u} \in \mathbf{H}^s(\Gamma) : \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\; d\mathbf{x} = 0\}, \\
\mathbf{V}^{-s}(\Gamma) \quad &: \quad = (\mathbf{V}^s(\Gamma))'.
\end{aligned}$$

We define by $\mathbf{u}_\tau := \mathbf{u} - \mathbf{u}_\mathbf{n}\mathbf{n}$ the tangent component for some $\mathbf{u}$ defined on Γ,

$$\begin{aligned}
\mathbf{G}^1(\Gamma_T) \quad : \quad = \Big\{ &\mathbf{u} \in L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) : \; \mathbf{u}_\tau \in H^{1/2}(0,T;\mathbf{V}^{-1/2}(\Gamma)), \\
&\mathbf{u}_\mathbf{n} \in H^{3/4}(0,T;\mathbf{V}^{-1}(\Gamma)\Big\},
\end{aligned}$$

$$\mathbf{S}^1(\Gamma_T) := \Big\{ \mathbf{u} \in L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma) \Big\}.$$

Statement of the Buckley-Leverett problem

We are concerned with the following initial-boundary value problem, denoted as **IBVP**:
Find a pair $(u, \mathbf{v}) = (u(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x})) : \Omega_T \to \mathbb{R} \times \mathbb{R}^d$ *solution to the generalized Buckley-Leverett system in the domain* Ω_T

$$\partial_t u + \operatorname{div}\big(\mathbf{v}\ g(u)\big) = 0, \tag{11}$$

$$\tau\, \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + h(u)\mathbf{v} = -\nabla p, \quad \operatorname{div}(\mathbf{v}) = 0, \tag{12}$$

satisfying the boundary conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T, \tag{13}$$

and the initial conditions

$$(u, \mathbf{v}) = (u_0, \mathbf{v}_0) \quad \text{in } \Omega. \tag{14}$$

We assume that our data satisfies the following regularity properties

$$\begin{aligned} g,\ h &\in W^{1,\infty}_{loc}(\mathbb{R}), && \text{such that} \quad 0 < h_0 \leq h(u), && (15)\\ u_b &\in L^\infty(\Gamma_T), && \text{such that} \quad 0 \leq u_b \leq 1 \quad \text{on } \Gamma_T, \\ u_0 &\in L^\infty(\Omega), && \text{such that} \quad 0 \leq u_0 \leq 1 \quad \text{in } \Omega, && (16) \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}_0 &\in \mathbf{V}^0(\Omega) \quad \text{and} \quad \mathbf{b} \in \mathbf{G}^1(\Gamma_T), \quad \text{such that} \\ \mathbf{b}(0)\cdot\mathbf{n} &= \mathbf{v}_0\cdot\mathbf{n} \quad \text{in} \quad H^{-1/2}(\Gamma). \end{aligned} \tag{17}$$

Now, since equation (11) is a hyperbolic scalar conservation law, the saturation function u may admit shocks. Therefore, in order to select the more correct physical solution, we need the entropy concept as given at the following

Definition

A pair $\mathbf{F}(u) := (\eta(u), q(u))$ *is called an entropy pair for* (11), *if* $\eta : \mathbb{R} \to \mathbb{R}$ *is a continuous differentiable and also convex function and* $q : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$q'(u) = \eta'(u)\, g'(u) \quad \text{for all } u \in \mathbb{R}. \tag{18}$$

Moreover, we say that $\mathbf{F}(u)$ *is a generalized convex entropy pair if it is a uniform limit of a family of convex entropy pairs over compact sets.*

Here, we consider the following parameterized family of entropy pairs for (11)

$$\mathbf{F}(u,v) = \Big(|u-v|, \operatorname{sgn}(u-v)\big(g(u)-g(v)\big)\Big) \tag{19}$$

for each $v \in \mathbb{R}$. Another two examples of parameterized family of entropy pairs for (11), which will be useful in the Kinetic formulation to be used latter are

$$\mathbf{F}^{\pm}(u,v) = \Big((u-v)^{\pm}, \operatorname{sgn}\big((u-v)^{\pm}\big)\big(g(u)-g(v)\big)\Big) \tag{20}$$

for each $v \in \mathbb{R}$, where

$$(u-v)^{+} := \max\{(u-v),0\}, \qquad (u-v)^{-} := \min\{(u-v),0\}.$$

Definition

A pair $(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0,T;\mathbf{V}^1(\Omega))$ *is called a generalized solution to the* **IBVP**, *if the pair* $(u, \mathbf{v})$ *satisfies, for all test functions* ϕ, ψ

$$\iint_{Q_T} \big(|u - v| \, \phi_t + \mathrm{sgn}(u-v)(g(u) - g(v)) \, \mathbf{v} \cdot \nabla \phi \big) \, d\mathbf{x} \, dt + M \int_{\Gamma_T} |u_b(r) - v| \phi(r) \, d\mathcal{H}^n(r) + \int_\Omega |u_0 - v| \phi(0) \, d\mathbf{x} \geq 0, \tag{21}$$

$$\int_{\Omega_T} \{ \tau \, \mathbf{v} \cdot \psi_t - \nu \, \nabla \mathbf{v} : \nabla \psi - h(u) \, \mathbf{v} \cdot \psi \} \, d\mathbf{x} dt + \tau \int_\Omega \mathbf{v}_0 \cdot \psi(0) \, d\mathbf{x} = 0. \tag{22}$$

The above definition is motivated by Theorem 4.1 in Chen and Frid [1]. Then, we have the following

Theorem

If the data g, h, u_b, u_0, $\mathbf{v_0}$, $\mathbf{b}$ fulfills the regularity properties (16)-(17), *then the initial-boundary value problem* **IBVP**: (11)-(14) *has a generalized solution* $(u, \mathbf{v})$, *satisfying*

$$\begin{aligned} 0 &\leqslant u \leqslant 1 \qquad a.\ a.\ in \quad \Omega_T, \\ \mathbf{v} &\in C(0,T;\mathbf{V}^0(\Omega)) \cap L^2(0,T;\mathbf{V}^1(\Omega)) \cap H^1(0,T;\mathbf{V}^{-1}(\Omega)). \end{aligned}$$

Parabolic approximation

In order to show existence of a generalized solution to (11)–(14), we study first the following approximated parabolic associated system

$$\partial_t u^\varepsilon + \operatorname{div}\big(\mathbf{v}^\varepsilon \, g(u^\varepsilon)\big) = \varepsilon \, \Delta u^\varepsilon, \tag{23}$$

$$\tau \, \partial_t \mathbf{v}^\varepsilon - \nu \Delta \mathbf{v}^\varepsilon + h(u^\varepsilon)\mathbf{v}^\varepsilon = -\nabla p^\varepsilon, \quad \operatorname{div}(\mathbf{v}^\varepsilon) = 0. \tag{24}$$

joint with the $(u_b^\varepsilon, \mathbf{b}^\varepsilon)$ and $(u_0^\varepsilon, \mathbf{v}_0^\varepsilon)$ respectively regularized boundary and initial data satisfying suitable compatibility conditions on Γ and at $t = 0$.

From well-known theory for parabolic and Navier-Stokes type equations, see Ladyzhenskaya et all [2], using Schauder's fixed point argument, based on the maximum principle for u^ε, i.e.

$$0 \leq u^\varepsilon \leq 1 \qquad \text{a.e. in } \Omega_T,$$

we get (as now a standard procedure) the solvability of the approximated system (23)–(23) and establish the following

Proposition

For each $\varepsilon > 0$, *there exists a unique solution* $(u^\varepsilon, \mathbf{v}^\varepsilon)$ *of the system* (23)–(23), *which satisfies the following properties*

$$\begin{aligned} u^\varepsilon \in L^\infty(0,T;H^1(\Omega)) \, \cap \, L^2(0,T;H^2(\Omega)), \quad \textit{such that} \\ \|\sqrt{\varepsilon}\,\nabla u^\varepsilon\|_{L^2(\Omega_T)} \leq C, \quad 0 \leq u^\varepsilon \leq 1, \quad \textit{a.e. on } \Omega_T, \end{aligned} \tag{25}$$

and

$$\mathbf{v}^\varepsilon \in L^2(0,T;\mathbf{V}^1(\Omega)) \, \cap \, H^1(0,T;\mathbf{V}^{-1}(\Omega)), \quad \textit{such that}$$

$$\|\sqrt{\tau}\,\mathbf{v}\|_{C([0,T];\mathbf{V}^0(\Omega))} + \|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \tag{26}$$

$$+\|\sqrt{\tau}\,\mathbf{v}\|_{H^1(0,T;\mathbf{V}^{-1}(\Omega))} \leqslant C, \tag{27}$$

where C *is a positive constant independent of* ε *and* τ.

The limit transition on $\varepsilon \to 0^+$

- The limit transition on $\varepsilon \to 0^+$ is obtained from the Kinetic Theory.
- One remarks that, the standard $W^{1,1}$ or BV estimates seems to be not possible to derive.
- Special attention should be done on the values of the kinetic function on the boundary of Γ_T.

The main idea. Sketch of the proof

Let $(\eta(u), q(u))$ be a entropy pair for (23). Then, we have in sense of distributions

$$\partial_t \eta(u^\varepsilon) + \mathrm{div}(\mathbf{v}^\varepsilon q(u^\varepsilon)) - \varepsilon\, \Delta\eta(u^\varepsilon) = -\varepsilon\, \eta''(u)\, |\nabla\eta(u)|^2.$$

Since η is a convex function, then

$$\partial_t \eta(u^\varepsilon) + \mathrm{div}(\mathbf{v}^\varepsilon q(u^\varepsilon)) - \varepsilon\, \Delta\eta(u^\varepsilon) \leq 0.$$

For instance, we could take the entropy pair $(\eta(u), q(u)) = \mathbf{F}^+(u, v)$ for all $v \in \mathbb{R}$, given in (20). Then, we have in sense of distributions

$$\partial_t (u-v)^+ + \mathrm{div}(\mathbf{v}^\varepsilon \,\mathrm{sgn}(u-v)^+ \big(g(u)-g(v)\big)) - \varepsilon\, \Delta(u-v)^+ = -m_+^\varepsilon, \tag{28}$$

where m_+^ε is a real nonnegative Radon measure given by

$$m_+^\varepsilon(t, \mathbf{x}, v) = \varepsilon\, |\nabla u^\varepsilon(t, \mathbf{x})|^2\, \delta_{v=u^\varepsilon(t,\mathbf{x})}([0, 1]).$$

Now, if we differentiate in the distribution sense (28) with respect to v, we get as now a standard procedure in the Kinetic formulation, the following linear transport equation

$$\partial_t f^\varepsilon + g'(v)\,\mathbf{v}^\varepsilon \cdot \nabla f^\varepsilon - \varepsilon\,\Delta f^\varepsilon = \partial_v m_+^\varepsilon, \tag{29}$$

where $f^\varepsilon(t,\mathbf{x},v) := \operatorname{sgn}(u^\varepsilon(t,\mathbf{x}) - v)^+$. Let us point out that, by definition

$$0 \leq f^\varepsilon(t,\mathbf{x},v) \leq 1 \qquad \text{in } \Omega_T \times \mathbb{R},$$

and since $0 \leq u^\varepsilon(t,\mathbf{x}) \leq 1$, if $v \notin [0,1]$, then $m_+^\varepsilon(t,\mathbf{x},v) \equiv 0$ in the distributional sense, that is, the support of m^ε is contained in $\Omega_T \times [0,1]$. Also, we have for each $v \in \mathbb{R}$

$$\iint_{\Omega_T} m_+^\varepsilon(t,\mathbf{x},v)\, d\mathbf{x}dt \leq \iint_{\Omega_T} |\sqrt{\varepsilon}\,\nabla u^\varepsilon|^2\, d\mathbf{x}dt \leq C,$$

i.e. m_+^ε is uniformly bounded with respect to $\varepsilon > 0$.

Passing to subsequences, if necessary, we obtain

$$\partial_t f + g'(v)\,\mathbf{v}\cdot\nabla f = \partial_v m_+ \qquad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \tag{30}$$

Moreover, we proved the strong traces for the initial-boundary values of the kinetic function. Indeed, we have the following

Proposition

The function $f = f(t, \mathbf{x}, v)$ *has the trace* f^0 *at the time* $t = 0$, *such that*

$$f^0 = f^0(\mathbf{x}, v) \equiv \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta f(s, \mathbf{x}, v)\, ds, \quad f^0 = \left(f^0\right)^2.$$

The function $f = f(t, \mathbf{x}, v)$ *has the trace* f^b *on* $\Gamma_T \times \mathbb{R}$, *such that*

$$f^b = f^b(t, \mathbf{x}, v) \equiv \lim_{\delta \to 0^+} \frac{1}{\delta} \int_0^\delta f(t, \mathbf{x} - s\, n(\mathbf{x}), v)\, ds, \quad f^b = \left(f^b\right)^2$$

for a. a. $(t, \mathbf{x}, v) \in \Gamma_T \times \mathbb{R}$, *where* $g'(v)\mathbf{b_n}(t, \mathbf{x}) \le 0$.

Similar result we obtain for $\eta(u) = (u - v)^-$, in particular

$$\partial_t(1-f) + g'(v)\,\mathbf{v}\cdot\nabla(1-f) = -\partial_v m_- \qquad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \quad (31)$$

Now, for each $\theta > 0$, let $\rho_\theta(\mathbf{x})$ be an even mollifier and denote by

$$z^\theta(\mathbf{x}) = (z * \rho_\delta)(\mathbf{x}) := \int_{\mathbb{R}^d} z(\mathbf{y})\,\rho_\theta(\mathbf{x}-\mathbf{y})\,d\mathbf{y}$$

be the regularization of $z = z(\mathbf{x})$ on the variable $\mathbf{x} \in \Omega$.

Then, mollifying (30) and (31), we obtain

$$\partial_t f^\theta + g'(v)\,\mathbf{v}\cdot\nabla f^\theta = \partial_v m_+^\theta + R_+^\theta \qquad \text{in } \mathcal{D}'(\Omega_T\times\mathbb{R}), \tag{32}$$

and

$$\partial_t(1-f^\theta)+g'(v)\,\mathbf{v}\cdot\nabla(1-f^\theta) = -\partial_v m_-^\theta + R_+^\theta \qquad \text{in } \mathcal{D}'(\Omega_T\times\mathbb{R}), \tag{33}$$

with $R_\pm^\theta \to 0$ in $L^1_{\rm loc}(\Omega_T\times(0,1))$ as $\theta\to 0^+$, in view of DiPerna-Lions' approach for Transport Equations (recall the regularity of the velocity field $\mathbf{v}$).

Now, we conveniently multiply (32), (33) respectively by $(1 - f^\theta)$, f^θ, and deduce that the function $F := f(1 - f)$, satisfies

$$\partial_t F + g'(v)\, \mathbf{v} \cdot \nabla F \leq 0 \qquad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \tag{34}$$

Therefore, applying the strong-trace results for the kinetic function, we obtain for almost all $t_0 \in [0, T]$

$$\iint_{\Omega \times \mathbb{R}} F \, d\mathbf{x} dv \leq 0.$$

Since $F \geq 0$, we deduce that $F \equiv 0$ a.e. on $\Omega_T \times \mathbb{R}$. Consequently,

$$f(t, \mathbf{x}, v) \in \{0, 1\} \qquad \text{a.e. on } \Omega_T \times \mathbb{R}. \tag{35}$$

The limit transition on $\tau \to 0^+$

For a given viscous parameter $\nu > 0$, we consider the following initial-boundary value problem, denoted as $\mathbf{IBVP}_{\tau=0}$:
Find a pair $(u, \mathbf{v}) = (u(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x})) : \Omega_T \to \mathbb{R} \times \mathbb{R}^d$ *solution to the quasi-stationary Stokes-Buckley-Leverett system in the domain* Ω_T

$$\partial_t u + \operatorname{div}\big(\mathbf{v}\, g(u)\big) = 0, \tag{36}$$

$$-\nu \Delta \mathbf{v} + h(u)\mathbf{v} = -\nabla p, \quad \operatorname{div}(\mathbf{v}) = 0, \tag{37}$$

satisfying the boundary conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T, \tag{38}$$

and the initial condition

$$u = u_0 \quad \text{in } \Omega. \tag{39}$$

Theorem

If the data g, h, u_b, u_0, $\mathbf{b}$ fulfills the regularity properties, then the $\mathbf{IBVP}_{\tau=0}$ *has a weak solution* $(u, \mathbf{v})$, *satisfying*

$$\begin{aligned} 0 &\leqslant u \leqslant 1 \qquad a.\ e.\ in \quad \Omega_T, \\ \mathbf{v},\ \partial_t \mathbf{v} &\in L^2(0, T; \mathbf{V}^1(\Omega)). \end{aligned}$$

Proposition

There exists a pair $(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0, T; \mathbf{V}^1(\Omega))$, *with* $\partial_t \mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, *and a subsequence of* $\{u^\tau, \mathbf{v}^\tau\}_{\tau>0}$, *such that*

$$u^\tau \rightharpoonup u \quad * \text{-weakly in } L^\infty(\Omega_T), \tag{40}$$

$$\mathbf{v}^\tau \to \mathbf{v} \quad \text{strongly in } L^2(\Omega_T). \tag{41}$$

The limit transition on $\nu \to 0^+$???

This is the last limit process to solve the Muskat Problem!
Recall the scalar conservation law, and using the incompressibility condition, we have

$$\partial_t u + \mathbf{v} \cdot \nabla_x g(u) = 0.$$

In fact, we should consider first the more simple case of Transport Equations, that is

$$\partial_t u + \mathbf{v} \cdot \nabla_x u = 0, \quad \mathrm{div}(\mathbf{v}) = 0, \tag{42}$$

to understand better the difficult involved.

- Chen G.-Q., Frid H., *Divergence measure fields and hyperbolic conservation laws.* Arch. Rational Mech. Anal. 147 (1999) 89–118.
- DiPerna R.J., Lions P.L., *Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math.* **98**, (1989) 511–547.
- Fursikov A., Simader M., Hou L., Trace theorems for the three-dimensional time-dependent solenoidal vector fields and their applications, Transactions of AMS, Vol. 354, N. 3, 1079-1116 (2001).

- GALDI G.P., SIMADER C.G., SOHR H., *On the stokes problem in Lipschitz domains, Annali di Matematica Pura ed Applicata*, Vol. 167, N. 1 (1994)
- HORNUNG U., *Homogenization and Porous Media*, Interdisciplinary Applied Math., Vol. 6, Springer, (1996).
- KUFNER A., JONH O., FUČIK S., *Function Spaces.* Noordholf Intern. Publishing, Leyden (1977).

- Ladyzhenskaya O.A., *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York-London, 1969.
- Ladyzhenskaya O.A., Solonnikov V.A., Ural'tseva N.N., *Linear and quasilinear equations of parabolic type.* American Mathematical Society, Providence RJ (1968).
- Ladyzhenskaya O.A., Uraltseva N.N., *Linear and quasilinear elliptic equations.* Academic Press, New York and London (1968).

- Lions P.-L., Perthame B., Tadmor E., *Kinetic formulation for isentropic gas dynamcs and p-systems.* Comm. Math. Phys. 163 (1994), 415–431.
- Lions P.-L., Perthame B., Tadmor E., *A kinetic formulation of multidimensional scalar conservation laws and related equations.* J. AMS 7 (1994), 169–191.
- Malek J., Necas J., Rokyta M., Ruzicka M., *Weak and measure-valued solutions to evolutionary PDEs.* Chapman&Hall, London (1996).

- NEVES W., *Scalar multidimensional conservation laws IBVP in noncylindrical Lipschitz domains,* Journal of Diff. Equations 192 (2003) 360–395.
- OTTO F., *Initial-boundary value problem for a scalar conservation law,* C.R. Acad. Sci. Paris 322 (1996) 729–734.
- PERTHAME B., DALIBARD A.-L., *Existence of solutions of the hyperbolic Keller-Segel model,* Trans. Amer. Math. Soc., **361**, 2319-2335 (2009).

- PERTHAME B., *Kinetic formulation of conservation laws,* Oxford University Press, 2002.
- RAYMOND J.P., *Stokes and Navier-Stokes equations with nonhomogeneous boundary conditions,* Ann. Inst. Henri Poincaré, Anal. Non Linéaire 24, 921-951 (2007).
- SCHEIDEGGER A.E., *Hydrodynamics in Porous Media*, Handbuch der Physik Vol. VIII/2, Flûgge, Springer, (1963).

- SCHEIDEGGER A.E., *The Physics of Flow Through Porous Media*, 3rd ed, University of Toronto Press, Toronto (1974).
- STRAUGHAN B., *Stability and Wave Motion in porous media,* Applied Math. Sciences Vol. 165, Springer, (2008).
- TEMAM R., *Navier-Stokes equations,* Theory and numerical analysis. AMS Chelsea publishing, Providence, Rhode Island (2001),