Boundary layers in homogenization theory

Nader Masmoudi (Courant Institute, NYU)

Joint work with David Gérard-Varet (Paris 7, IMJ)

Padova, June 2012

1. Setting of the problem

Motivation:
Physically : To compute accurate and effective properties of mixtures.
Mathematically: To compute solutions of homogenization problems.
These problems come from :

- diffusion of heat or electricity,
- equilibrium of elastic bodies, ...

Classical problem of elliptic homogenization: In a bounded domain Ω of $\mathbb{R}^d$, $d \geq 2$:

$$\boxed{\begin{cases} \nabla \cdot (A(\cdot/\varepsilon)\nabla u^\varepsilon) = f \ \text{ in } \Omega, \\ u^\varepsilon|_{\partial\Omega} = \phi. \end{cases}} \qquad (\mathrm{S}^\varepsilon)$$

- $u^\varepsilon = u^\varepsilon(x)$, ϕ and f take values in $\mathbb{R}^N$ for some $N \geq 1$.
- $A = A(y)$ takes values in $M_d\,(M_N(\mathbb{R}))$.

Usual notation: $\boxed{\nabla \cdot (A(\cdot/\varepsilon)\nabla u^\varepsilon) \ := \ \partial_\alpha\,(A_{\alpha\beta}(\cdot/\varepsilon)\partial_\beta u)}$

where $A_{\alpha\beta}(y) \in M_N(\mathbb{R})$ for all $1 \leq \alpha, \beta \leq d$.

Assumptions:

(H1) *Coercivity:* There exists $\lambda > 0$, , s.t. for all family $(\xi_\alpha)_{1\leq\alpha\leq d}$ of vectors in $\mathbb{R}^N$ and all y in $\mathbb{R}^d$.

$$A_{\alpha\beta}(y)\,\xi_\alpha \cdot \xi_\beta \;\geq\; \lambda\,\xi_\alpha \cdot \xi_\alpha$$

(H2) *Periodicity:* $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$,

$$A(y+h) \;=\; A(y), \quad f(y) \;=\; f(y+h)$$

(H3) *Smoothness:* A, f and Ω are smooth.

Question: Behavior of the solutions u^ε as $\varepsilon \to 0$?

Classical approach: *two-scale asymptotic expansion*:

$$\boxed{u^\varepsilon_{app} \;=\; u^0(x) \,+\, \varepsilon u^1(x, x/\varepsilon) \,+\, \ldots \,+\, \varepsilon^n u^n(x, x/\varepsilon)}$$

with $u^i = u^i(x, y)$ periodic in y.

Use formal asymptotics to determine the u^i inductively.

Case without boundary

Proposition: *There exists smooth (non trivial)* $u^0, \; u^1, \; \dots, \; u^n$ *such that*

$$\nabla \cdot (A(\cdot/\varepsilon)\nabla u^{\varepsilon}_{app}) \; = \; O(\varepsilon^{n-2}) \text{ in } L^2(\Omega).$$

- The construction of the u^i's involves the famous *cell problem*

$$\boxed{-\nabla \cdot \; (A\nabla\chi^{\gamma})\,(y) \; = \; \nabla_{\alpha} \cdot A^{\alpha\gamma}\,(y), \quad y \text{ in } \mathbb{T}^d}$$

with solution $\chi^{\gamma} \in M_N(\mathbb{R})$.

- The first term u^0 does not depend on y.
- u^1 is given by $u^1 = -\chi^{\gamma}\partial_{x_{\gamma}} u^0(x) + \overline{u}^1$.

The solvability condition for u^2 yields the equation satisfied by u^0.

u^0 necessarily satisfies

$$\nabla \cdot A^0 \nabla u^0 \;=\; 0$$

where the constant homogenized matrix is given by

$$\boxed{A^{0,\alpha\beta} \;=\; \int_{\mathbb{T}^d} A^{\alpha\beta}(y)\, dy \;+\; \int_{\mathbb{T}^d} A^{\alpha\gamma}(y) \partial_{y_\gamma} \chi^{\beta}(y)\, dy.}$$

Case with boundary

Problem: The two-scale expansion (computed as in the case without boundaries) provides a poor approximation of the solution !

Reason: The boundary condition is far from being satisfied.

The error term $e^\varepsilon = u^\varepsilon - u^\varepsilon_{app}$ satisfies

$$\begin{cases} \nabla \cdot (A(\cdot/\varepsilon)\nabla e^\varepsilon) \approx 0 \text{ in } \Omega, \\ \qquad\qquad e^\varepsilon|_{\partial\Omega} \approx -\varepsilon u^1(\cdot, \cdot/\varepsilon). \end{cases}$$

The boundary data is $O(\sqrt{\varepsilon})$ in $H^1(\partial\Omega)$, $O(\varepsilon)$ in $L^2(\partial\Omega)$.

The error is $O(\sqrt{\varepsilon})$ in $H^1(\Omega)$, $O(\varepsilon)$ in $L^2(\Omega)$.

Better approximation: Requires to study systems in which both the coefficients and the boundary data oscillate.

Our main model problem

Model problem:

$$\boxed{\begin{cases} \nabla \cdot (A(\cdot/\varepsilon)\nabla u) = 0 \ \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi(\cdot/\varepsilon). \end{cases}} \qquad (\mathrm{S}^{\varepsilon})$$

We keep assumptions (H1)-(H2)-(H3).

Question: Behavior of the solutions u^ε as $\varepsilon \to 0$?

Much harder than the original homogenization problem !

In the original problem, energy estimates yield $\|u^\varepsilon\|_{H^1(\Omega)} \leq C$.
Here, $\|u^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{-1/2}$.

Classical compactness methods fail.

We shall really need (H1)-(H2)-(H3).

Remark: Under these assumptions, we can use results of Avellaneda and Lin: the solution of (S^ε) satisfies

$$\|u^\varepsilon\|_{L^p(\Omega)} \;\le\; C\|\varphi(\cdot/\varepsilon)\|_{L^p(\partial\Omega)} \le C', \quad \forall 1 < p \le \infty.$$

From there: $\|u^\varepsilon\|_{H^1(\omega)} \;\le\; C''$, for all $\omega \Subset \Omega$.

Suggests that singularities are stronger near the boundary: *boundary layer.*

Difficulty: the periodic structure of the oscillations breaks down in the boundary layer. No simple two-scale expansion.

> *A large number of questions remain open. Of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem.*

Bensoussan et al, Asymptotic analysis for periodic structures

Existing results:

(Moscow and Vogelius [97], Allaire and Amar [99], Neuss [01], Sarkis [08])

Obtained under some restrictions on the domain:

Ω *is a polyhedron whose sides have normal vectors in* $\mathbb{Q}^d$.

Case $d = 2$: polygons with sides of rational slope.

The work with David Gérard-Varet:

- Extension to generic polyhedrons *(J. Eur. Math. Soc. 2010)*
- Extension to smooth domains *(Acta Math. 2012)*

2. Statement of the result

Theorem: *Let Ω be uniformly convex. Assume (H1)-(H2)-(H3). The solution u^ε of (S^ε) converges in $L^2(\Omega)$ to the solution u^0 of*

$$\boxed{\begin{cases} \nabla \cdot (A^0 \nabla u) = 0 \quad in\, \Omega, \\ u|_{\partial\Omega} = \varphi^0, \end{cases}} \qquad (S^0)$$

where the matrix A^0 is constant, and the boundary data φ^0 is in $L^p(\partial\Omega)$ for all p. Moreover,

$$\boxed{\|u^\varepsilon - u^0\|_{L^2(\Omega)} = O(\varepsilon^\alpha) \quad \text{for some } \alpha > 0.}$$

Remarks:

- A^0 and φ^0 are "explicit".
- Strong convergence with a rate. The optimal rate is an interesting open problem.

- φ^0 comes from solving a half-space problem (boundary layer)
- No smoothness on φ^0. This may be intrinsic.
- $u^0 \in L^2(\Omega)$, but is smooth inside Ω.
- Possible generalizations:
  $\varphi(x, y)$ instead of $\varphi(y)$, less constraints on Ω.
- The proof of the theorem simplifies a little for scalar equations (maximum principle).

From now on:

$$\boxed{N = 1, \quad d = 2, \quad \Omega = D(0, 1).}$$

3. Ideas from the proof

a) Explanation for the homogenization

Idea:

$$u^\varepsilon \approx \underbrace{u^{\varepsilon,int}}_{\text{interior part}} + \underbrace{u^{\varepsilon,bl}}_{\text{boundary layer corrector}}$$

The Homogenized system will be understood if we have some explicit approximation for these interior and boundary layer terms.

- The interior term

Classical *two-scale asymptotic expansion* is OK :

$$u^{\varepsilon,int} = u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \ldots + \varepsilon^n u^n(x, x/\varepsilon)$$

Question: What is the boundary value φ^0 of u^0 ?

- Boundary layer corrector

Difficulty: no clear structure for the boundary layer.

Guess: The boundary layer has typical scale ε. No curvature effect:

1. Near a point $x_0 \in \partial\Omega$, replace $\partial\Omega$ by the tangent plane at x_0:

$$T_0(\partial\Omega) := \{x,\ x \cdot n_0 = x_0 \cdot n_0\} :$$

2. Dilate by a factor ε^{-1}.

Formally, for $x \approx x_0$, one looks for

$$\boxed{u^{\varepsilon,bl}(x) \approx U_0(x/\varepsilon)}$$

The profile $U_0 = U_0(y)$ is defined in the half plane

$$H_0^\varepsilon \;=\; \{y,\; y \cdot n_0 > \varepsilon^{-1} x_0 \cdot n_0\}.$$

It satisfies the system:

$$\boxed{\left\{\begin{array}{c} \nabla_y \cdot (A \nabla_y U_0) = 0 \quad \text{in } H_0^\varepsilon, \\ U_0|_{\partial H_0^\varepsilon} = \varphi - \varphi^0(x_0). \end{array}\right.}$$

Remark: x_0 is just a parameter in this system.

How to determine $\varphi^0(x_0)$??

We need to understand the properties of the following system :

Auxiliary boundary layer system

$$\boxed{\begin{cases} \nabla_y \cdot (A\nabla_y U) = 0 \quad \text{in } H, \\ U|_{\partial H} = \phi. \end{cases}} \tag{BL}$$

where $H := \{y, \quad y \cdot n > a\}$.

Idea: *The solution U of* (BL) *satisfies:*

$$U \to U_\infty, \quad \text{as } y \cdot n \to +\infty,$$

for some constant U_∞ that depends linearly on ϕ.

Back to U_0, one can derive the homogenized boundary data φ^0. Indeed:

- On one hand, one wants $U_0 \to 0$ (localization property).
- On the other hand,

$$U_0 \to U_\infty(\varphi - \varphi^0(x_0)) \;=\; U_\infty(\varphi) - \varphi^0(x_0).$$

  so that:

$$\varphi^0(x_0) \;:=\; U_\infty(\varphi)\ldots$$

…*This formal reasoning raises many problems !*

- *Well-posedness of* (BL) *is unclear.*

  - No natural functional setting (no decay along the boundary).
  - No Poincaré inequality.
  - No maximum principle.

- *Existence of a limit U_∞ for* (BL) *is unclear.*
  Underlying problem of ergodicity.

- *U_∞ depends also on H, that is on n and a.*

  - No regularity of U_∞ with respect to n.

  - Back to the original problem, our definition of $\varphi^0(x_0)$ depends on x_0, but also on ε.

  Possibly many accumulation points as $\varepsilon \to 0$.

b) Polygons with sides of rational slopes

In such cases, the boundary layer systems of type (BL) can be fully understood.

- Well-posedness: *the coefficients of the systems are periodic tangentially to the boundary*.

  After rotation, they turn into systems of the type

$$\boxed{\begin{cases} \nabla_z \cdot (B\nabla_z V) = 0, \quad z_2 > a, \\ V|_{z_2=a} = \psi, \end{cases}} \qquad \text{(BL1)}$$

  with coefficients and boundary data that are periodic in z_1.

  This yields a natural variational formulation.

- Existence of the limit : *Saint-Venant estimates* on (BL1).

  One shows that $F(t) := \int_{z_2>t} |\nabla_z V|^2 \, dz$ satisfies the differential inequality.

$$\boxed{F(t) \le -CF'(t)}$$

  From there, one gets exponential decay of all derivatives, and:

$$V \to V_\infty, \text{ exponentially fast, as } z_2 \to +\infty$$

  or

$$U \to U_\infty, \text{ exponentially fast, as } y \cdot n \to +\infty$$

  <u>Key</u>: Poincaré for functions periodic in z_1 with zero mean.

- In polygonal domains, the regularity of U_∞ with respect to n does not matter.

- *For rational slopes, the limit U^∞ does depend on a.*

Back to (S^ε) (in polygons with rational slopes):

The analogue of our thm is only available up to subsequences in ε.

The homogenized system may depends on the subsequence.

There are examples with a continuum of accumulation points.

Conclusion: Far from enough to handle general domains.

Need to know more on (BL), getting rid of the "rationality" assumption.
Ref : Moscow and Vogelius [97], Allaire and Amar [99].

c)More general treatment of (BL)

Remark: One can not be fully general: the existence of U_∞ requires some ergodicity property.

Simpler example: $\boxed{\Delta U = 0 \quad \text{in } \{y_2 > 0\}, \quad U|_{y_2=0} = \phi}$.

- If ϕ 1-periodic, then $U(0, y_2) \to \int_0^1 \phi$ exponentially fast.
- *But there exists $\phi \in L^\infty$ such that $U(0, y_2)$ has no limit.*

Remarks:

- Explicit formula: $U(0, y_2) = \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}} \frac{y_2}{y_2^2 + t^2} \phi(t)\, dt.$
- For ϕ with values in $\{+1, -1\}$, the asymptotics relates to *coin tossing*.

In our problem, we have some ergodicity property ! For general half planes, the coefficients of (BL) or (BL1) are not periodic, *but they are quasiperiodic in the tangential variable.*

Reminder: A function $F = F(z_1)$ is quasiperiodic if it reads

$$\boxed{F(z_1) = \mathcal{F}(\lambda z_1)},$$

where $\lambda \in \mathbb{R}^D$ and $\mathcal{F} = \mathcal{F}(\theta)$ is periodic over $\mathbb{R}^D$ ($D \geq 1$).

Example: For (BL1), $D = 2$, and $\lambda = n^\perp$ (the tangent vector).

Previous results: $n \in \mathbb{R}\,\mathbb{Q}^2$.

Idea: Replace this by the small divisor assumption:

$$\boxed{\text{(H)} \quad \exists \kappa > 0,\ |n \cdot \xi| \geq \kappa |\xi|^{-2}, \quad \forall \xi \in \mathbb{Z}^2 \setminus \{0\}}$$

Remarks:

- Assumption (H) is generic in the normal n: satisfied for a set of full measure in $\mathbb{S}^1$.
- Does not include the previous result.

Theorem:

If n satisfies (H), the system (BL) *is "well-posed", with a smooth solution U that converges fast to some constant U_∞.*

Moreover, U_∞ does not depend on a.

Proof of the proposition:

- Well-posedness: involves quasiperiodicity. One has:

$$\boxed{\begin{cases} \nabla_z \cdot (B \nabla_z V) = 0, \quad z_2 > a, \\ V|_{z_2 = a} = \psi, \end{cases}}$$

where $B(z) = \mathcal{B}(\lambda z_1, z_2)$, $\psi(z) = \mathcal{P}(\lambda z_1, z_2)$.

Functions $\mathcal{B} = \mathcal{B}(\theta, t)$ and $\mathcal{P} = \mathcal{P}(\theta, t)$ are periodic in $\theta \in \mathbb{T}^2$.

Idea: consider an enlarged system in θ, t, of unknown $\mathcal{V} = \mathcal{V}(\theta, t)$:

$$\boxed{\begin{cases} D \cdot (\mathcal{B} D \mathcal{V}) = 0, \quad t > a, \\ \mathcal{V}|_{t=a} = \mathcal{P} \end{cases}} \tag{BL2}$$

where D is the "degenerate gradient" given by $D = (\lambda \cdot \nabla_\theta, \partial_t)$

Advantage: Back to a periodic setting ($\theta \in \mathbb{T}^2$).

Drawback: degenerate elliptic equation.

- Variational formulation with a unique weak solution $\mathcal{V}$.

- One can prove through energy estimates than $\mathcal{V}$ is smooth.

- Allows to recover V through the formula $V(z) = \mathcal{V}(\lambda z_1, z_2)$.

- Convergence to a constant:

  Relies on Saint-Venant estimates, adapted to (BL2). Thanks to (H), we prove that $F(t) := \int_{t'>t} |D\mathcal{V}|^2 \, d\theta \, dt'$ satisfies

$$\boxed{F(t) \le C(-F'(t))^\alpha, \quad \forall \alpha < 1.}$$

Conclusion: Much better understanding of the auxiliary boundary layer systems. Allows to handle generic polygonal domains.

d) Extension to smooth domains

Main problem: No smoothness of U_∞ with respect to n is known. It is only defined almost everywhere (diophantine assumption).

Idea: *For any $\kappa > 0$, U_∞ is Lipschitz in restriction to*

$$A_\kappa := \left\{ n \in \mathbb{S}^1,\ |n \cdot \xi| \geq \frac{\kappa}{|\xi|^2},\ \forall \xi \in \mathbb{Z}^2 \setminus \{0\} \right\}.$$

Remark: One has $|A_\kappa^c| = O(\kappa)$.

Idea: The construction of the boundary layer corrector can be performed in the vicinity of points x such that $n(x) \in A_\kappa$.

Contribution of the remaining part of the boundary is negligible when $\kappa \ll 1$.

Broadly, optimizing in κ and ε yields a rate.

Thank you