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Università di Padova - June 25-29, 2012

Can one obtain numerically a non-existent solution for a viscous system of conservation laws?

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HYP 2012 – Padova

System of 2 conservation laws with diffusion

$$U_t + F(U)_x = kU_{xx}$$

where $U = (u, v)^T$ and $F(U) = \frac{1}{2}\begin{pmatrix} 3u^2 + v^2 + \sigma u + 2v \\ 2uv + \sigma v \end{pmatrix}$

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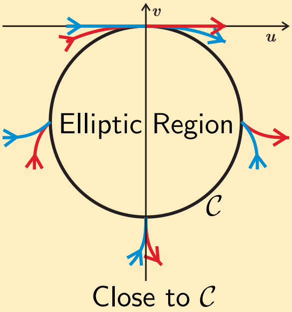
Rarefactions and Elliptic region

1-Rarefaction
2-Rarefaction



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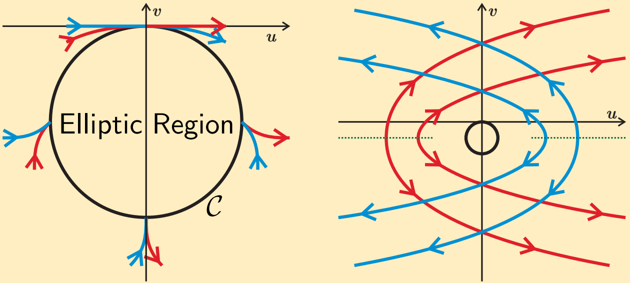

Close to $\mathcal{C}$

Far from $\mathcal{C}$
Schaeffer-Shearer's Case IV

The Riemann problem (chosen from an open set)

$U(x < 0, t = 0) = L = (-0.80; -0.39)$,

$U(x > 0, t = 0) = R = (-1.32; -1.15)$

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Choices in numerical simulations

Non-linear Crank-Nicolson

$k = 2$

Simulation	A	B	C	D
Total Time	10e3	10e3	10e3	10e3
Δt	68.0e-3	34.1e-3	17.0e-3	8.5e-3
N. Time Steps	0.15e6	0.29e6	0.59e6	1.18e6
x interval	12e3	12e3	12e3	12e3
Δx	0.40	0.20	0.10	0.05
N. Grid points	30e3	60e3	120e3	240e3
σ	1.95-2.15	1.95-2.15	1.95-2.15	1.95-2.15

Numerical simulation A

Time simulation: 10.0e3

Δt: 17.7e-3

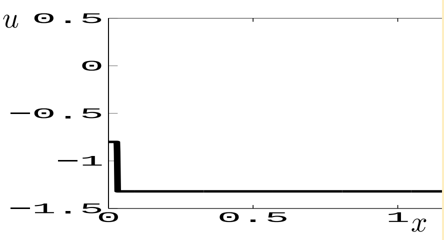
Time steps: 0.57e6

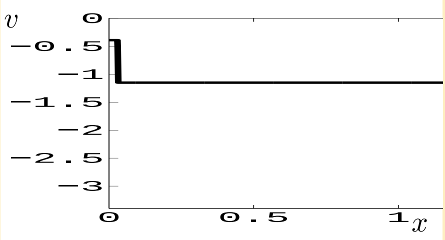
Grid size: 0.4

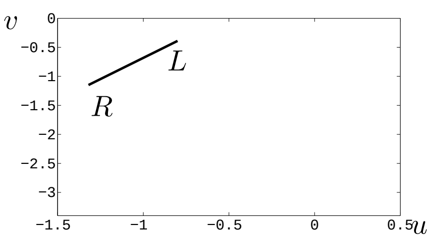
Number of Grid Points: 30e3

Numerical simulation **A**: Grid points 30.0e3

Initial time; step $0.0e3$

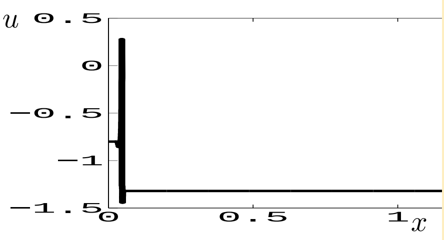


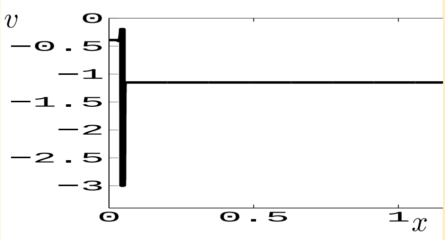


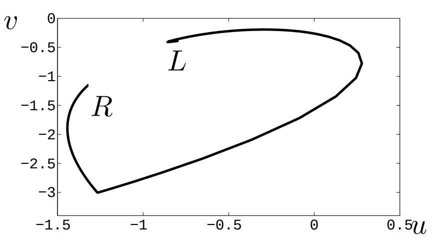


Numerical simulation **A**: Grid points 30.0e3

Time = 300; step $4.4e3$







Numerical simulation **A**: Grid points 30.0e3

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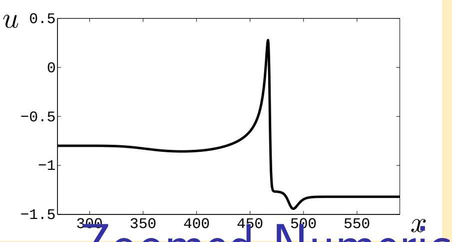
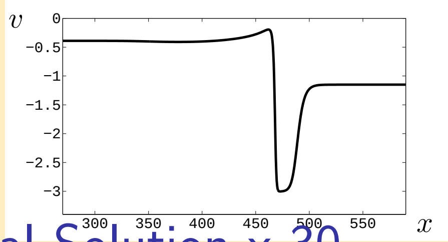
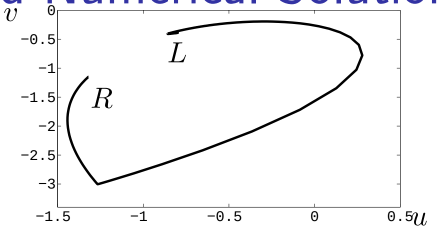




Zoomed Numerical Solution x 30



Numerical simulation **A**: Grid points 30.0e3

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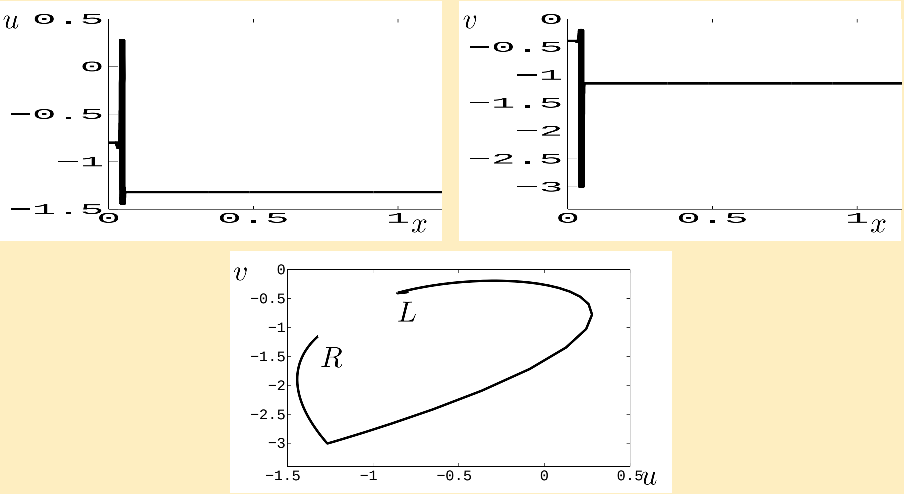
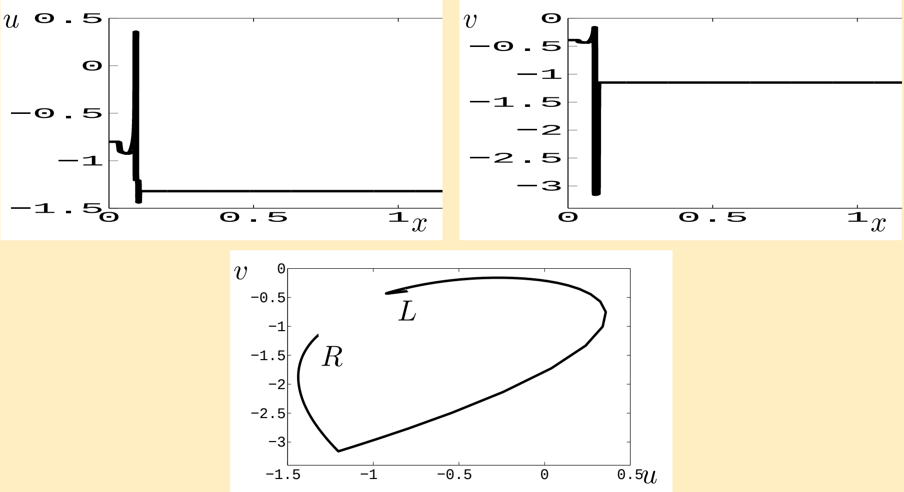


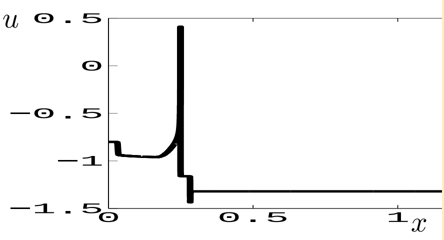
Numerical simulation **A**: Grid points 30.0e3

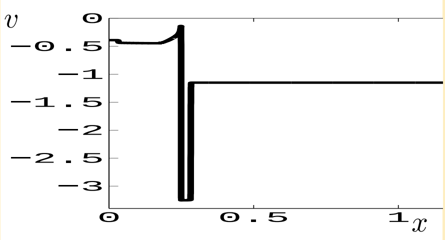
Time = 1000; step $14.7e3$

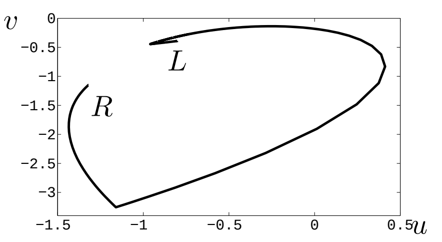


Numerical simulation **A**: Grid points 30.0e3

Time = 3000; step $44.1e3$

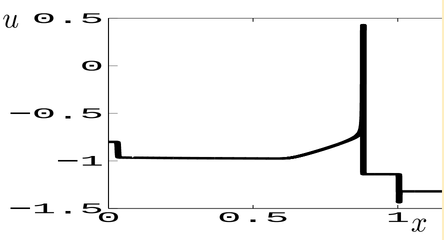


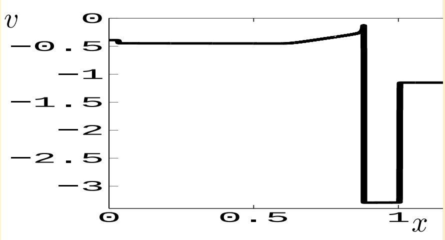


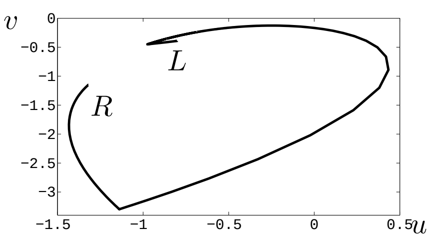


Numerical simulation **A**: Grid points 30.0e3

Time = 10000; step $147e3$



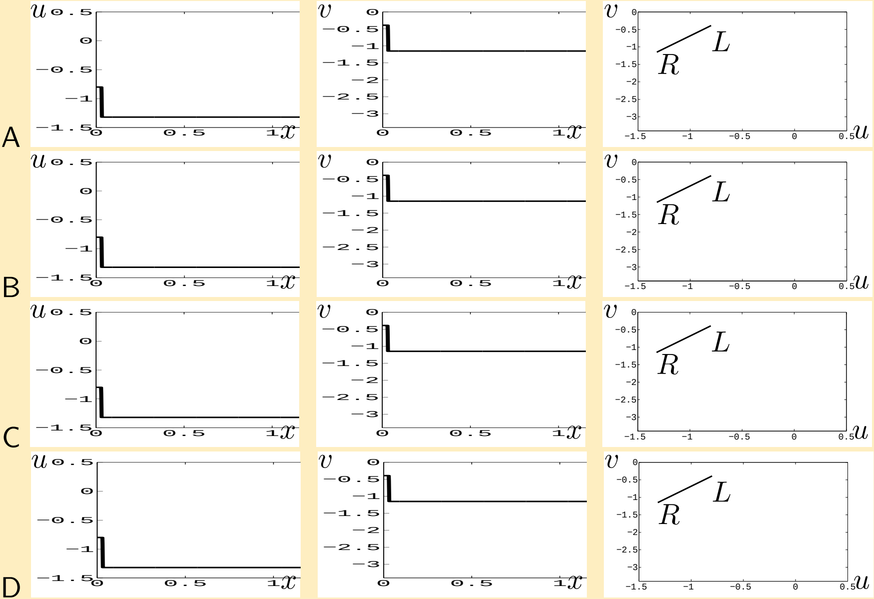




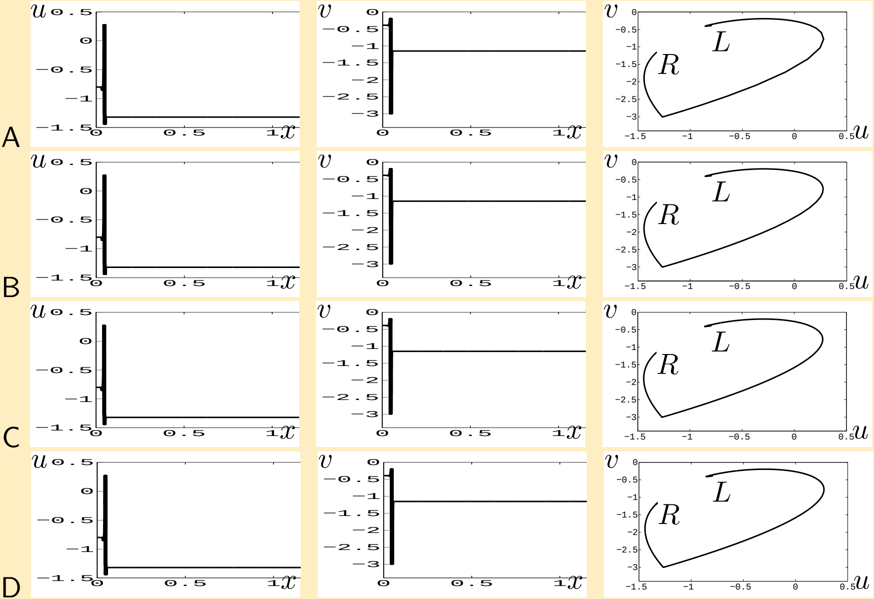
Simulations A, B, C and D.

Comparison of the simulations.

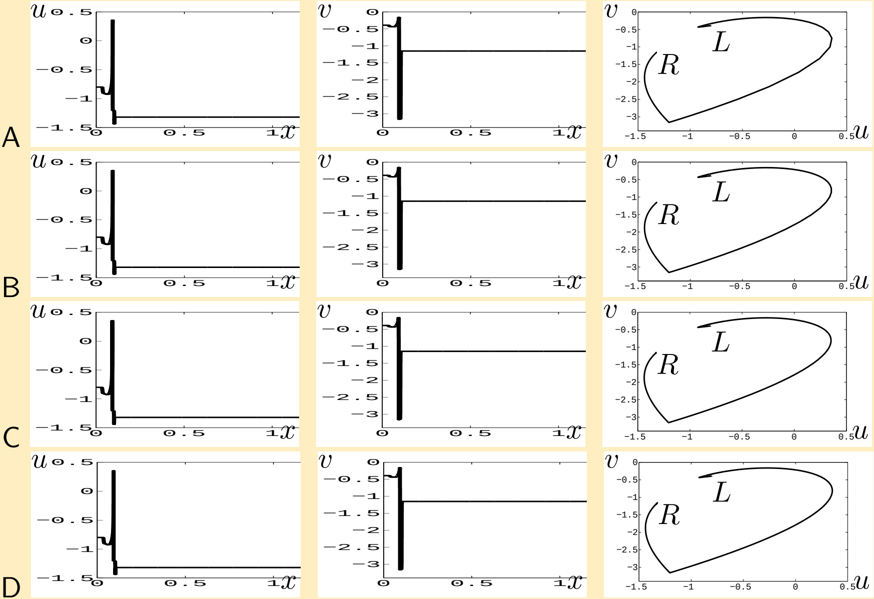
Initial time



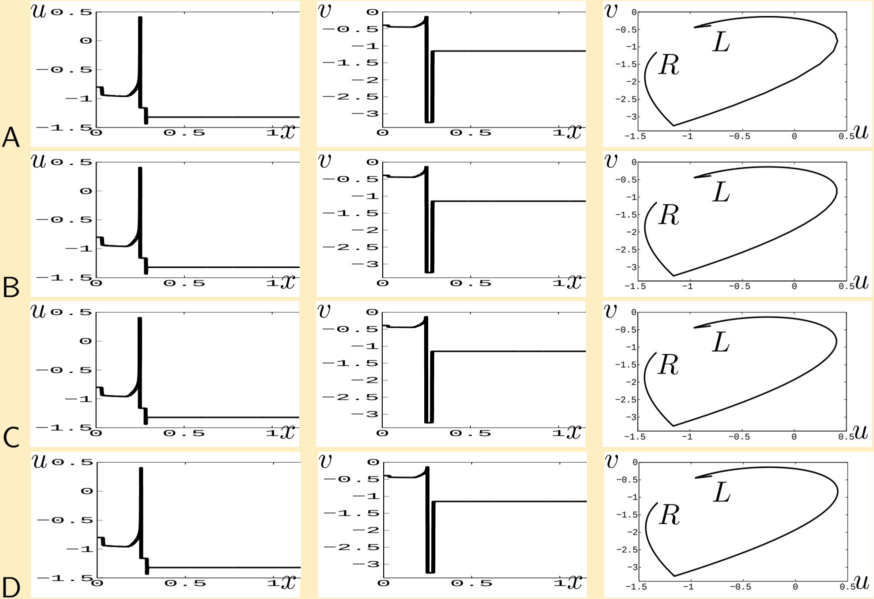
Time = 300



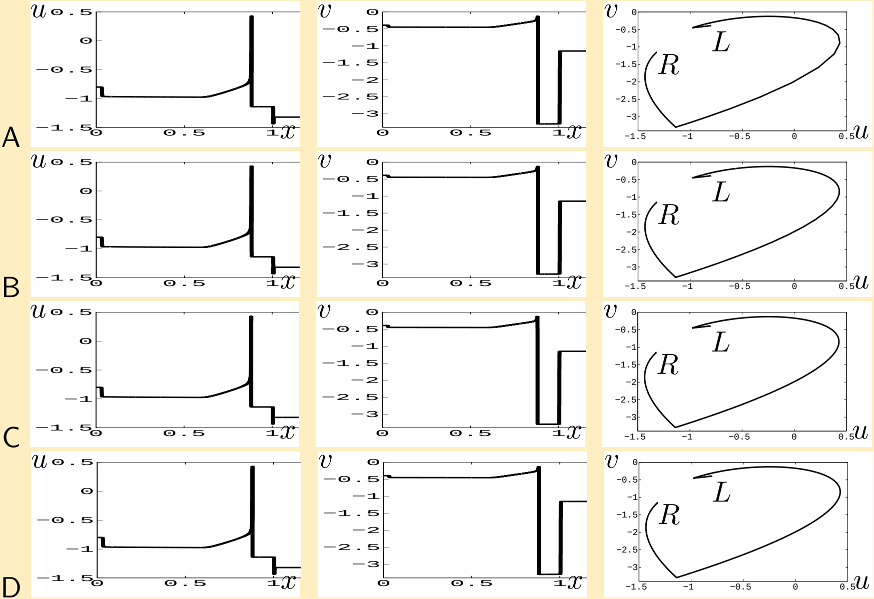
Time = 1000



Time = 3000

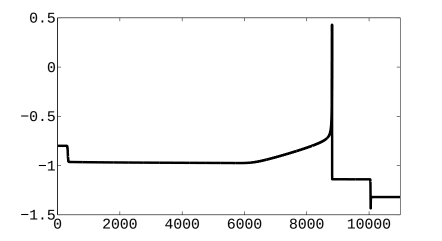


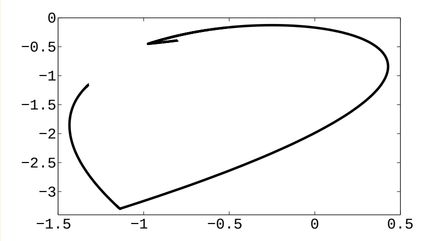
Time = 10000



Structure of the solution.

The numerical solution contains four waves:





Structure of the solution.

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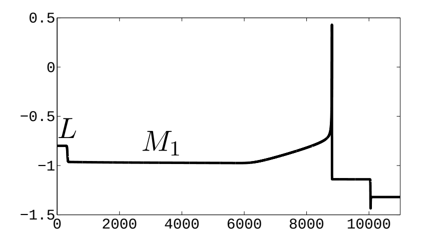

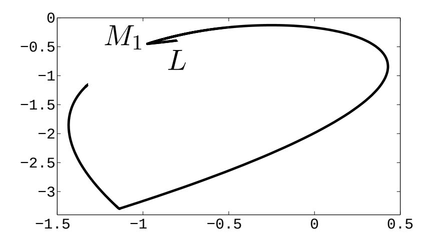

- 1-Shock: $L - M_1$

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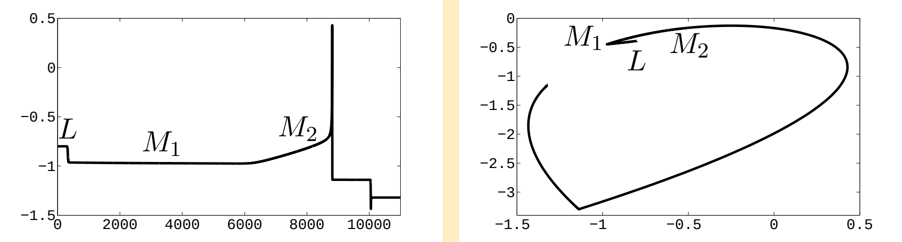

- 1-Shock: $L - M_1$
- 2-Rarefation: $M_1 - M_2$

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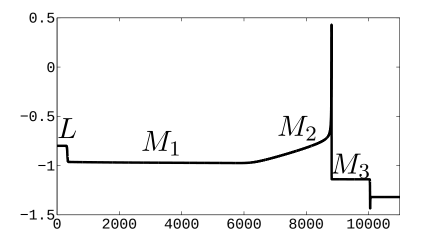

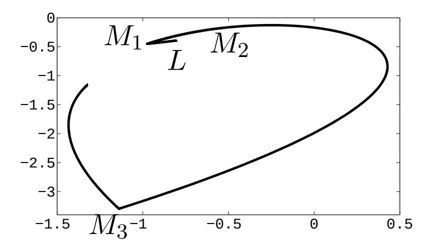

- 1-Shock: $L - M_1$
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- 2L-Characteristic "crossing" shock: $M_2 - M_3$

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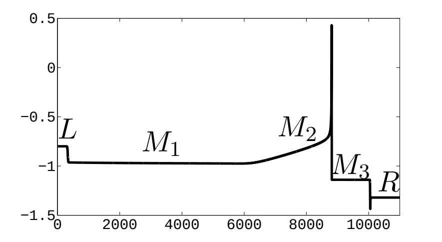

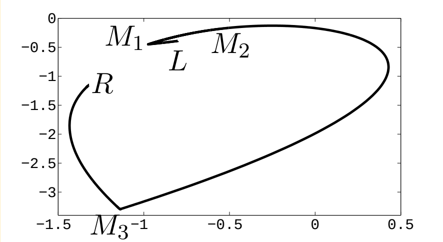

- 1-Shock: $L - M_1$
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- 2L-Characteristic "crossing" shock: $M_2 - M_3$
- 2-shock: $M_3 - R$

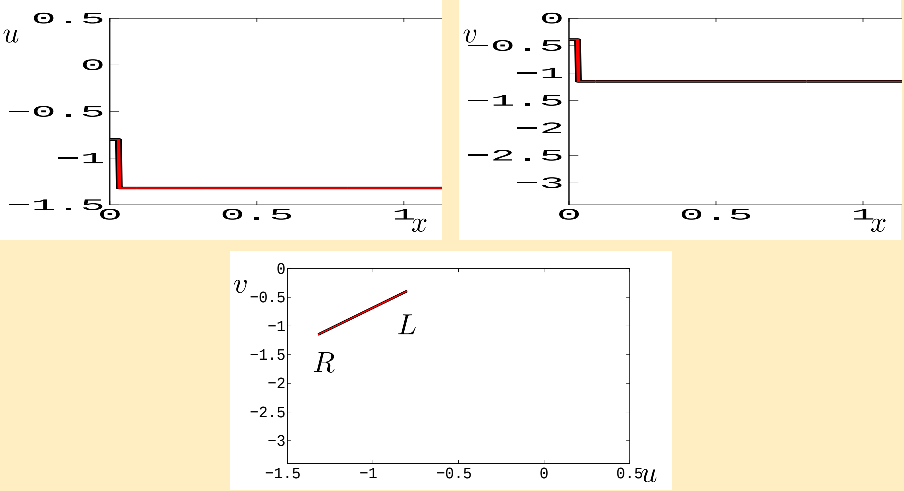
Simulation **A** and **D**

Solutions A and D overlapped.

Shocks zoomed.

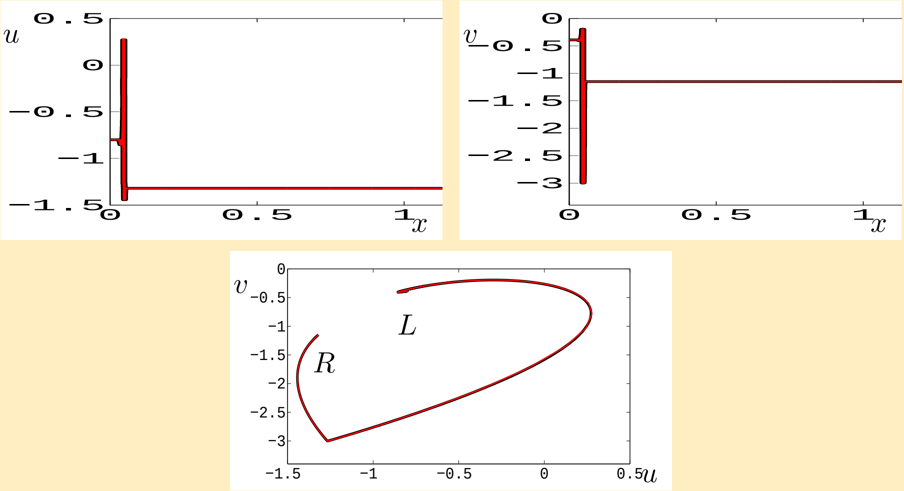
Simulation A in red; Simulation D in black.

Initial time



Simulation A in red; Simulation D in black.

Time = 300



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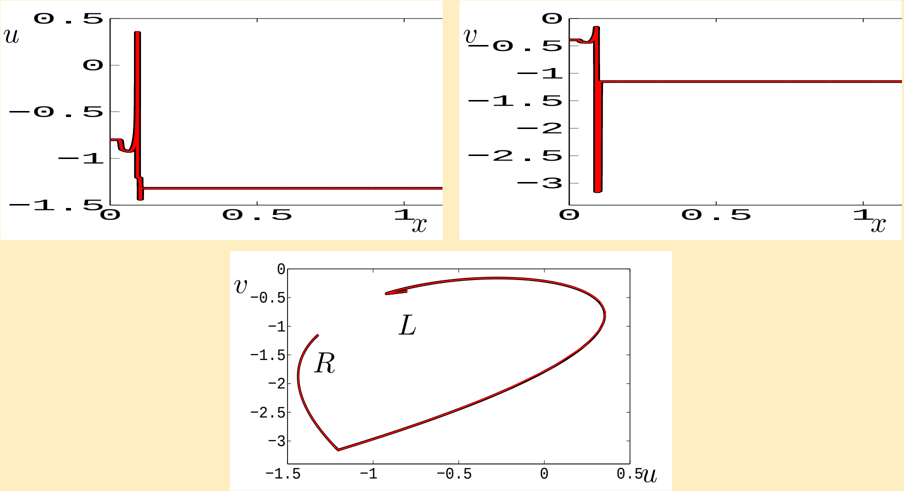
Time = 1000



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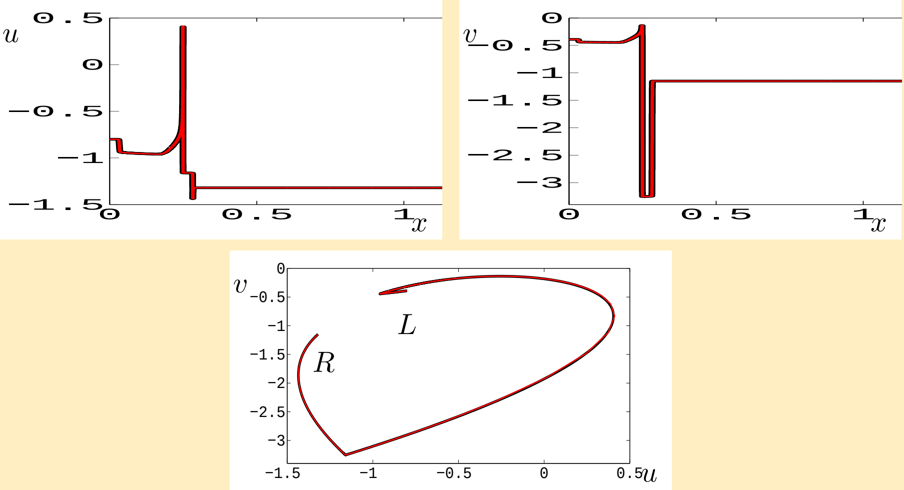
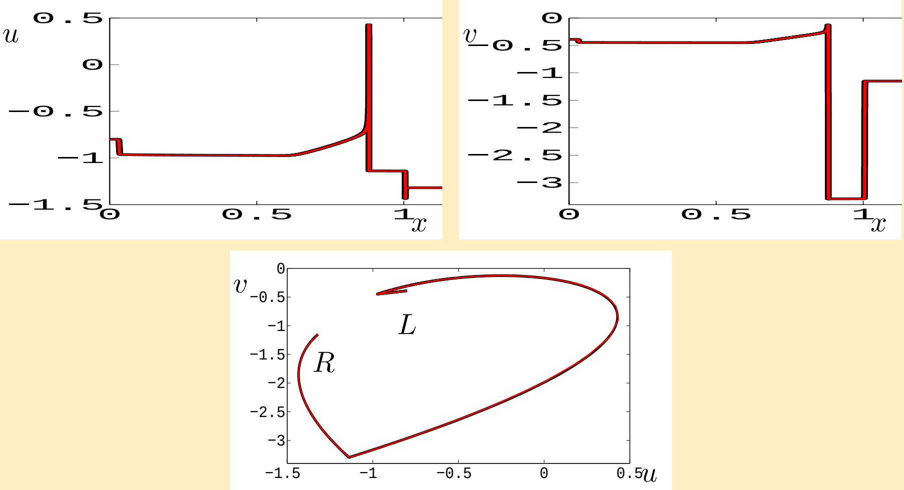


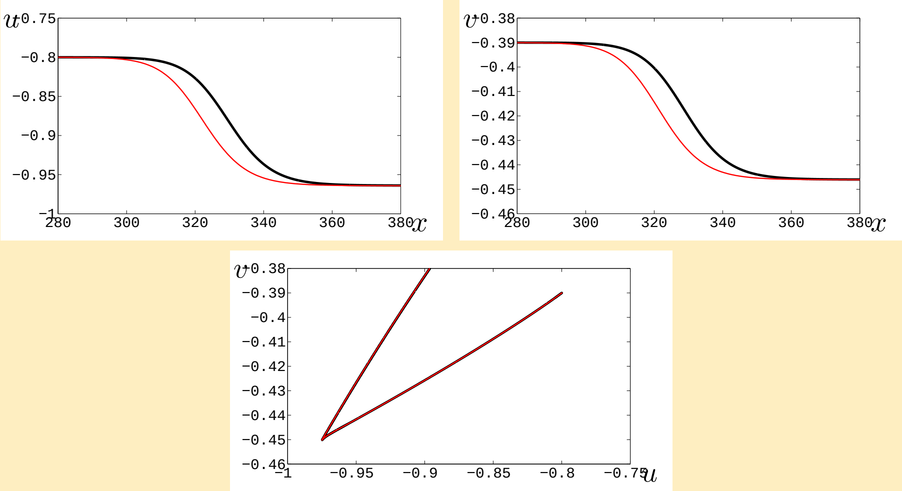
Simulation A in red; Simulation D in black.

Time = 10000



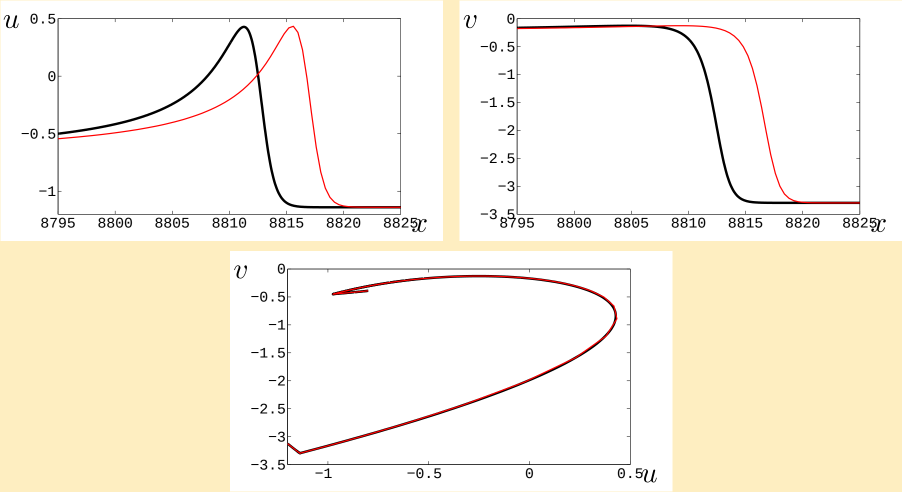
Simulation A in red; Simulation D in black.

Time = 10000 **First Shock:** $L - M_1$.



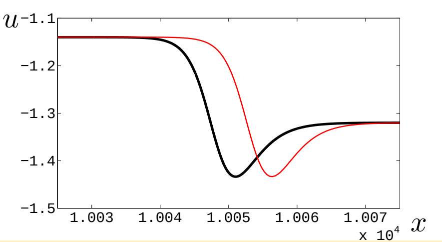
Simulation A in red; Simulation D in black.

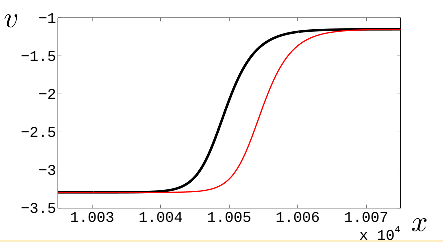
Time = 10000 **Second Shock:** $M_2 - M_3$.

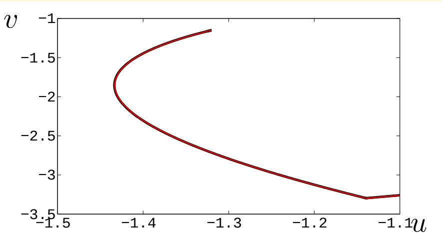


Simulation A in red; Simulation D in black.

Time = 10000 **Third Shock:** $M_3 - R$.



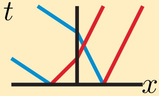




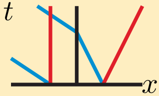
What is a crossing shock?

1-characteristic

2-characteristic



Crossing shock



2L-Characteristic crossing shock

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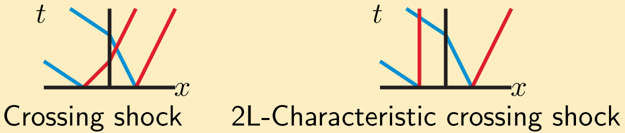
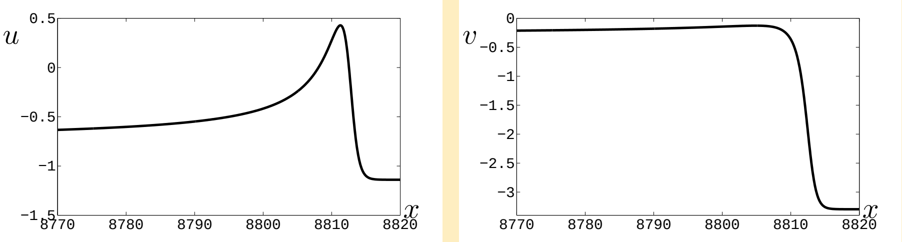


The 2L-characteristic crossing shock profile:



Simulation E: 1000 grid points out of 240e3.

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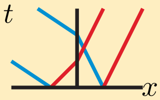
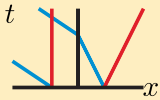
2-characteristic



Crossing shock



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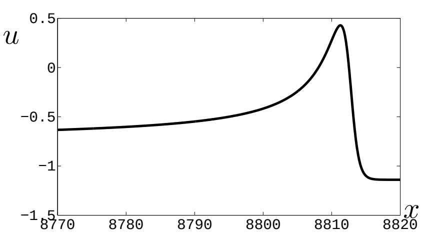

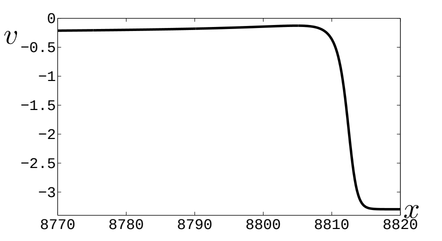

Simulation E: 1000 grid points out of 240e3.

However, this shock in the PDE does not have a viscous profile.

Background

Traveling wave:

A solution of $U_t + F(U)_x = kU_{xx}$ depending only on $\xi = \frac{x-st}{k}$ satisfies

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Integrating once over $(-\infty, \xi)$ with boundary conditions $\lim_{\xi\to-\infty} U = U_-$ and $\lim_{\xi\to-\infty} U_\xi = 0$, we obtain the ODE:

$$\frac{dU}{d\xi} = F(U) - F(U_-) - s(U - U_-) \tag{1a}$$

$$\dot{U} = G(U; U_-, s) \tag{1b}$$

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$$\dot{U} = G(U; U_-, s) \quad (1b)$$

Admissible shock:

A shock connecting U_- to U_+ is admissible iff there is an orbit of (**??**) connecting the equilibrium U_- to the equilibrium U_+.

Phase diagram and Numerical Solution

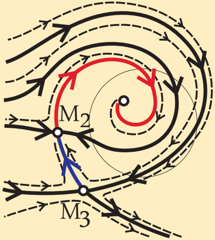
For the 2L-characteristic crossing shock between M_2 and M_3 the ODE is

$$\dot{U} = F(U) - F(M_2) - s(U - M_2) = G(U; M_2, s)$$

(s is the shock speed.)

M_2 is a saddle-node equilibrium and M_3 is a saddle equilibrium.

ODE phase portrait:



There is no orbit from M_2 to M_3 (incorrect orientation).

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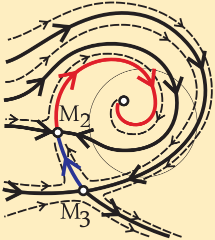
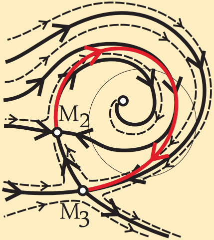


Numerical profile:



There is no orbit from M_2 to M_3 (incorrect orientation).

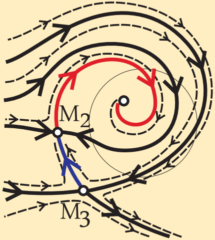
Phase diagram and Numerical Solution

For the 2L-characteristic crossing shock between M_2 and M_3 the ODE is

$$\dot{U} = F(U) - F(M_2) - s(U - M_2) = G(U; M_2, s)$$

(s is the shock speed.)
M_2 is a saddle-node equilibrium and M_3 is a saddle equilibrium.

ODE phase portrait:



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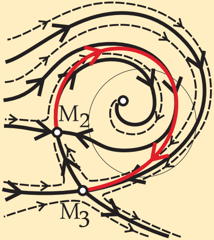

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The ODE phase portrait is compatible with the degenerate Bogdanov-Takens and Transcritical bifurcations that occur.

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We consider $\dot{U} = G(U; L, s) = G(U; L)$ with fixed s, thus the two bifurcation parameters are L.

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where $\mathcal{C}$ is the coincidence locus ($DF(U^*)$ has a double real eigenvalue).

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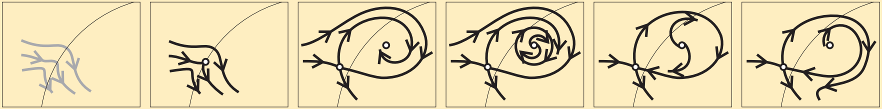
Bogdanov-Takens bifurcation on a fold

The map $(U; L) \mapsto (H(U; L), \operatorname{tr} D_U G(U; L), \det D_U G(U; L))$ is not regular at (U^*, L^*).

Bogdanov-Takens bifurcation

ODE phase portraits observed

(Gray is not possible: L is a trivial equilibrium)



(Partial) Saddle-Node   Andronov-Hopf   Homoclinic

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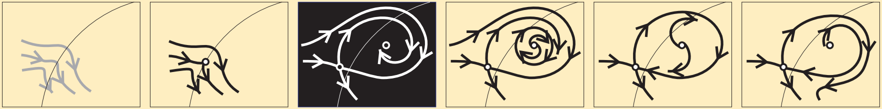

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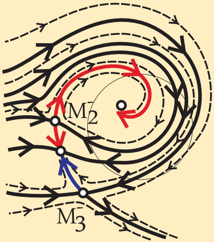
Transcritical bifurcation

There exist two extra equilibria of $\dot{U} = G(U; L)$, one of them near M_2.

A transcritical bifurcation occurs near the Bogdanov-Takens bifurcation.



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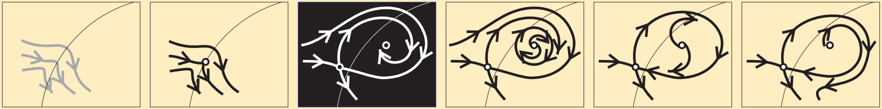

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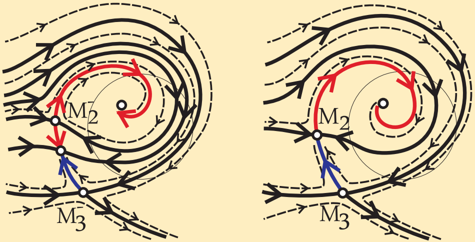

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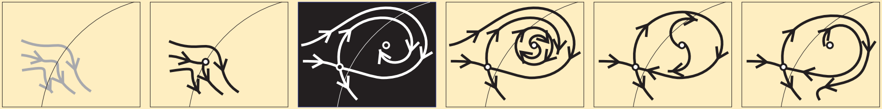

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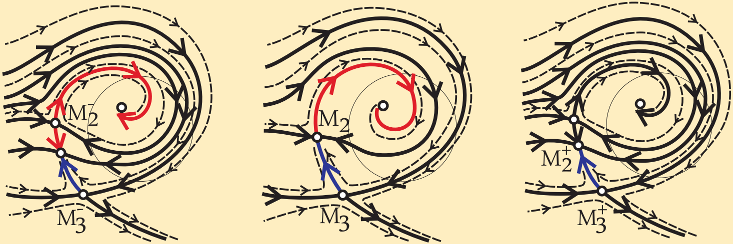

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What are we obtaining from the computer?

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$$\left(\frac{\Delta t^2 s^2}{4k^2} + \frac{\Delta x^2}{6k^2} \right) \frac{d^2 F(U)}{d\xi^2} = \frac{dU}{d\xi} - (F(U) - F(U_-) - s(U - U_-))$$

Crossing shock revisited

If we examine the vector field in $\mathbb{R}^4$ induced by:

$$\delta \frac{d^2U}{d\xi^2} = \frac{dU}{d\xi} - (F(U) - F(U_-) - s(U - U_-)) \quad (3)$$

where the parameter δ depends on the scheme and the discretization parameter.

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The frustrated revisiting of the numerical method

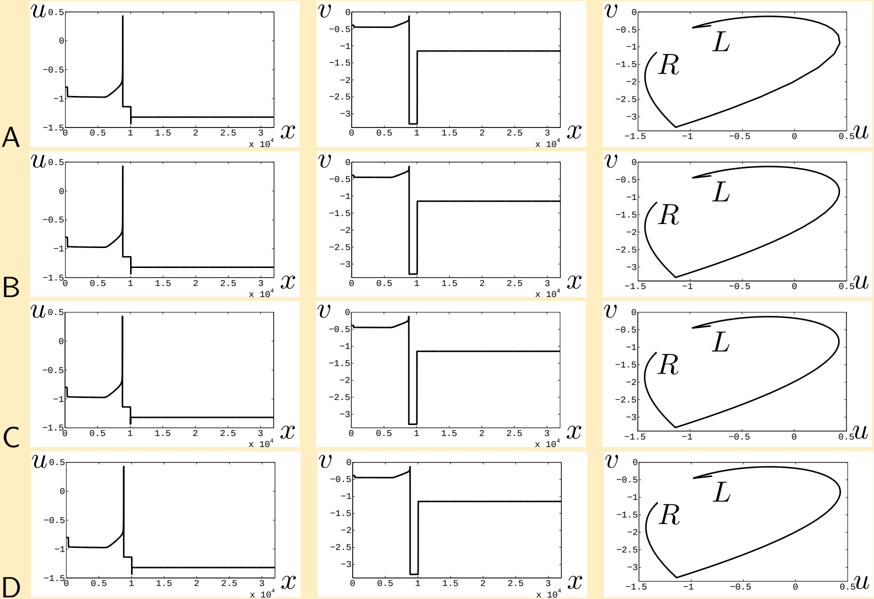
We see numerical evidence that there is an "orbit" connecting U_- a U_+.

But the "orbit" is independent of δ; the "orbit" is not explained by (**??**).

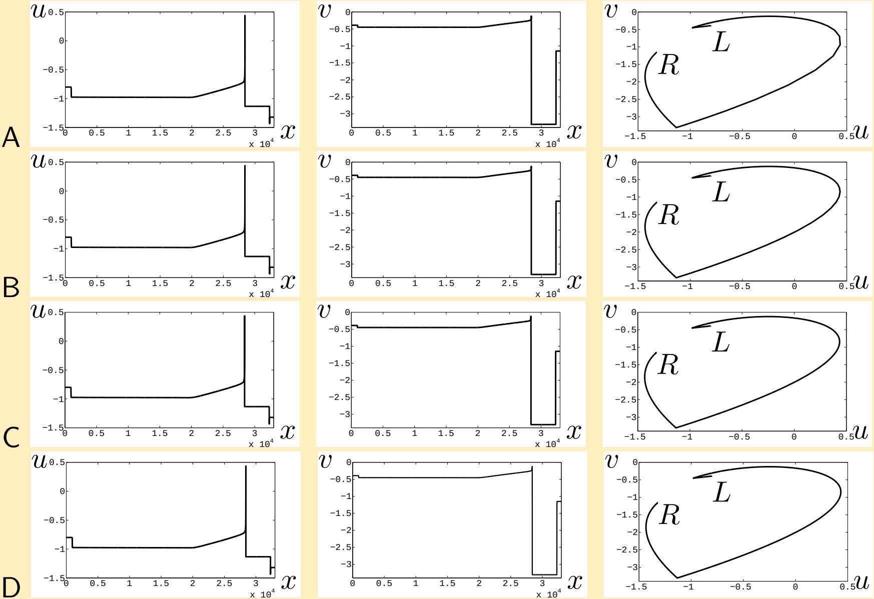
Looking to even longer simulations

Simulation	A	B	C	D
Total Time	30e3	30e3	30e3	30e3
Δt	68.0e-3	34.1e-3	17.0e-3	8.5e-3
N. Time Steps	0.44e6	0.88e6	1.76e6	3.53e6
x interval	33e3	33e3	33e3	33e3
Δx	0.40	0.20	0.10	0.05
N. Grid points	82.5e3	165e3	330e3	660e3
σ	1.95-2.15	1.95-2.15	1.95-2.15	1.95-2.15

Time = 10000

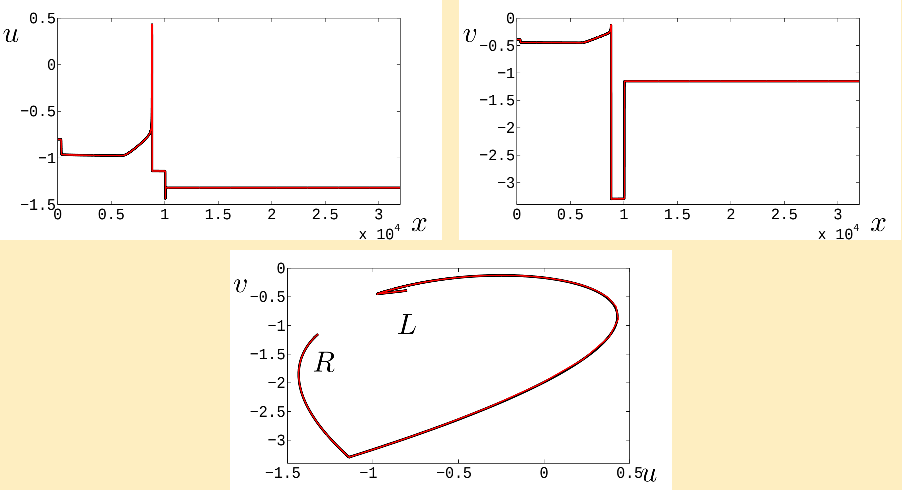


Time = 30000



Simulation A in red;   Simulation D in black.

Time = 10000



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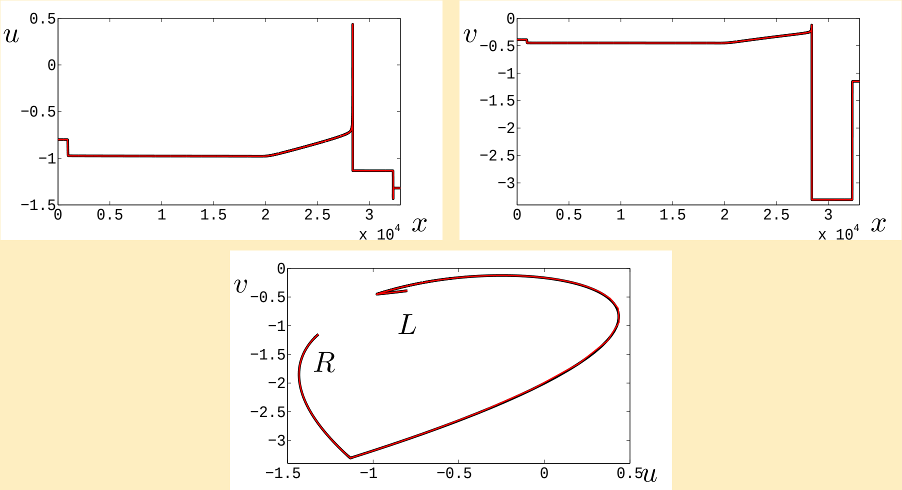

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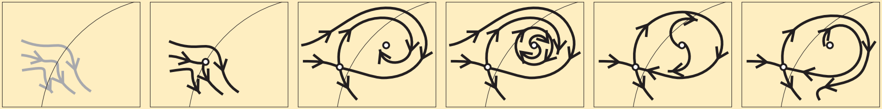
Despite the small change of the location, the stability of the third equilibrium changes from attractor to repeller.

Bogdanov-Takens bifurcation



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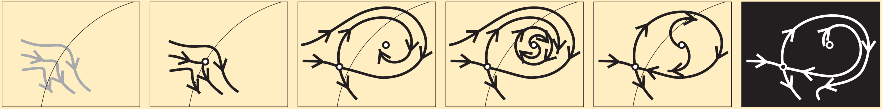

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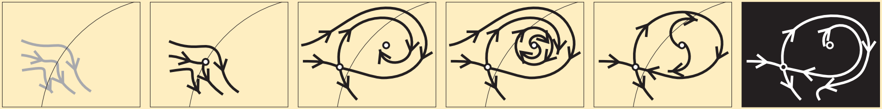
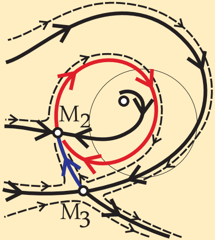


EDO phase portrait



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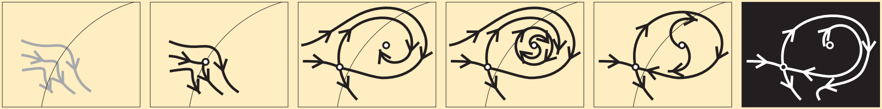

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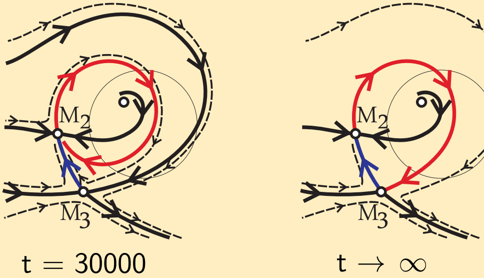

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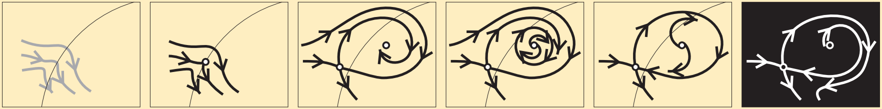

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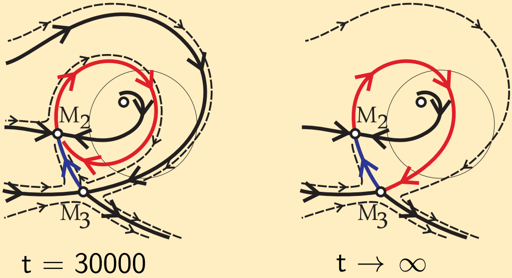

Conclusion

We find a Riemann problem that converges so slowly that, for long time, seems to converge to a non viscous solutions.

Thank You