Singular solutions of a fully nonlinear 2x2 system of conservation law

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General system

Consider first a general 2x2 system of conservation laws

$$\begin{cases} \partial_t u + \partial_x f(u,v) = 0 \\ \partial_t v + \partial_x g(u,v) = 0 \end{cases} \tag{1}$$

with the Riemann initial data

$$u|_{t=0} = U_0(x) = \begin{cases} u_1, & x < 0 \\ u_2, & x > 0 \end{cases}, \quad v|_{t=0} = V_0(x) = \begin{cases} v_1, & x < 0 \\ v_2, & x > 0. \end{cases} \tag{2}$$

If the system is *genuinely nonlinear*, then the latter Riemann problem has a unique solution consisting of rarefaction waves and compressive shock waves (Lax admissible waves) if the states (u_1, v_1) and (u_2, v_2) are *close to each other*.

No solutions to certain Riemann problems without δ-solution concept.

B. L. Keyfitz, H. C. Krantzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Differential Equations 118 (1995) 420-451.

Glimm scheme blows up

C. Tsikkou, *Hyperbolic conservation laws with large initial data. Is the Cauchy problem well-posed?*, Quart. Appl. Math. **68** (2010), 765–781.

If the functions f and g are linear with respect to v, it has been confirmed that the system naturally admits δ-shock type solutions.
More precisely, the vanishing viscosity approximation to (1), (2) converges to a distribution fo the form:

$$\begin{aligned} u(x,t) &= U_0(x-ct), \\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct), \end{aligned} \tag{3}$$

for an appropriate constant c.
For details, see (K.T.Joseph (1992), Keyfitz and Kranzer (1995), Shelkovich (2006) etc...
Also, it can be shown that δ-shock naturally arises from smooth initial data along characteristics (Danilov and Mitrovic (2008)).

Solution concept

Question:
If we have a distribution of the form:

$$u(x,t) = U(x,t) + \alpha(t)\delta(x-ct),$$

when we can say that it is a solution to an equation

$$\partial_t u + \partial_x F(u) = 0. \tag{4}$$

- If $u \in C^1(\mathbb{R}^2)$ it is clear since all the operations in the previous expression are well defined;
- If $u \in L^1_{loc}(\mathbb{R}^2)$, it can still be a solution to (4) in a weaker sense (weak solution concept). It is defined so that the differentiation of the function u is avoided;
- If u contains δ (i.e. $e \neq 0$), we need to apply even weaker concept in which we shall avoid nonlinear operations on δ;

Proper generalization of the classical weak solution concept on δ-shock solution concept for system (1) was provided by Danilov and Shelkovich (2005) but for systems which are linear with respect to one of the variables. There are no obstacles for extending the definition on an arbitrary system of form (1) (Kalisch and Mitrovic (2012), to appear in PEMS).

Suppose $\Gamma = \{\gamma_i \mid i \in I\}$ is a graph in the closed upper half plane, containing Lipschitz continuous arcs γ_i, $i \in I$, where I is a finite index set. Let I_0 be the subset of I containing all indices of arcs that connect to the x-axis, and let $\Gamma_0 = \{x_k^0 \mid k \in I_0\}$ be the set of initial points of the arcs γ_k with $k \in I_0$. Define the singular part by $\alpha(x,t)\delta(\Gamma) = \sum_{i\in I} \alpha_i(x,t)\delta(\gamma_i)$. Let (u, v) be a pair of distributions represented in the form

$$\begin{aligned} u(x,t) &= U(x,t) + \alpha(x,t)\delta(\Gamma) \\ v(x,t) &= V(x,t) + \beta(x,t)\delta(\Gamma), \end{aligned} \tag{5}$$

and where $U, V \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Finally, the expression $\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}$ denotes the tangential derivative of a function φ on the graph γ_i, and $\int_{\gamma_i}$ connotes the line integral over the arc γ_i.

Definition

The pair of distributions (5) is called a generalized δ-shock solution of (1) with the initial data $U_0(x) + \sum_{I_0} \alpha_k(x_0^k, 0)\delta(x - x_k^0)$ and $V_0(x) + \sum_{I_0} \beta_k(x_0^k, 0)\delta(x - x_k^0)$ if it satisfies

$$\int_{\mathbb{R}_-}\int_{\mathbb{R}} (U\partial_t\varphi + f(U,V)\partial_x\varphi)\ dxdt \tag{6}$$

$$+\sum_{i\in I}\int_{\gamma_i} \alpha_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}} + \int_{\mathbb{R}} U_0(x)\varphi(x,0)\ dx + \sum_{k\in I_0}\alpha_k(x_k^0,0)\varphi(x_k^0,0) = 0,$$

$$\int_{\mathbb{R}_-}\int_{\mathbb{R}} (V\partial_t\varphi + g(U,V)\partial_x\varphi)\ dxdt \tag{7}$$

$$+\sum_{i\in I}\int_{\gamma_i} \beta_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}} + \int_{\mathbb{R}} V_0(x)\varphi(x,0)\ dx + \sum_{k\in I_0}\beta_k(x_k^0,0)\varphi(x_k^0,0) = 0,$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$.

Heuristics

Assume that $\Gamma = \{x = ct\}$ for a constant c. It holds

$$\partial_t u = ([U] + \alpha'(t))\,\delta(x - ct) - c\alpha(t)\delta'(x - ct)$$
$$\partial_t v = ([V] + \beta'(t))\,\delta(x - ct) - c\beta(t)\delta'(x - ct).$$

No matter how we define nonlinear operations f and g for the distributions u and v from (5), we must do it so that it holds

$$\partial_x f(u, v) \sim [f]\delta(x - ct) + c\alpha(t)\delta'(x - ct)$$
$$\partial_x g(u, v) \sim [g]\delta(x - ct) + c\beta(t)\delta'(x - ct).$$

In the framework of the last definition, we can prove that the Riemann problem (1), (2) always admits δ-shock solution.

a) If $u_1 \neq u_2$ then the pair of distributions

$$\begin{aligned} u(x,t) &= U_0(x-ct), \\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct), \end{aligned}$$

where

$$c = \frac{[f(U,V)]}{[U]} = \frac{f(u_2,v_2)-f(u_1,v_1)}{u_2-u_1}, \quad \text{and} \quad \alpha(t) = (c[V]-[g(U,V)])t,$$

represents the δ-shock wave solution of (1), (2).

b) If $v_1 \neq v_2$ then the pair of distributions

$$\begin{aligned} u(x,t) &= U_0(x-ct) + \alpha(t)\delta(x-ct), \\ v(x,t) &= V_0(x-ct), \end{aligned}$$

where

$$c = \frac{[g(U,V)]}{[V]} = \frac{g(u_2,v_2)-g(u_1,v_1)}{v_2-v_1}, \quad \text{and} \quad \alpha(t) = (c[U]-[f(U,V)])t$$

represents the δ-shock solution of (1), (2).

Justification of the definition

In order to justify the previous definition, in principle, we need to find a family of smooth solutions which approximately solve (1), and which tends to (3) in the sense of distributions (e.g. vanishing viscosity approximation). We formalize this demand via the weak asymptotic concept.

Definition

We say that the families of smooth complex-valued distributions (u_ε) and (v_ε) represent a weak asymptotic solution to (1) if there exist real-valued distributions $u, v \in C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}))$, such that for every fixed $t \in \mathbb{R}_+$

$$u_\varepsilon \rightharpoonup u, \quad v_\varepsilon \rightharpoonup v \ \text{ as } \ \varepsilon \to 0,$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$, and

$$\left.\begin{array}{rcl} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon, v_\varepsilon) & = & o_{\mathcal{D}'}(1), \\ \partial_t v_\varepsilon + \partial_x g(u_\varepsilon, v_\varepsilon) & = & o_{\mathcal{D}'}(1). \end{array}\right\} \tag{8}$$

Remark in particular that $\varepsilon u_{xx} = o_{\mathcal{D}'}(\varepsilon)$ i.e. the vanishing viscosity

Brio system

Let us consider the particular system, stemming from the MHD equations:

$$\begin{aligned} \partial_t u + \partial_x\left(\tfrac{u^2+v^2}{2}\right) &= 0, \\ \partial_t v + \partial_x(v(u-1)) &= 0. \end{aligned} \tag{9}$$

The system is strictly hyperbolic; it is genuinely nonlinear at $\{(u,v):\ u \in \mathbb{R},\ v > 0\}$ and $\{(u,v):\ u \in \mathbb{R},\ v < 0\}$, but not on the whole of $\mathbb{R}^2$.

For the Riemann initial data (2) such that $v_1 < 0 < v_2$, the system does not admit Lax admissible solutions. Moreover, for certain combination of the Riemann inital data, no weak solutions exist.

Using the complex corrections, we can obtain the weak asymptotic solution to the Brio system for any combination of the Riemann initial data. The weak asymptotic solution converges toward the δ-shock solution.

a) If $u_1 \neq u_2$ then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system, such that the families (u_ε) and (v_ε) have distributional limits

$$\begin{aligned} u(x,t) &= U_0(x-ct), \\ v(x,t) &= V_0(x-ct) + \alpha(t)\delta(x-ct), \end{aligned}$$

where $c = \frac{u_1^2+v_1^2-u_2^2-v_2^2}{2(u_1-u_2)}$ and
$\alpha(t) = \frac{1}{2}\left(c(v_2-v_1) + (v_1(u_1-1) - v_2(u_2-1))\right)t.$

b) If $v_1 \neq v_2$ then there exist weak asymptotic solutions (u_ε), (v_ε) of the Brio system, such that the families (u_ε) and (v_ε) have distributional limits

$$\begin{aligned} u(x,t) &= U_0(x-ct) + \alpha(t)\delta(x-ct), \\ v(x,t) &= V_0(x-ct), \end{aligned}$$

where $c = \frac{v_1(u_1-1)-v_2(u_2-1)}{v_1-v_2}$ and
$\alpha(t) = \left(c(u_2-u_1) + \frac{u_1^2+v_1^2-u_2^2-v_2^2}{2}\right)t.$

Uniqueness issues

As usual when passing to weaker solution concept, problem of uniqueness arises and additional demands must be imposed on the solution.

Usual additional demand for δ-shock solution is the overcompressivity condition:

$$\lambda_i(u_2, v_2) \leq c \leq \lambda_i(u_1, v_1), \quad i = 1, 2,$$

where λ_i are characteristic speeds corresponding to the system, and c is speed of the δ-shock connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$.

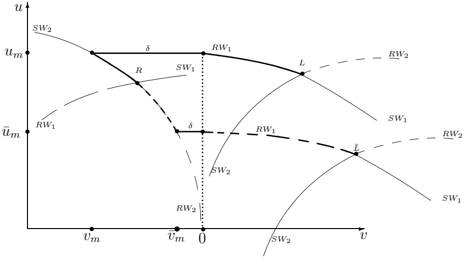
In the case of the Brio system, we are able to obtain only compressivity conditions, i.e.

Definition
A δ-shock solution of (9), connecting a left state $L = (u_1, v_1)$ and a right state $R = (u_2, v_2)$ is i-admissible if

$$\lambda_i(u_2, v_2) \le c \le \lambda_i(u_1, v_1), \tag{10}$$

for $i = 1$ or $i = 2$.

Theorem
Given any Rieman initial data (2) such that $v_2 < 0 < v_1$, there exists a δ-shock solution of (9) which consists of a combination of the classical Lax admissible simple waves (shock or rarefaction) and overcompressive 1-admissible δ waves.



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More on uniqueness

Compressivity does not provide uniqueness in the case of the Brio system. Moreover, a scalar conservation law admits infinitely many overcompressive δ-type solutions.

A natural way to avoid this kind of improperness is to demand minimal number of δ-shocks. More precisely:

Definition (Kalisch and Mitrovic (2012), to appear in IMA Journal of Applied Mathematics)
We say that the distribution of form (3) represents an admissible δ-type solution to (1) if

- the jumps of V and U from (3) (i.e. the regular part of (3)) satisfy the Lax admissibility conditions;
- the number of δ-supporting curves in (u, v) is minimal.

The End

Thank you for listening.