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Università di Padova

Classification of the umbilic point for general immiscible three-phase flow in porous media

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Summary

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- The goal: Classify the models
- Results from Schaeffer and Shearer
- A new result on quadratic fluxes
- The classification of the models

Physical constants and functions, $i \in w, o, g$

s_i, saturation of phase i; ρ_i, density of phase i;

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The system (1) has an isolated umbilic point in the triangle.

(See Medeiros or Schaeffer, Shearer, Marchesin, Paes-Leme)

The isolated umbilic point lies at $(s_w, s_o) = (\alpha, \beta)$ where:

$$\frac{\mathrm{d}m(s_i)}{\mathrm{d}s_i} = \frac{\mathrm{d}m(s_j)}{\mathrm{d}s_j} = \xi, \quad \text{for } i \neq j \in \{w, o, g\}$$

(At the umbilic point each liquid hampers the motion of all others.)

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Classification of the $F(S)$ near the umbilic point according to Schaeffer and Shearer depending on its location within the saturation triangle.
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The real Goal

Where does the singularity change from type I to type II?

Review of Schaeffer and Shearer's Results

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Definition: Umbilic point

Let H be a C^2 function $H : \mathbb{R}^2 \to \mathbb{R}^2$, $U \to H(U)$.
If DH has two equal eigenvalues and is diagonalizable at U^* then U^* is an umbilic point of H.

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Definition: SS-isolated umbilic point and SS-flux

Let G be the second order expansion of the flux H around U^*.
If U^* is an isolated umbilic point of G then U^* is a SS-isolated umbilic point of H and H is a SS-flux.

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Theorem: Schaeffer and Shearer's normal form

There exists a and b such that G is equivalent to

$$Q(U) = \begin{bmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{bmatrix}.$$

Non degenerate G

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We need another way to distinguish Type I from Type II

New result on quadratic fluxes

Definition of N_G

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Therefore, $\det(DQ(U)) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 4a - 4b^2 & -2b \\ -2b & -4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

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$N_Q = \begin{bmatrix} 4a - 4b^2 & -2b \\ -2b & -4 \end{bmatrix}$ and $\det(N_Q) = -4\left(4a - 3b^2\right)$. Thus,

$$\det(N_Q) > 0 \Leftrightarrow a < \tfrac{3}{4}b^2 \Leftrightarrow Q \text{ is Type I};$$

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\Rightarrow \det\left(N_Q\right) = \det\left(M^T N_G M\right)
\end{gathered}
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Proof: Equivalent flux

If G is equivalent to some Q then there is an invertible matrix M such that

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We have $\det^2(M) > 0$, therefore Q and G are type I if and only if:

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Remark

If G is in degenerate border case between type I and type II then

$$\det(N_G) = 0.$$

Classification of the flux F

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The original flux

$$f_i = \frac{m_i}{m} = \frac{b_i s_i^{a_i}}{\sum_j b_j s_j^{a_j}};\ b_i > 0\,;\ a_i > 1.$$

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Quadratic expansion of the $F(S)$ about (α, β), where $\gamma = 1 - \alpha - \beta$.

$$\bar{f}_w(u,v) = \frac{q_w\left(\bar{m}_o + \bar{m}_g\right) - q_g\bar{m}_w}{m^2}\frac{u^2}{2} - \frac{q_g\bar{m}_w}{m^2}vu - \frac{\left(q_o + q_g\right)\bar{m}_w}{m^2}\frac{v^2}{2},$$

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where:

$u = s_w - \alpha$; $\quad v = s_o - \beta$; $\quad m = \bar{m}_w + \bar{m}_o + \bar{m}_g$;

$\bar{m}_w = m_w(\alpha)$; $\quad \bar{m}_o = m_0(\beta)$; $\quad \bar{m}_g = m_g(\gamma)$;

$$q_w = \frac{\mathrm{d}^2 m_w}{\mathrm{d}s_w^2}(\alpha);\quad q_o = \frac{\mathrm{d}^2 m_o}{\mathrm{d}s_o^2}(\beta);\quad q_g = \frac{\mathrm{d}^2 m_g}{\mathrm{d}s_g^2}(\gamma).$$

The constants A, B and C

$$q_w = A_w^2 b_w \alpha^{a_w - 2}; \quad q_o = A_o^2 b_o \beta^{a_o - 2}; \quad q_g = A_g^2 b_g \gamma^{a_g - 2}.$$

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Calculation of N_F

$$N_F = \begin{bmatrix} \dfrac{-\bar{m}_2 q_1 q_3}{(\bar{m}_1 + \bar{m}_2 + \bar{m}_3)^3} & \dfrac{\bar{m}_3 q_1 q_2 - \bar{m}_1 q_2 q_3 - \bar{m}_2 q_1 q_3}{2\,(\bar{m}_1 + \bar{m}_2 + \bar{m}_3)^3} \\[2ex] \dfrac{\bar{m}_3 q_1 q_2 - \bar{m}_1 q_2 q_3 - \bar{m}_2 q_1 q_3}{2\,(\bar{m}_1 + \bar{m}_2 + \bar{m}_3)^3} & \dfrac{-\bar{m}_1 q_2 q_3}{(\bar{m}_1 + \bar{m}_2 + \bar{m}_3)^3} \end{bmatrix}.$$

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The sign of $\det(N_F)$

$$\mathcal{R} = \frac{4A^2 B^2 C^2 \alpha^2 \beta^2 \gamma^2 (\bar{m}_1 + \bar{m}_2 + \bar{m}_3)^6}{q_1 q_2 q_3 \bar{m}_1 \bar{m}_2 \bar{m}_3} det(N_F)$$

$\mathcal{R}$ is a quartic function on α and β. Remembering $\gamma = 1 - \alpha - \beta$.

$-\mathcal{R}$ is the product of the four linear functions:

$+BC\alpha + AC\beta - AB\gamma$; $+BC\alpha - AC\beta + AB\gamma$;

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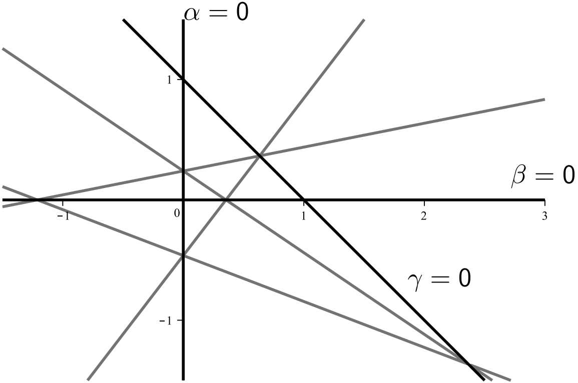
Zero level of $\det(N_F)$ and $\mathcal{R}$

If $A \neq B \neq C \neq A$ then $\det(N_F)$ vanishes on the four straight lines:



The classification of F within the triangle

The four straight lines cross the border of the triangle in three points:

$(\alpha, \beta) = \left(0, \frac{AB}{CA+AB}\right)$ and $\gamma = \frac{CA}{CA+AB}$;

$(\alpha, \beta) = \left(\frac{AB}{AB+BC}, 0\right)$ and $\gamma = \frac{BC}{AB+BC}$;

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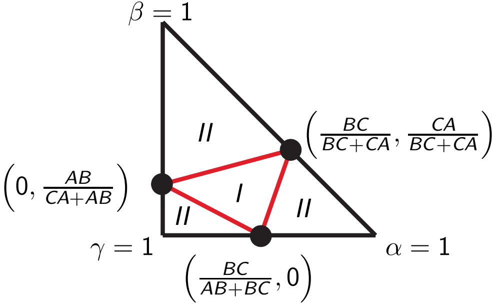

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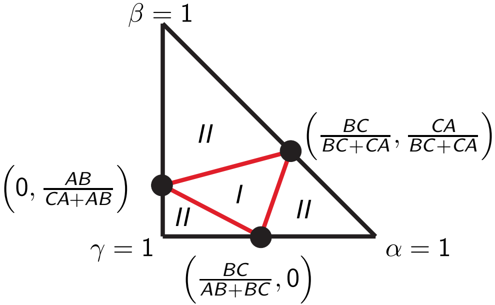

Thank you.