Inflow-Implicit/Outflow-Explicit Finite Volume Methods for Solving Advection Equations

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Joint work with

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• we present second-order scheme for solving equations

$$u_t + \mathbf{v} \cdot \nabla u = 0$$

$u \in \mathbb{R}^d \times [0, T]$ is an unknown function and $\mathbf{v}(x)$ is a vector field.

• basic idea - a second order (e.g. finite volume) scheme can be writen in a cell through the "forward and backward diffusion" contributions

• forward diffusion - inflow coefficients - implicit treatment, backward difusion - outflow coefficients - explicit treatment

• possible interpretation - we know what is flowing out from a cell at an old time step, but we leave the method to resolve a system of equations determined by the inflows to obtain a new value in the cell - outflow is treated explicitly while inflow is treated implicitly.

$$u_t + \mathbf{v} \cdot \nabla u = 0$$

- matrix of the system is determined by the inflow coefficients - it is **diagonally dominant M-matrix** yielding favourable solvability and stability properties

- method is **exact for any choice of time step** on uniform rectangular grids in the case of constant velocity transport of any quadratic function in any dimension

- it is formally (and also in numerical experiments) **second order accurate in space and time** for smooth solutions for any choice of time step

- **high-resolution stabilized versions** has accuracy at least 2/3 for solutions with shocks for any choice of time step

- it can be extended to $\mathbf{v} = \mathbf{v}(x, u, \nabla u)$ - level set methods, nonlinear hyperbolic equations, intrinsic PDEs for motion of curves and surfaces

Outline of the talk

- derivation of the scheme
- theoretical properties
- stabilization techniques - high resolution versions
- numerical experiments and comparisons
- applications - **image segmentation** by the level-set approach and **forest fire simulations** by the Lagrangean approach allowing topological changes

Derivation of IIOE scheme

- let us consider equation $u_t + \mathbf{v} \cdot \nabla u = 0$ either in 1D interval or in a bounded polygonal domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and time interval $[0, T]$

- let p be a finite volume (cell) with measure m_p and let e_{pq} be an edge between p and q, $q \in N(p)$, where $N(p)$ is a set of neighbouring finite volumes

- let us denote by u_p a (constant) value of the solution in a finite volume p computed by the scheme.

- let us rewrite the equation in the formally equivalent form with conservative and non-conservative parts

$$u_t + \nabla \cdot (\mathbf{v} u) - u \nabla \cdot \mathbf{v} = 0.$$

$$u_t + \nabla \cdot (\mathbf{v}u) - u\nabla \cdot \mathbf{v} = 0$$

$$\int_p u_t \; dx + \int_p \nabla \cdot (\mathbf{v}u) \; dx - \int_p u\nabla \cdot \mathbf{v} \; dx = 0$$

$$m_p \frac{\mathrm{d}\bar{u}_p}{\mathrm{dt}} + \sum_{q \in N(p)} \bar{u}_{pq} \int_{e_{pq}} \mathbf{v} \cdot n_{pq} \; ds - \bar{u}_p \sum_{q \in N(p)} \int_{e_{pq}} \mathbf{v} \cdot n_{pq} \; ds = 0$$

- where constant representations of the solution on the cell p is denoted by $\bar{u}_p$ and on the cell interfaces e_{pq} by $\bar{u}_{pq}$. Let us denote fluxes in the inward normal direction to the finite volume p by

$$\bar{v}_{pq} = -\int_{e_{pq}} \mathbf{v} \cdot n_{pq} \; ds$$

- we arrive at the equation

$$m_p \frac{\mathrm{d}\bar{u}_p}{\mathrm{dt}} + \sum_{q \in N(p)} \bar{v}_{pq}(\bar{u}_p - \bar{u}_{pq}) = 0$$

$$m_p \frac{\mathrm{d}\bar{u}_p}{\mathrm{d}t} + \sum_{q \in N(p)} \bar{v}_{pq}(\bar{u}_p - \bar{u}_{pq}) = 0$$

- influence of neighbours on $\bar{u}_p$ in the form of discretization of a diffusion equation

- $\bar{v}_{pq} > 0$ - forward diffusion - inflow - implicit scheme natural

- $\bar{v}_{pq} < 0$ - backward diffusion - outflow - explicit scheme natural

- novelty of our scheme - splitting of the fluxes to the cell p into the corresponding inflow and outflow parts by defining

$$a_{pq}^{in} = \max(\bar{v}_{pq}, 0), \quad a_{pq}^{out} = \min(\bar{v}_{pq}, 0)$$

$$m_p \frac{\mathrm{d}\bar{u}_p}{\mathrm{d}t} + \sum_{q \in N(p)} \bar{v}_{pq}(\bar{u}_p - \bar{u}_{pq}) = 0$$

$$a_{pq}^{in} = \max(\bar{v}_{pq}, 0), \quad a_{pq}^{out} = \min(\bar{v}_{pq}, 0)$$

- we approximate $\frac{\mathrm{d}\bar{u}_p}{\mathrm{d}t} \approx \frac{\bar{u}_p^n - \bar{u}_p^{n-1}}{\tau}$ and take inflow parts implicitly and outflow parts explicitly - the **general IIOE scheme**:

$$\bar{u}_p^n + \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{in} \, (\bar{u}_p^n - \bar{u}_{pq}^n) = \bar{u}_p^{n-1} - \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{out} \, (\bar{u}_p^{n-1} - \bar{u}_{pq}^{n-1})$$

- straightforward reconstructions $\bar{u}_p^m = u_p^m, \quad \bar{u}_{pq}^m = \frac{1}{2}(u_p^m + u_q^m)$ lead to the **basic IIOE scheme**:

$$u_p^n + \frac{\tau}{2m_p} \sum_{q \in N(p)} a_{pq}^{in}(u_p^n - u_q^n) = u_p^{n-1} - \frac{\tau}{2m_p} \sum_{q \in N(p)} a_{pq}^{out}(u_p^{n-1} - u_q^{n-1})$$

Theoretical properties

Theorem 1. Let us consider advection equation with constant velocity vector $\mathbf{v}$ and IIOE scheme on a uniform rectangular grid. If the initial condition is given by a second order polynomial, then IIOE scheme gives the exact solution for any choice of time step τ.

- for a constant $v > 0$ the 1D IIOE scheme takes the form

$$u_i^n + \frac{\tau v}{2h}(u_i^n - u_{i-1}^n) = u_i^{n-1} - \frac{\tau(-v)}{2h}(u_i^{n-1} - u_{i+1}^{n-1})$$

$u_0(x) = ax^2 + bx + c$, $u(x,\tau) = u^0(x - v\tau)$, if we plug the exact values

$$u_i^{n-1} = ax_i^2 + bx_i + c, u_{i+1}^{n-1} = a(x_i + h)^2 + b(x_i + h) + c,$$
$$u_i^n = a(x_i - v\tau)^2 + b(x_i - v\tau) + c, u_{i-1}^n = a(x_i - h - v\tau)^2 + b(x_i - h - v\tau) + c$$

into the scheme, we get true identity.

- for a constant $v > 0$ the 1D IIOE scheme takes the form

$$u_i^n + \frac{\tau v}{2h}(u_i^n - u_{i-1}^n) = u_i^{n-1} - \frac{\tau(-v)}{2h}(u_i^{n-1} - u_{i+1}^{n-1})$$

Theorem 2. Local conservativity

$$\begin{aligned} u_i^n &= u_i^{n-1} + \frac{\tau}{h}\, v\, \frac{1}{2}\,(u_i^{n-1} + u_{i-1}^n) - \frac{\tau}{h}\, v\, \frac{1}{2}\,(u_{i+1}^{n-1} + u_i^n) \\ &= u_i^{n-1} - \frac{\tau}{h}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right), \\ F_{i-\frac{1}{2}} &= v\, \frac{1}{2}\,(u_i^{n-1} + u_{i-1}^n),\ F_{i+\frac{1}{2}} = v\, \frac{1}{2}\,(u_{i+1}^{n-1} + u_i^n) \end{aligned}$$

- the same holds in higher dimensions **for polygonal grids and divergence free velocity fields**

Theorem 3. Considering 1D problem with constant v and periodic boundary conditions (cyclic tridiagonal matrices) the scheme is L_2 stable (and stabilized versions are L_∞ stable)

Theorem 4. Let us consider 1D advection equation with variable velocity $v(x)$ and the corresponding IIOE scheme on a uniform rectangular grid. Then the scheme is formally second order and the consistency error is of order

$$\mathcal{O}(h^2) + \mathcal{O}(\tau h) + \mathcal{O}(\tau^2).$$

Stabilization techniques

- **general IIOE scheme**:

$$\bar{u}_p^n + \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{in} \, (\bar{u}_p^n - \bar{u}_{pq}^n) = \bar{u}_p^{n-1} - \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{out} \, (\bar{u}_p^{n-1} - \bar{u}_{pq}^{n-1})$$

- **basic IIOE scheme**:

$$u_p^n + \frac{\tau}{2m_p} \sum_{q \in N(p)} a_{pq}^{in}(u_p^n - u_q^n) = u_p^{n-1} - \frac{\tau}{2m_p} \sum_{q \in N(p)} a_{pq}^{out}(u_p^{n-1} - u_q^{n-1})$$

- $\bar{u}_p^m = u_p^m, \quad \bar{u}_{pq}^m = \frac{1}{2}(u_p^m + u_q^m)$ - implicit part not always "dominates" the explicit part and (non-unboundedly growing) oscillations can occur

- 1st stabilization: $\bar{u}_{pq}^{n-1} = \frac{1}{2}(u_p^{n-1} + u_q^{n-1})$, $\bar{u}_p^{n-1} = \frac{1}{|N(p)|} \sum_{q \in N(p)} \bar{u}_{pq}^{n-1}$ in outflow part

- **general IIOE scheme**:

$$\bar{u}_p^n + \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{in} \, (\bar{u}_p^n - \bar{u}_{pq}^n) = \bar{u}_p^{n-1} - \frac{\tau}{m_p} \sum_{q \in N(p)} a_{pq}^{out} \, (\bar{u}_p^{n-1} - \bar{u}_{pq}^{n-1})$$

- 2nd stabilization is based on **adaptive upstream weighted choice for the averages** at the cell interfaces

$$\overline{u}_p^m = u_p^m, \quad \overline{u}_{pq}^m = (1 - \theta_{pq}^m) u_p^m + \theta_{pq}^m u_q^m$$

- weighting parameter $\theta_{pq}^m \in [0, 1]$, $\theta_{pq}^m = 1/2$ - the basic scheme, $\theta_{pq}^m = 1$ - full up-wind for inflows, $\theta_{pq}^m = 0$ - full up-wind for outflows

- **stabilized IIOE scheme**

$$u_p^n + \frac{\tau}{m_p} \sum_{q \in N(p)} \theta_{pq}^{in,n} a_{pq}^{in} (u_p^n - u_q^n) = u_p^{n-1} - \frac{\tau}{m_p} \sum_{q \in N(p)} \theta_{pq}^{out,n-1} a_{pq}^{out} (u_p^{n-1} - u_q^{n-1})$$

- **stabilized IIOE scheme**

$$u_p^n + \frac{\tau}{m_p}\sum_{q\in N(p)} \theta_{pq}^{in,n} a_{pq}^{in}(u_p^n - u_q^n) = u_p^{n-1} - \frac{\tau}{m_p}\sum_{q\in N(p)} \theta_{pq}^{out,n-1} a_{pq}^{out}(u_p^{n-1} - u_q^{n-1})$$

- $\theta_{pq}^{out,n-1}$ are chosen according to the so-called flux-corrected transport (FCT) methodology (Boris-Book, Zalesak) and

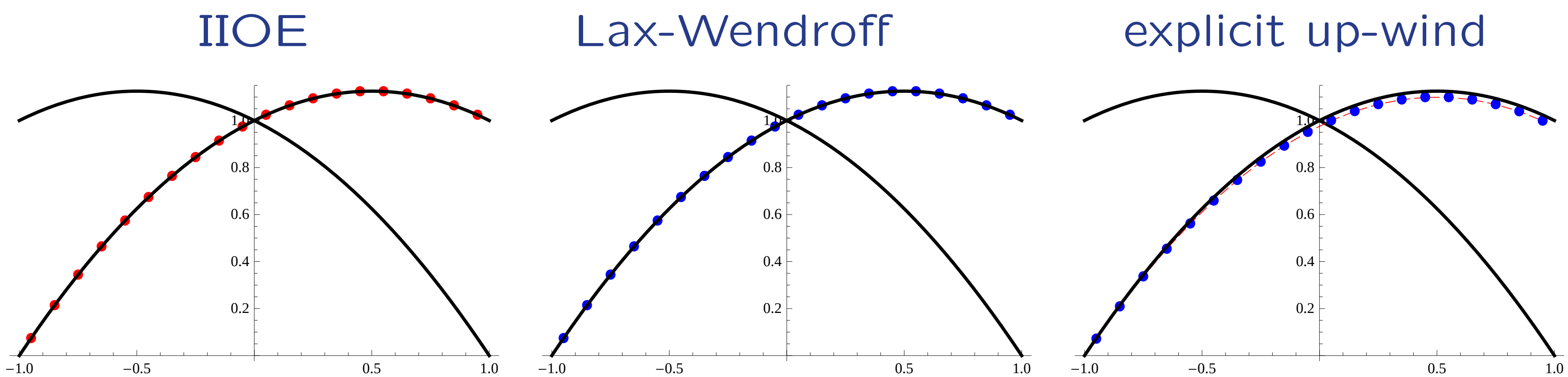
$$\theta_{pq}^{in,n} = 1 - \theta_{qp}^{out,n-1}$$

- for outflow faces we always have $\theta_{pq}^{out,n-1} \in [0, 1/2]$ and for inflow faces $\theta_{pq}^{in,n} \in [1/2, 1]$ - the relaxation may only shift the reconstruction at cell interfaces towards an upstream average.

- **S^{1}IIOE** scheme - relaxation coefficients are computed for every finite volume

- **S^{2}IIOE** scheme - two steps procedure - first the basic scheme is applied and only in points where discrete minimum-maximum principle is violated the relaxation coefficients are computed

Numerical experiments and comparisons



- **Constant advection of a quadratic polynomial in 1D** - comparison of the exact and numerical solutions in case of quadratic initial function, $n = 20, h = 0.1$ and $\tau = h/2$. The green solid curves represent the initial condition and the exact solution at time $T = 1$, respectively.

- **Constant advection of a quadratic polynomial in 1D - comparison with explicit schemes**

n	$\tau = h$	NTS	IIOE error	Lax-Wendroff error	up-wind error
20	0.1	10	$1.8\ 10^{-16}$	0.0	0.0
40	0.05	20	$3.5\ 10^{-16}$	0.0	0.0
80	0.025	40	$7.5\ 10^{-16}$	0.0	0.0
160	0.0125	80	$1.4\ 10^{-15}$	0.0	0.0

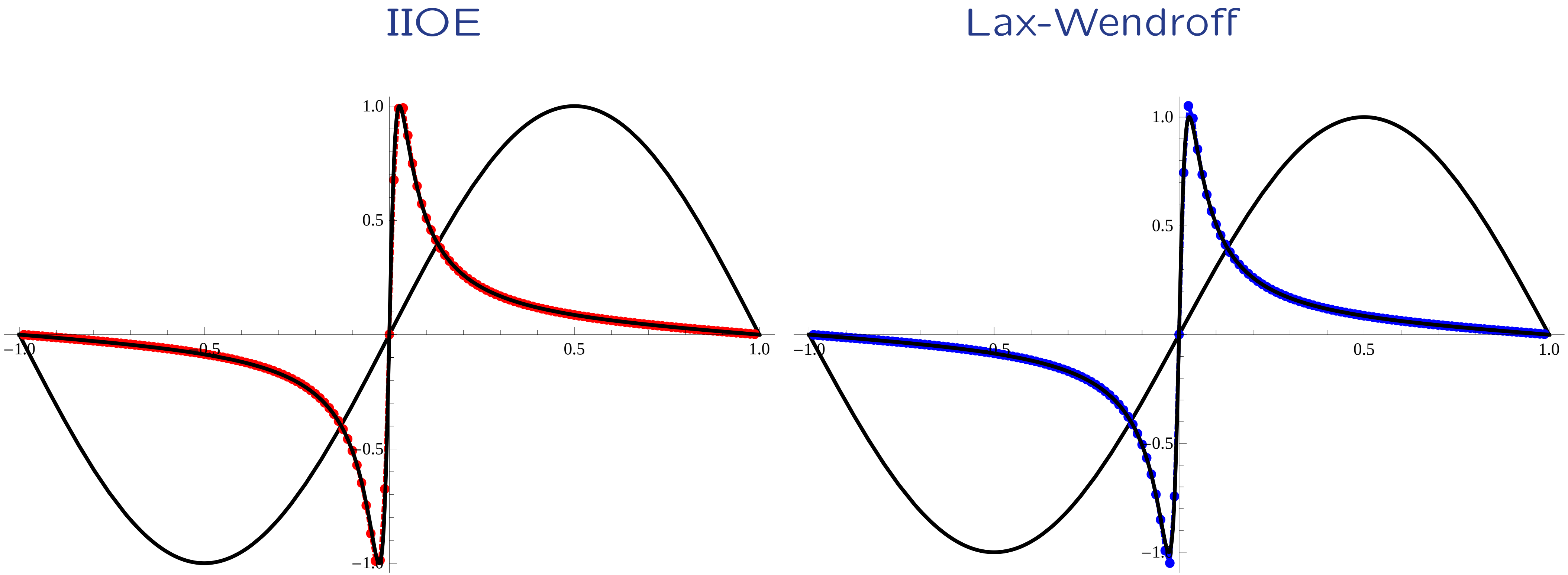
n	$\tau = h/2$	NTS	IIOE	Lax-Wendroff	up-wind
20	0.05	20	$3.7\ 10^{-16}$	$5.1\ 10^{-17}$	$1.83\ 10^{-2}$
40	0.025	40	$8.0\ 10^{-16}$	$7.5\ 10^{-17}$	$8.99\ 10^{-3}$
80	0.0125	80	$1.1\ 10^{-15}$	$8.3\ 10^{-17}$	$4.45\ 10^{-3}$
160	0.00625	160	$2.4\ 10^{-15}$	$9.9\ 10^{-17}$	$2.22\ 10^{-3}$

- **Constant advection of a quadratic polynomial in 1D**

n	$\tau = 2h$	NTS	IIOE	Lax-Wendroff	up-wind
20	0.2	5	$2.1\ 10^{-16}$	$1.1\ 10^{-11}$	$5.02\ 10^{-2}$
40	0.1	10	$2.1\ 10^{-16}$	$1.4\ 10^{-9}$	0.641
80	0.05	20	$3.9\ 10^{-16}$	0.466	$3.8\ 10^{+3}$
160	0.025	40	$5.7\ 10^{-16}$	$1.6\ 10^{+16}$	$1.3\ 10^{+12}$

n	$\tau = 10h$	NTS	IIOE	Lax-Wendroff	up-wind
20	1	1	$4.3\ 10^{-16}$	–	–
40	0.5	2	$1.0\ 10^{-15}$	–	–
80	0.25	4	$1.5\ 10^{-15}$	–	–
160	0.125	8	$2.5\ 10^{-15}$	–	–
160	$\tau = 40h = 0.5$	2	$1.7\ 10^{-15}$	–	–
160	$\tau = 80h = 1$	1	$2.6\ 10^{-15}$	–	–

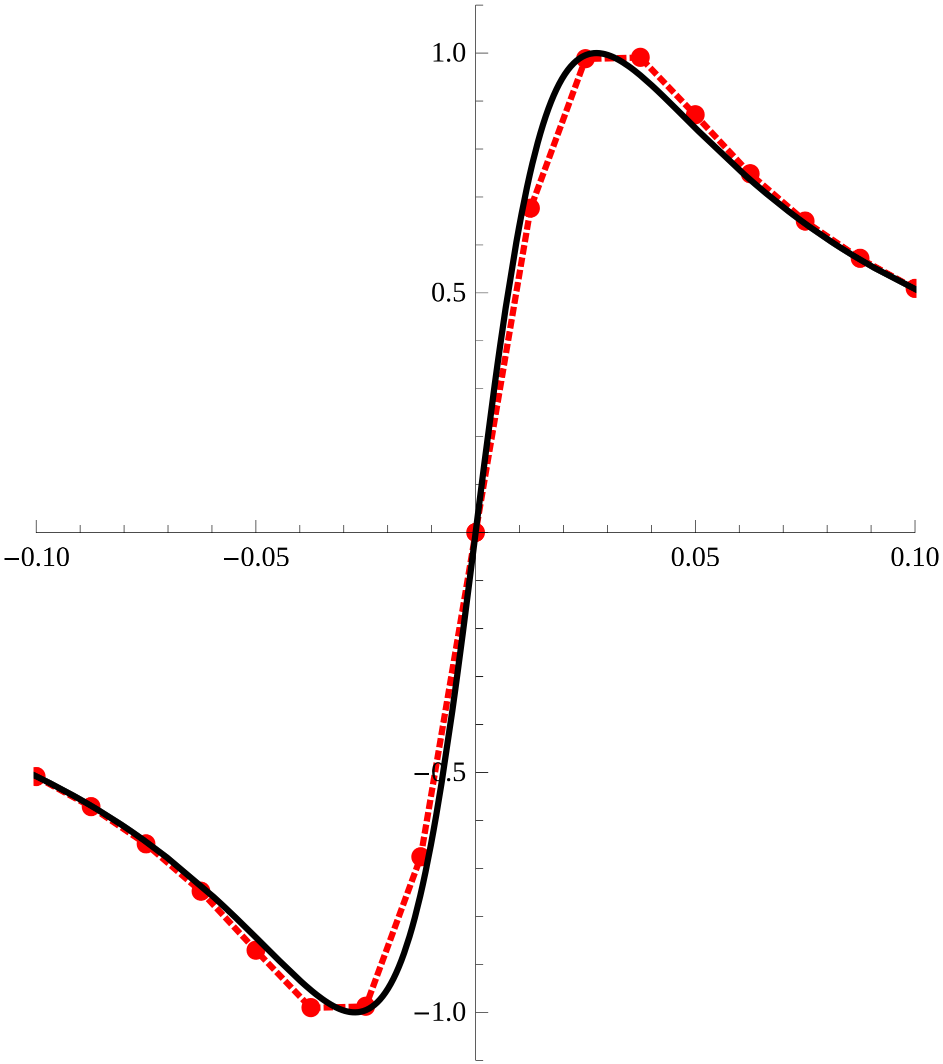
• Advection with variable velocity - comparison with the Lax-Wendroff scheme



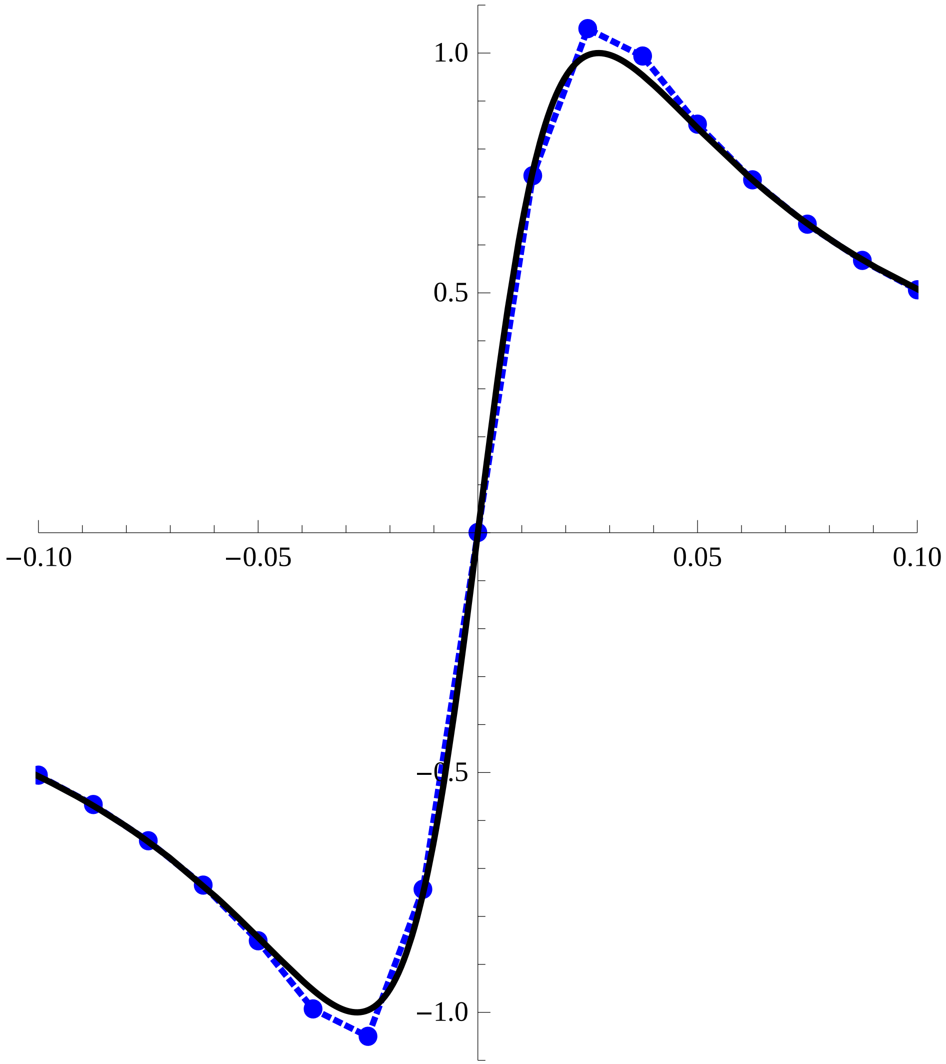
• $v(x) = -\sin(x)$, $u_0(x) = \sin(x)$, $\Omega = (-1, 1)$, $I = (0, T)$, $T = 1$, $n = 160$, $\tau = h$

- **Advection with variable velocity**

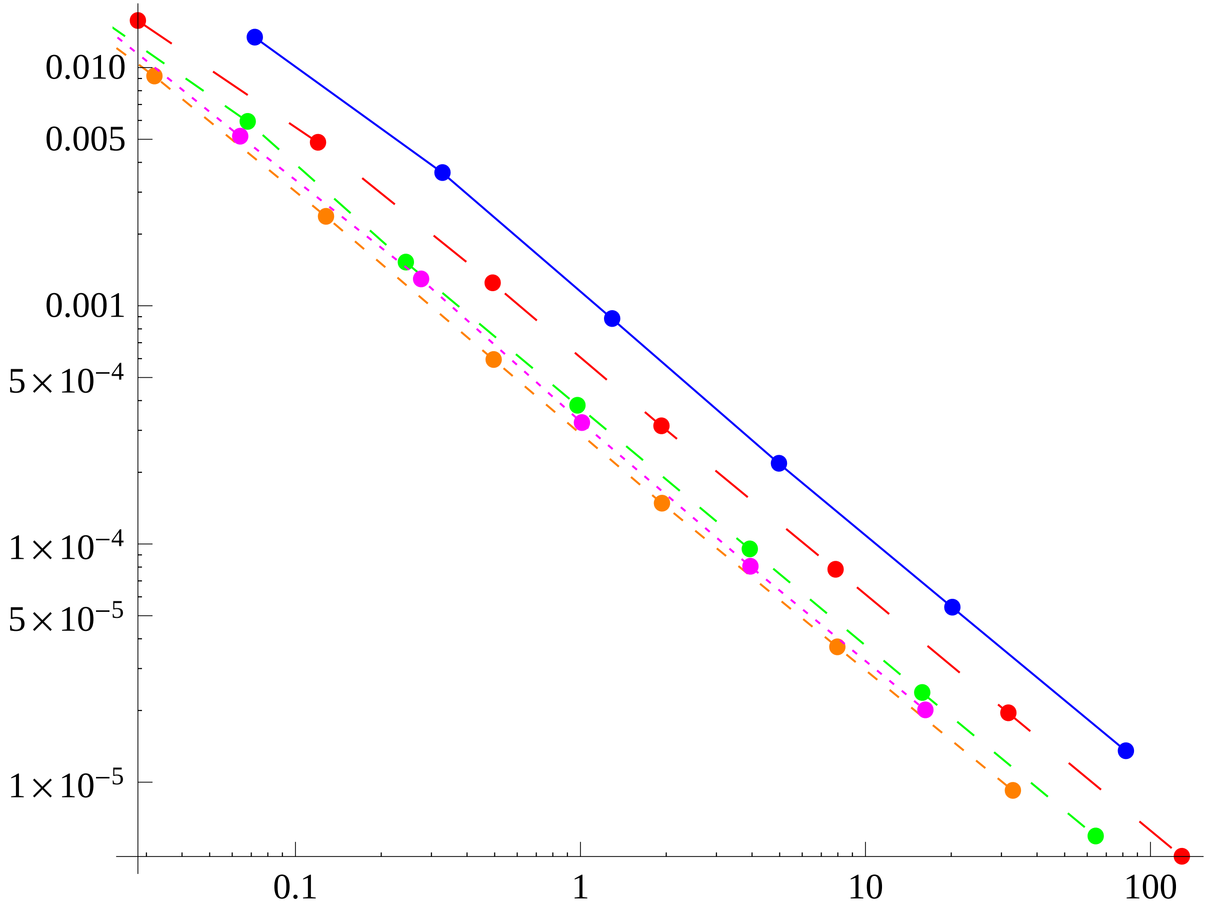
IIOE



Lax-Wendroff

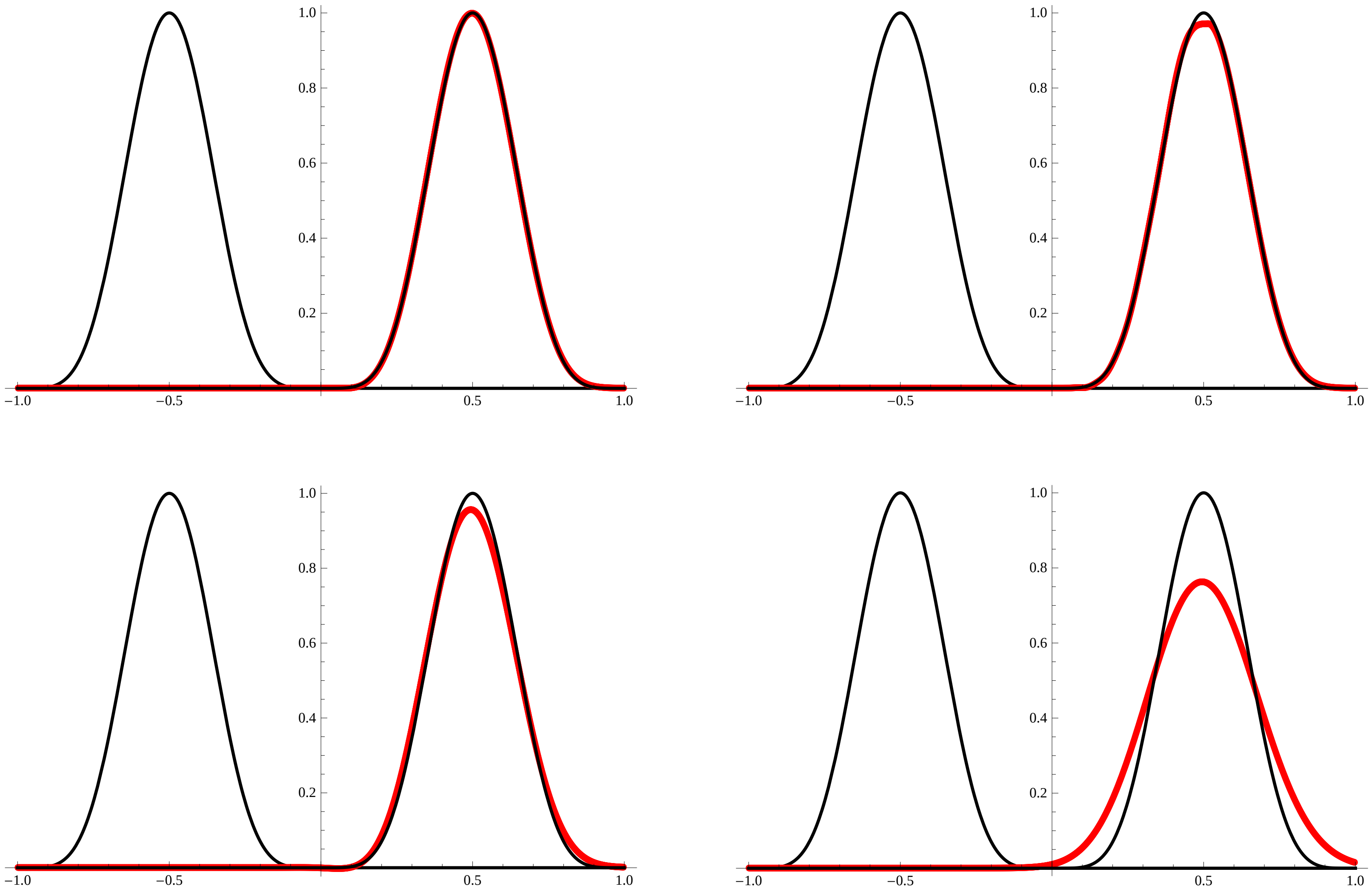


- **Advection with variable velocity**



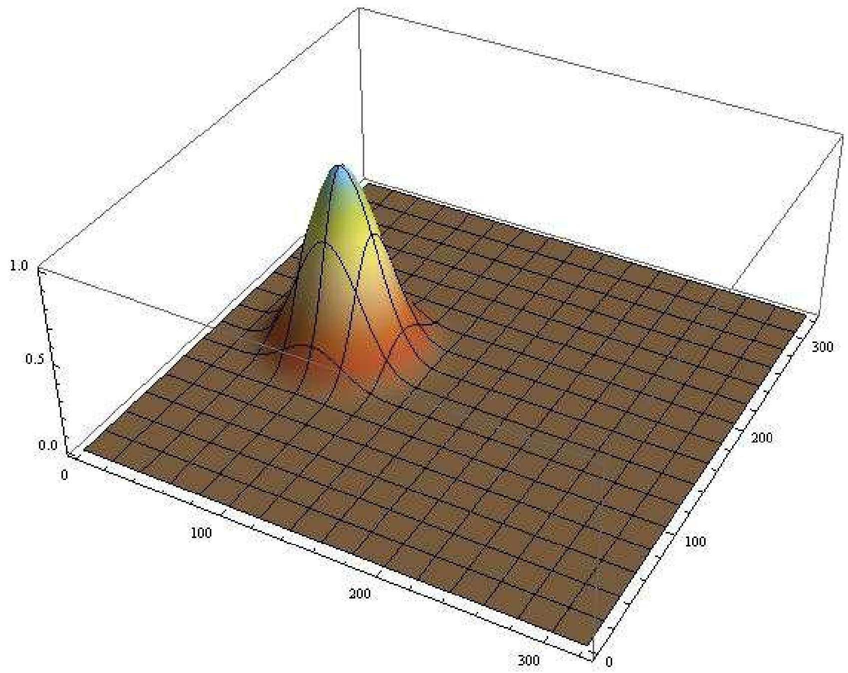
CPU versus $L_2(I,L_2)$**-error** for the Lax-Wendroff method (blue solid line) and for the IIOE scheme with CFL=1 (red large dashing), CFL=2 (green medium dashing), CFL=4 (orange small dashing) and CFL=8 (magenta tiny dashing).

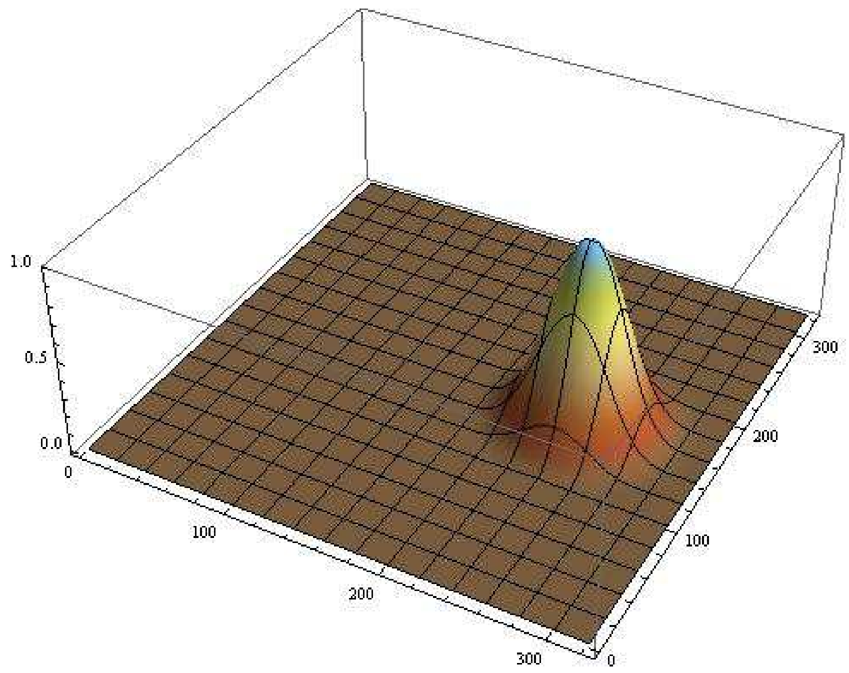
• Advection of smooth hump - comparison with implicit schemes



S^{2}IIOE scheme (top left), **S^{1}IIOE** scheme (top right), Gear's up-wind scheme (bottom left) and implicit up-wind scheme (bottom right), CFL=8.

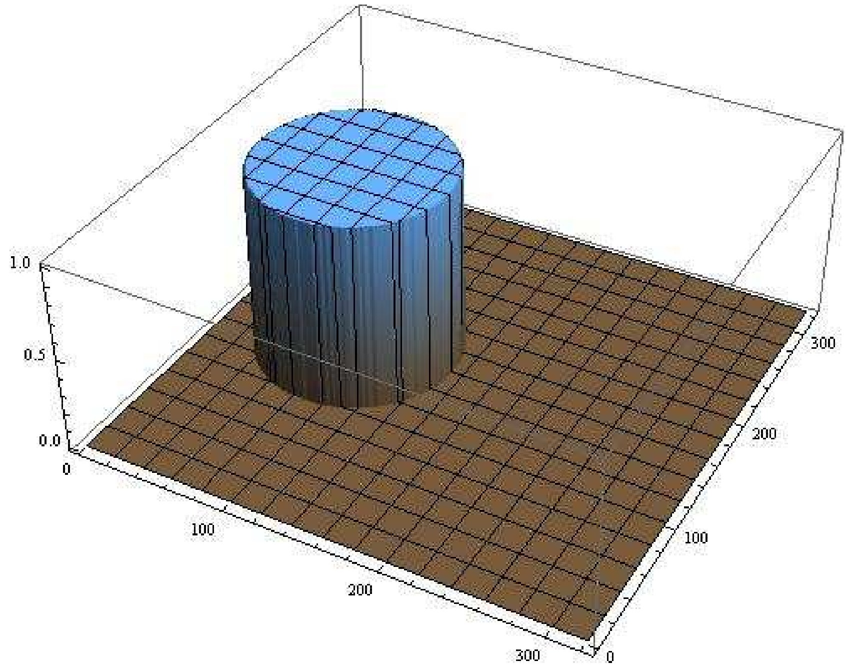
- **Rotation of a smooth hump in 2D**

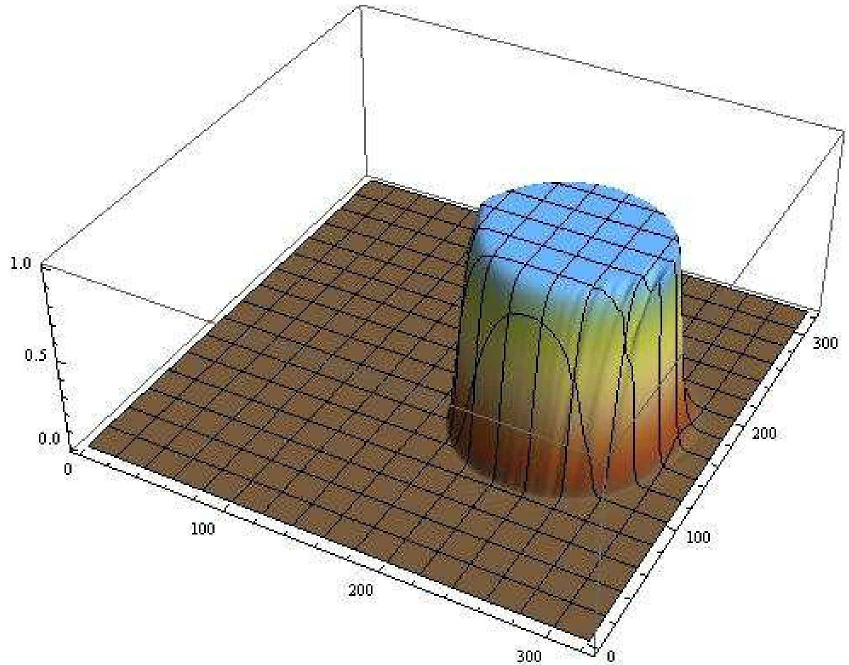




- S^2IIOE scheme - EOC=2 for any choice of time step

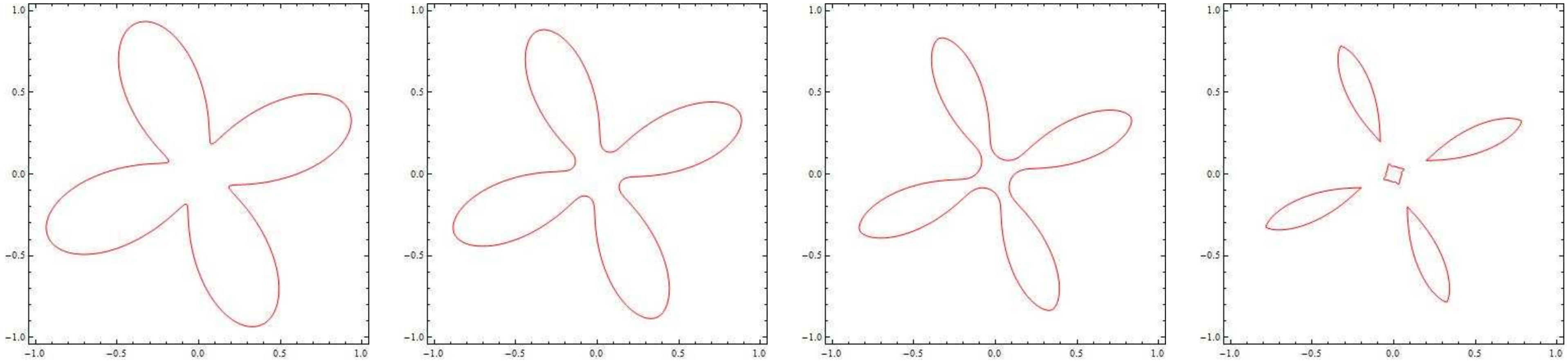
- **Rotation of a cylinder in 2D**

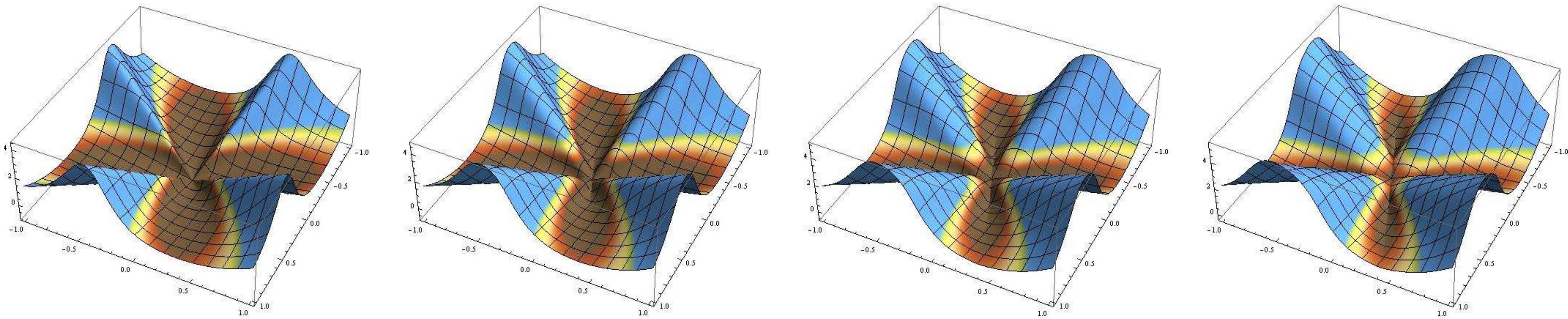




- IIOE stabilized scheme - EOC=2/3 for any choice of time step while for other implicit schemes (Gear's, upwind) EOC=1/2

- **Motion of level sets in normal direction - shrinking quatrefoil**



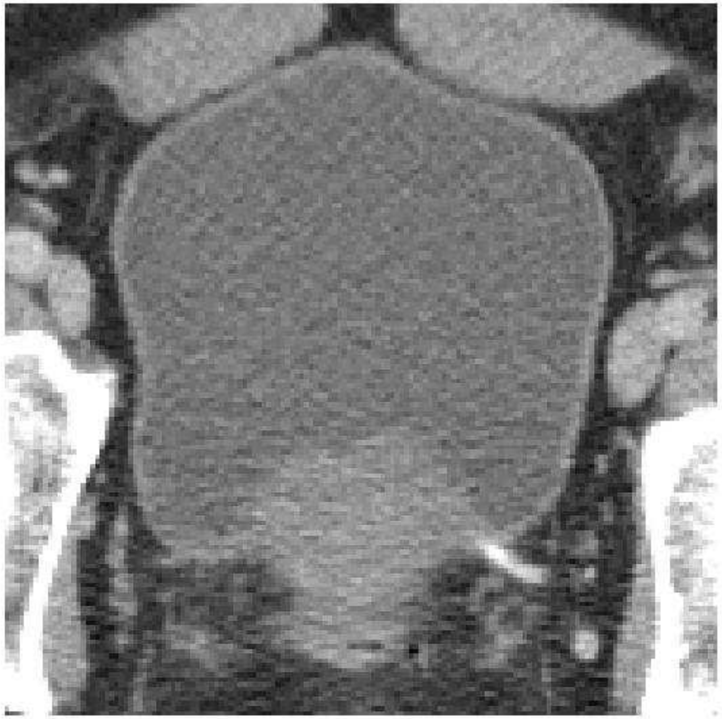


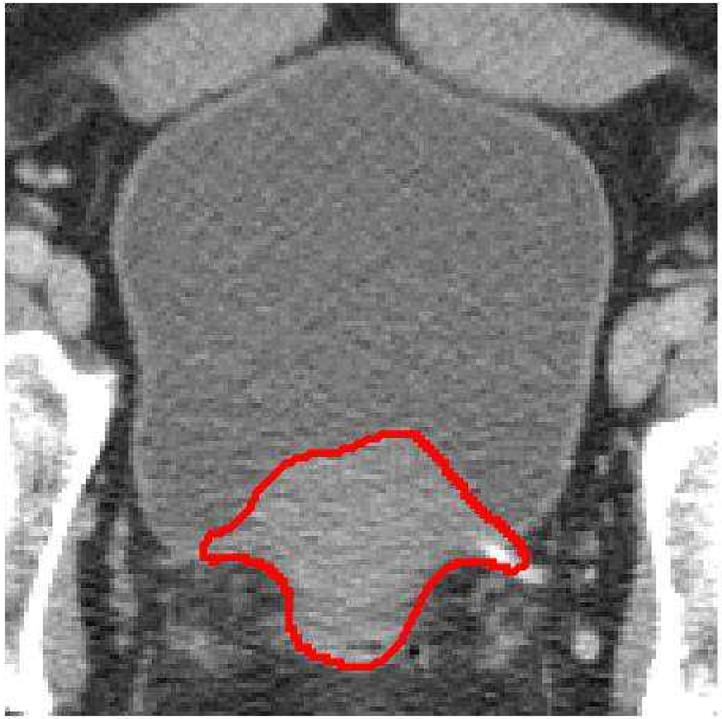
- velocity depends on gradient of solution $\mathbf{v}(x, \nabla u) = F \frac{\nabla u}{|\nabla u|}$, F=-1

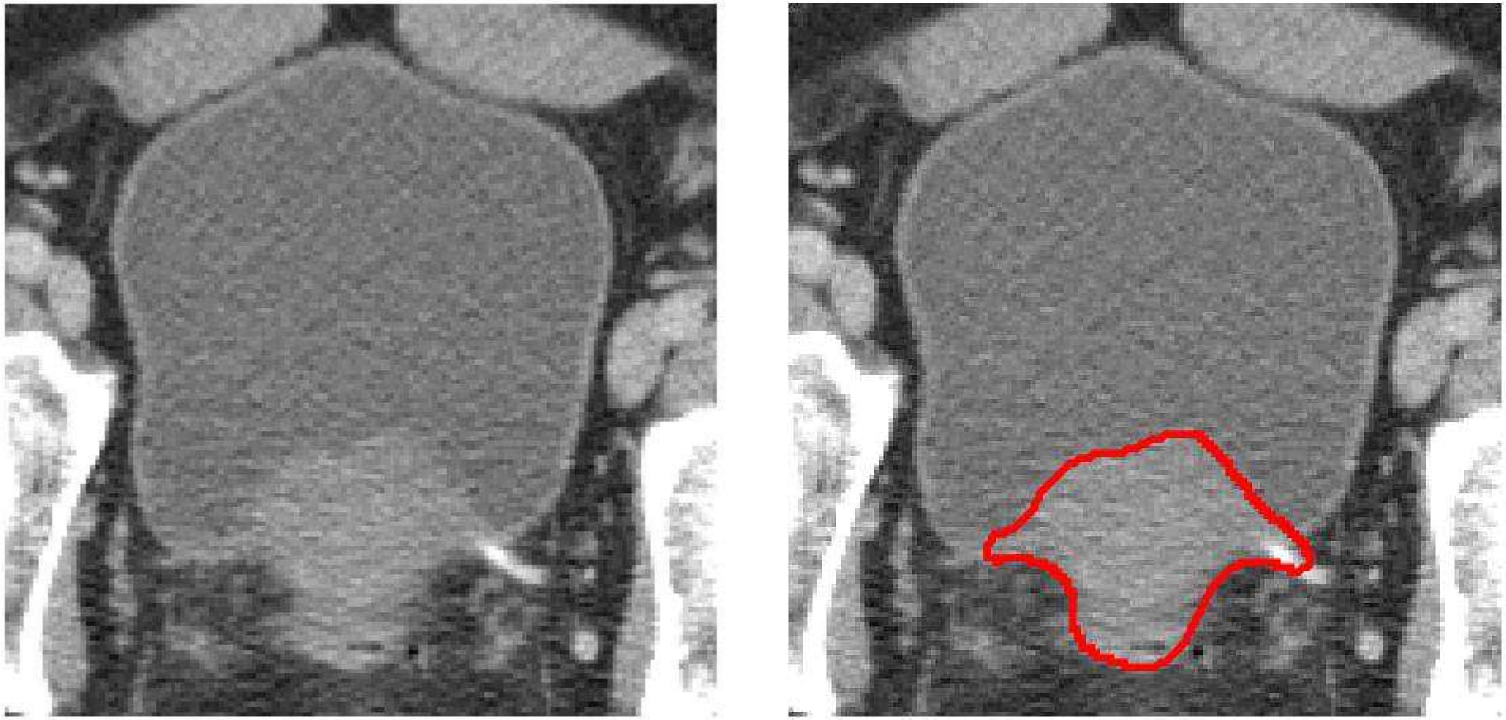
Applications - medical image segmentation

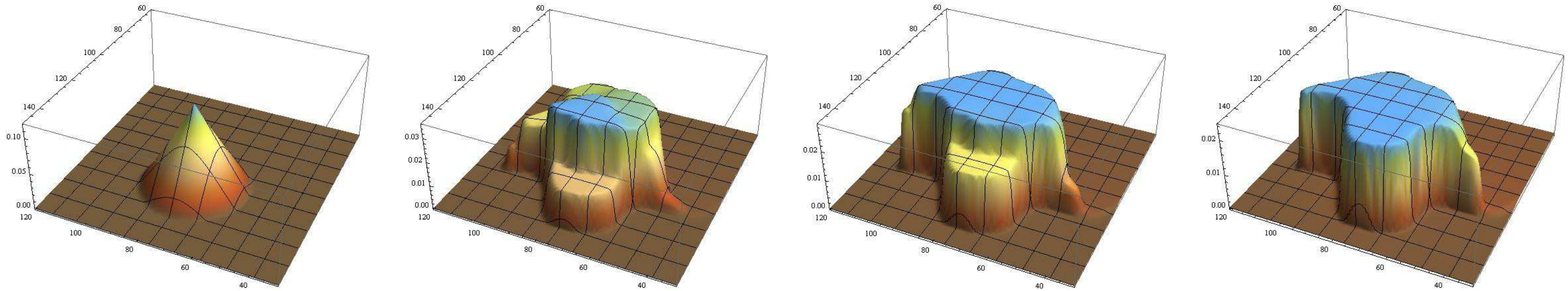
- S^2IIOE scheme in medical image segmentation - extraction of prostate by generalized subjective surface (GSUBSURF) model

$$u_t - w_a \nabla g \cdot \nabla u - w_d g \sqrt{\varepsilon^2 + |\nabla u|^2} \nabla \cdot \left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|}} \right) = 0$$









- J.Urbán, TatraMed Bratislava - building into software TomoCon

Application - forest fire front propagation

• fire front is modelled by an evolving closed plane curve with outer normal velocity

$$\beta = f(x) e^{\lambda(\mathbf{V}.\mathbf{N})}(1 - \varepsilon k)$$

x - position vector, $f(x)$ - speed of fire front in nonhomogeneos forest (age, density, humidity), $\mathbf{V}$ - wind velocity (slope of terrain), $\mathbf{N}$ - outer normal to the curve, k - curvature

• Lagrangean approach - intrinsic PDE for evolving curve position vector ($\mathbf{T}$ - tangent to the curve)

$$\partial_t x = \beta \mathbf{N} + \alpha \mathbf{T} = f(x) e^{\lambda(\mathbf{V}.\mathbf{N})}(\mathbf{N} + \varepsilon \partial_{ss} x) + \alpha \partial_s x$$

• treatment of tangential velocity (intrinsic advection term) α yielding uniform grid point redistribution by **IIOE method** is crucial for stable numerical solution allowing topological changes

The last video

Thanks for your attention

- motion of level sets in normal direction by speed F

$$u_t + F|\nabla u| = 0, \quad u_t + F\frac{\nabla u}{|\nabla u|} \cdot \nabla u = 0$$

$$u_t + \mathbf{v} \cdot \nabla u, \quad \mathbf{v} = F\frac{\nabla u}{|\nabla u|}$$

$$u_t + \nabla \cdot (\mathbf{v}u) - u\nabla \cdot \mathbf{v} = 0.$$

$$u_t + \nabla \cdot \left(Fu\frac{\nabla u}{|\nabla u|}\right) - u\nabla \cdot \left(F\frac{\nabla u}{|\nabla u|}\right) = 0.$$