Renormalization and universality of blowup in hydrodynamic flows and conservation laws

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Outline

- Blowup:
  - shell models of turbulence,
  - inviscid Burgers equation
  - scalar conservation laws
  - incompressible Euler equations

- Renormalization and universal structure of blowup in Fourier and physical spaces

Blowup in incompressible Euler equations

Navier–Stokes equations $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \Delta \mathbf{v} + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0$$

In Fourier space k $$\mathbf{v}(\mathbf{k}) = (2\pi)^{-3} \int \mathbf{v}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3 x$$

$$\frac{\partial \mathbf{v}}{\partial t} + i\mathbf{P} \int \mathbf{k} \cdot \mathbf{v}(\mathbf{p}) \mathbf{v}(\mathbf{k}-\mathbf{p}) d^3 p = -\mu k^2 \mathbf{v} + \mathbf{f}, \quad \mathbf{k} \cdot \mathbf{v} = 0$$

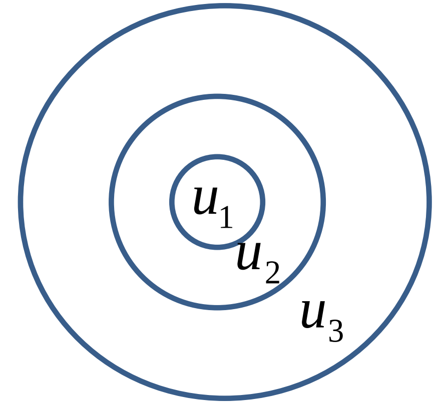
Euler equation: $\mu \to 0$

Long-standing open problems:

- statistics of veclosity field of developed turbulence
(anomalous scaling of structure functions in homogeneous isotropic turbulence)

- finite time singularity

Shell models of turbulence



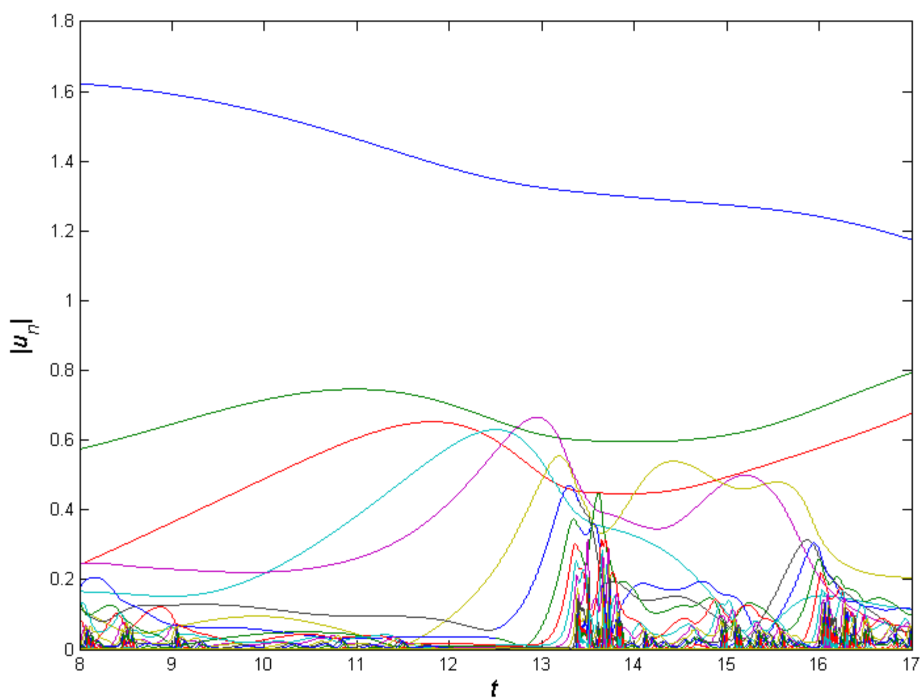
Descrete variables: $\mathbf{k} \to k_n = k_0 \lambda^n, \qquad \mathbf{v} \to u_n$

$\lambda = 2: \quad k_1 = 2k_0, \quad k_2 = 4k_0, \quad k_3 = 8k_0, \quad k_4 = 16k_0 \ldots$

Gledzer-Okhitani-Yamada (GOY) model:

$$\left(\frac{d}{dt} + \mu k_n^2\right) u_n = i(k_n u_{n+2}^* u_{n+1}^* - b k_{n-1} u_{n+1}^* u_{n-1}^* + c k_{n-2} u_{n-1}^* u_{n-2}^*) + f_n$$

(quadratic nonlinearity, conservation of energy and helicity)



Structure functions:

$$S_p(k_n) = \left\langle |u_n|^p \right\rangle \propto k_n^{-\varsigma_n}$$

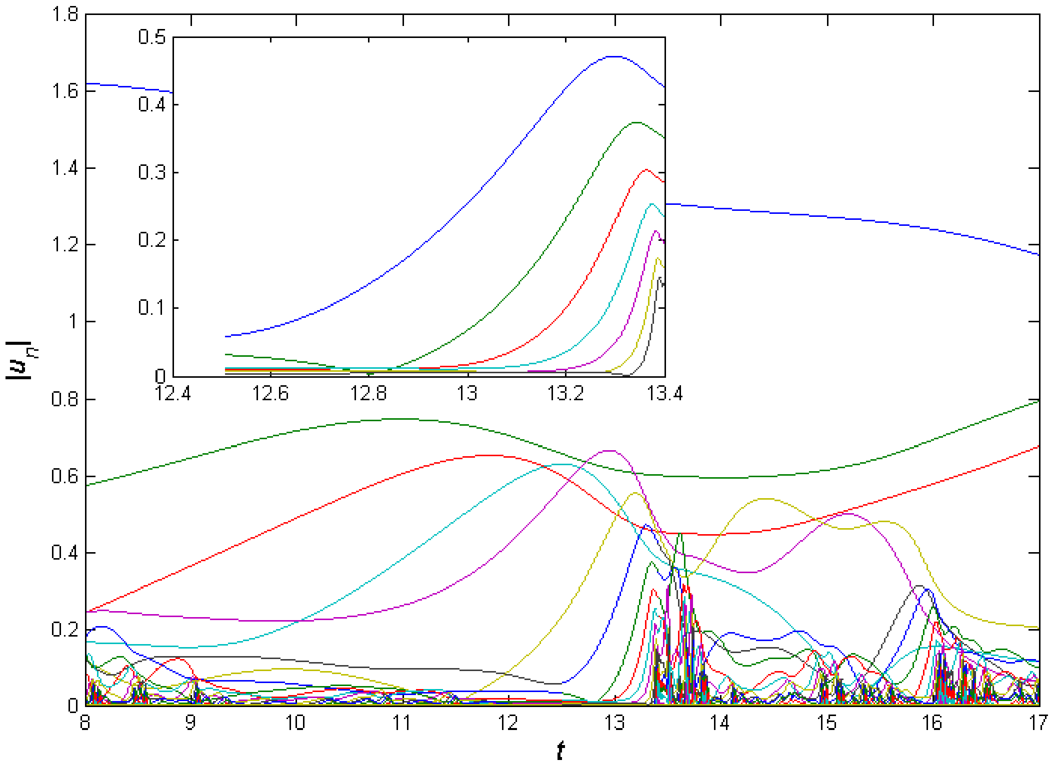
Kolmogorov (K41 theory) $\quad \varsigma_n = p/3$

Anomalous scaling

$$GOY: \ \varsigma_2 = 0.71, \ \varsigma_3 = 1,$$
$$\varsigma_4 = 1.27, \ \varsigma_5 = 1.52$$

$$NS: \ \varsigma_2 = 0.7(2), \ \varsigma_3 = 1,$$
$$\varsigma_4 = 1.27(3), \ \varsigma_5 = 1.53(4)$$

Scaling universality of blow-up in shell model



Self-similar solution
(for vanishing viscosity):

$$u_n(t) = -ik_n^{-y} f((t - t_n)k_n^z)$$

$$k_n = k_0 \lambda^n$$

$$u_n \propto k_n^{-y}$$

$$t_n - t_{n-1} = T\lambda^{-zn} \propto k_n^{-z}$$

Self-similarity of blow-up under scaling in time, space and state

$$z = 0.719, \quad y = 1 - z = 0.281$$

Kolmogorov theory K41 gives
z = 2/3 = 0.666

Numerical description: E.D.Siggia (1978), T.Nakano (1988)

Relation to turbulence statistics: V.S.L'vov et al. (2001), AM (2012)

Dombre-Gilson approach (1998)

Renormalized variables (blowup $t \to t_c$ or $\tau \to \infty$):

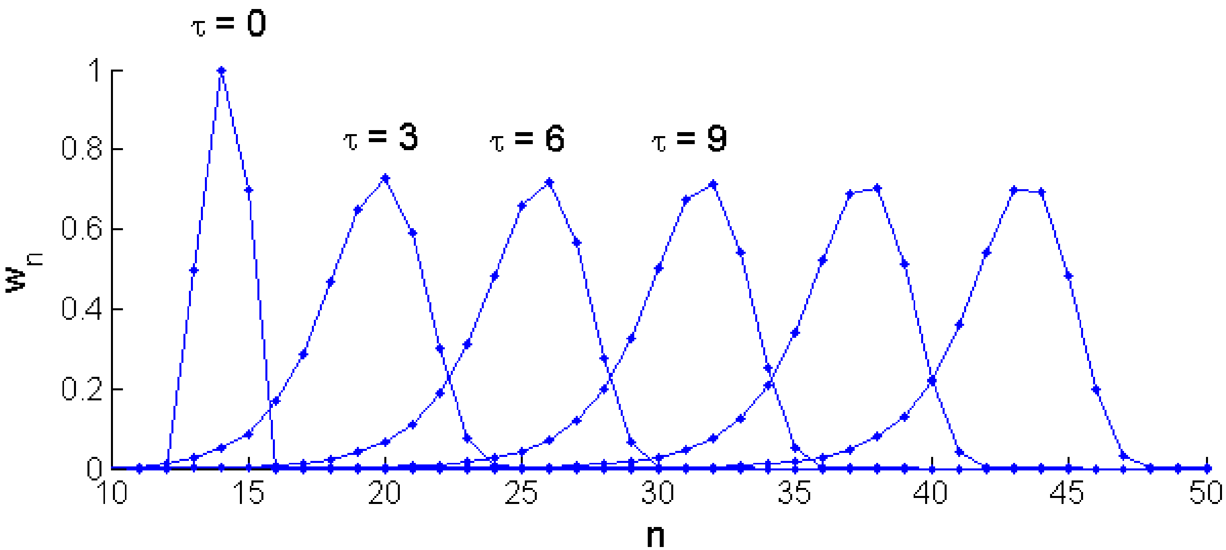
$$u_n = -ik_n^{-1} \exp\left[\int_0^{\tau} A(\tau')d\tau'\right] w_n, \quad t = t_c - \int_{\tau}^{\infty} \exp\left[-\int_0^{\tau'} A(\tau'')d\tau''\right] d\tau'$$

Renormalized equations:

$$\frac{dw_n}{d\tau} = N_n[w] - Aw_n, \quad N_n[w] = -w_{n+2}w_{n+1}^*/4 + w_{n+1}w_{n-1}^*/2 + 2w_{n-1}w_{n-2}$$

Conservation law:

$$A = \sum_n w_n N_n[w] \Big/ \sum_n w_n^2 \quad \Rightarrow \quad \sum_n w_n^2 = const$$



Solitary wave:

$$w_n(\tau) = W(n - s\tau)$$

self-similar blowup in original variables

Blowup in the inviscid Burgers (Hopf) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Solution (method of characteristics):

$$x = x_0 + u_0(x_0)(t - t_0), \quad u = u_0(x_0)$$

First derivative:

$$\frac{\partial u}{\partial x} = \frac{\partial u / \partial x_0}{\partial x / \partial x_0} = \frac{u_0'(x_0)}{1 + u_0'(x_0)(t - t_0)}$$

blow-up (formation of a shock) at $t = t_0 - \dfrac{1}{u_0'(x_0)}$

Initial condition (adjusted using symmetries of the model):

$$u_0(0) = 0, \quad t_0 = 1/u_0'(0) < 0, \quad u_0''(0) = 0, \quad u_0'''(0) > 0$$

Blow-up occurs at $t = x = u = 0$

Dombre-Gilson approach for inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \xrightarrow[\text{transform}]{\text{Fourier}} \frac{\partial u}{\partial t} = -i\int (k-p)u(p)u(k-p)dp$$

Renormalized variables:

$$u(k) = ik^{-2}\exp\left[\int_0^{\tau} A(\tau')d\tau'\right]w(k), \quad t = t_c - \int_{\tau}^{\infty}\exp\left[-\int_0^{\tau'} A(\tau'')d\tau''\right]d\tau'$$

Renormalized equations:

$$\frac{\partial w}{\partial \tau} = N[w] - Aw, \quad N[w] = \sum_{\sigma=\pm}\int w(\eta)w(\varsigma)\frac{k^2}{pq}d\eta$$

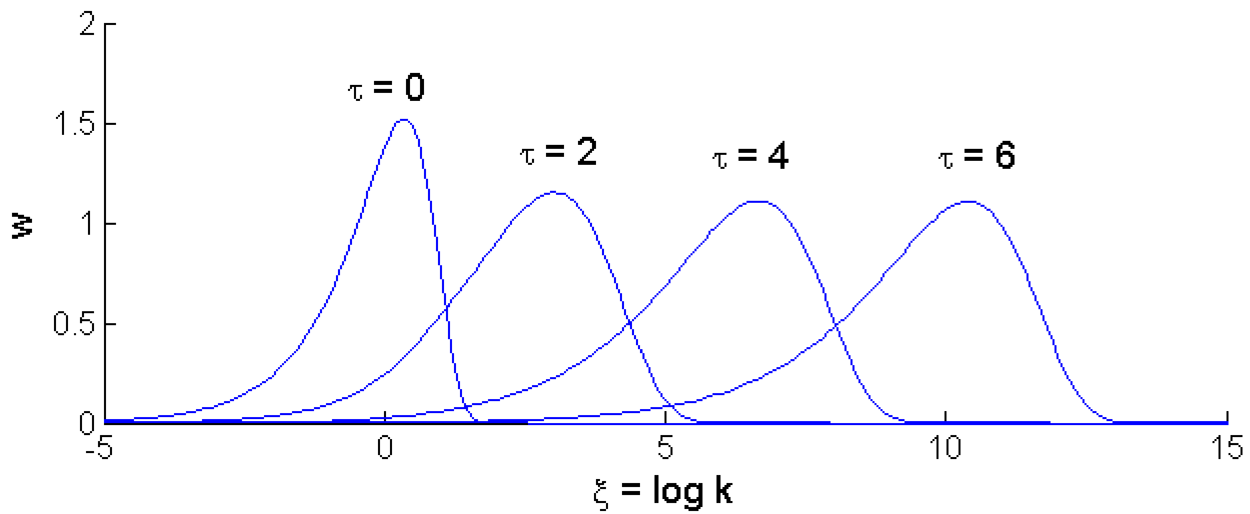
$$k = e^{\xi}, \quad p = \sigma e^{\eta}, \quad q = |k-p| = e^{\varsigma}, \quad A = \int wN[w]d\xi \Big/ \int w^2 d\xi$$

Translational invariance and conservation law:

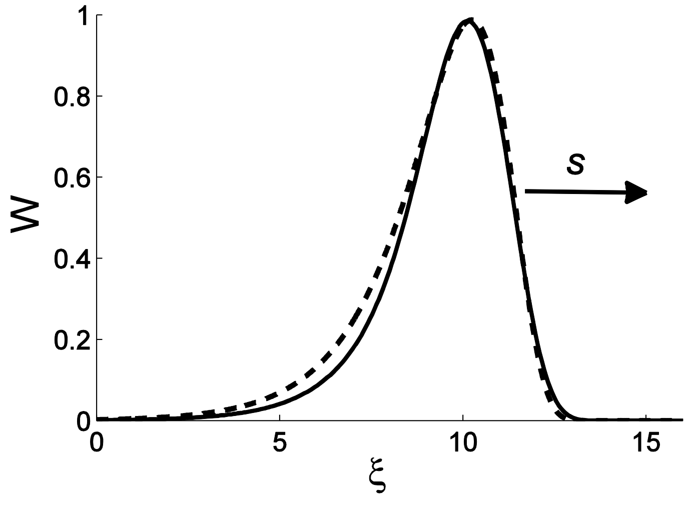
$$\xi \to \xi + \Delta\xi, \quad \int w^2 d\xi = const$$

Solitary wave in the inviscid Burgers equation

$$w(\tau,\xi) = W(\xi - s\tau)$$



Comparison with the blowup
in the shell model of turbulence



Renormalization and universality of blowup in scalar conservation laws

Inviscid Burgers equation

Renormalized solution: $u_\lambda(t,x) = \lambda^{1/3} u(\lambda^{-2/3} t, \lambda^{-1} x)$

Small scale limit: $w(t,x) = \lim_{\lambda\to\infty} u_\lambda(t,x)$

Universal function: $x = wt - cw^3$ Y.Pomeau et al. (2008), J.Eggers & M.A.Fontelos (2009)

General scalar conservation law

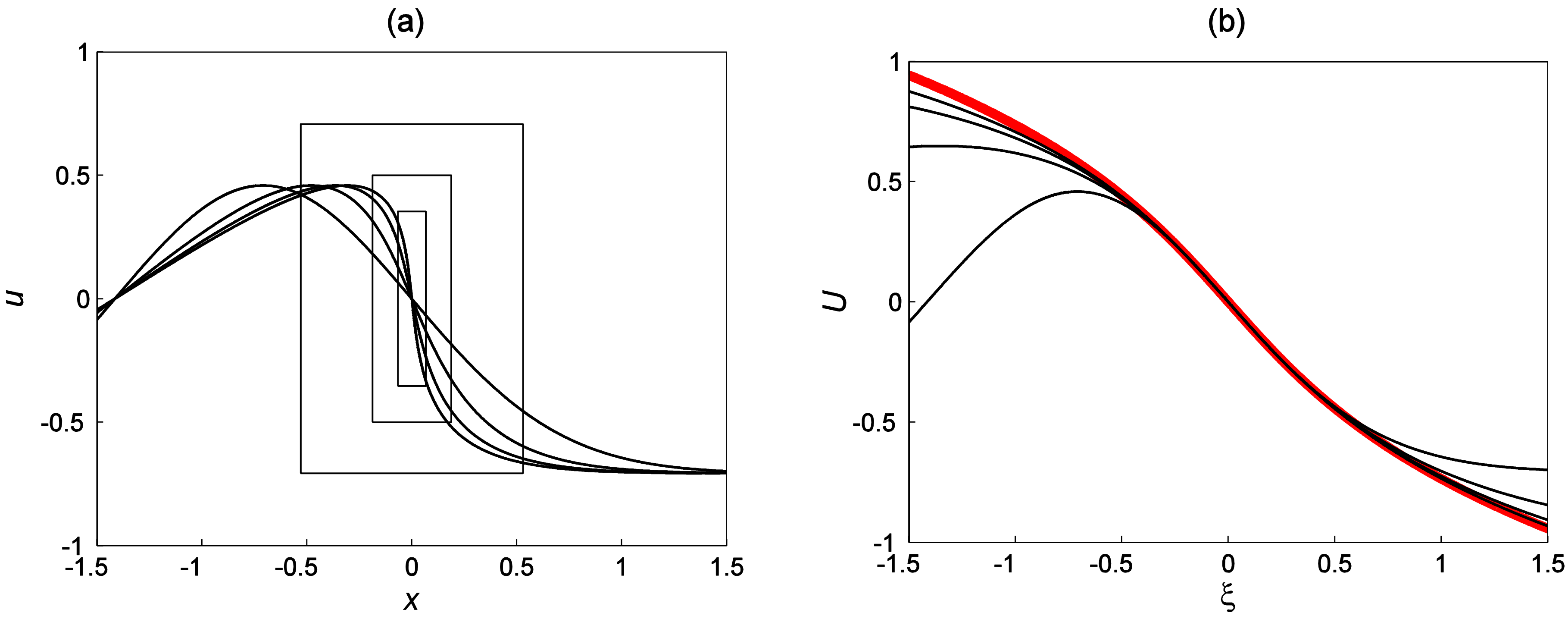
$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0$$

Renormalization of flux function: $F_\lambda(u) = \lambda^{2/3} F(\lambda^{-1/3} u)$

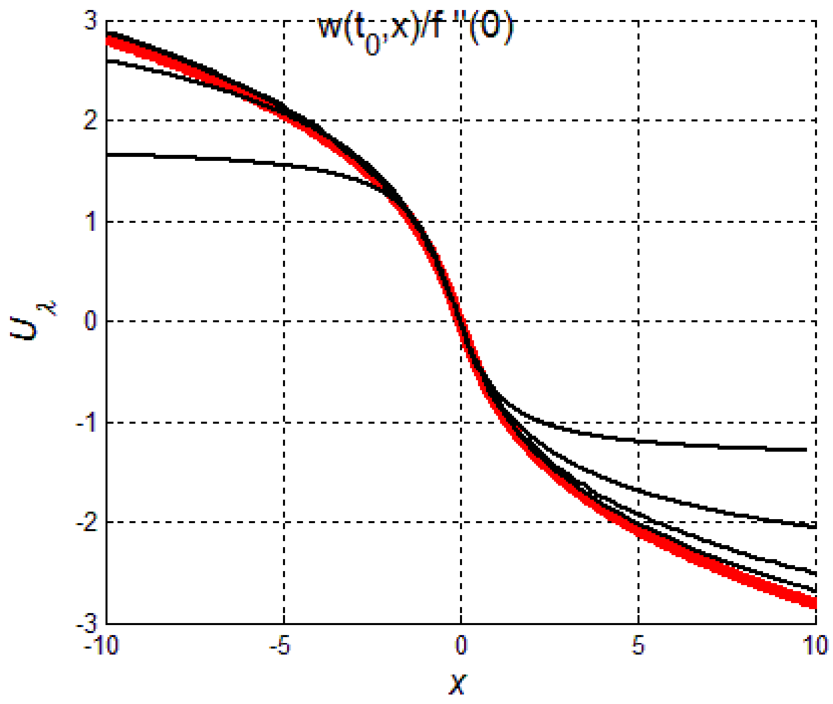
Small scale limit: $w(t,x) = \lim_{\lambda\to\infty} u_\lambda(t,x), \quad u^2/2 = \lim_{\lambda\to\infty} F_\lambda(u)$

Fourier transform of the universal function yields a solitary wave in logarithmic coordinates and scaled variables.

Renormalization and universality of blowup in inviscid Burgers equation (numerical example)



Renormalization and universality of blowup in ideal gas (numerical example)



Convergence of scaled profiles $u_\lambda(t_0,x)$ corresponding to the density variation (thin black curves) to the universal function $w(t_0,x)$ (bold red curve) for blow-up of a simple wave in ideal polytropic gas.

λ = 1, 10, 100, 1000.

Proof for the Burgers equation

$$x = x_0 + u_0(x_0)(t - t_0), \quad u = u_0(x_0) = u_0'(0)x_0 + \frac{u_0'''(0)}{6}x_0^3 + o(x_0^3), \quad t_0 = \frac{1}{u_0'(0)}$$

Asymptotics near the origin:

$$x = ut - \frac{u_0'''(0)}{6u_0'(0)}x_0^3 + o(x_0^3), \quad u = u_0'(0)x_0 + o(x_0)$$

$$x = ut - cu^3 + o(u^3), \quad c = \frac{u_0'''(0)}{6(u_0'(0))^4} > 0$$

Renormalization function: $u_\lambda(t,x) = G_\lambda u \equiv \lambda^{1/3}u(\lambda^{-2/3}t, \lambda^{-1}x)$

$u_\lambda(t,x)$ is a new solution (scaling symmetry of Burgers equation)

$$x = u_\lambda t - cu_\lambda^3 + \lambda o((\lambda^{-1/3}u_\lambda)^3)$$

Small scale limit: $w(t,x) = \lim_{\lambda\to\infty} u_\lambda(t,x)$

Universal function: $x = wt - cw^3 \qquad (G_\lambda w = w)$

Proof for a general conservation law

$$U = F'(u) \quad \rightarrow \quad \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = 0$$

$$u = G(U) = G(0) + G'(0)U + G''(0)U^2/2 + \cdots$$
$$= U/F''(0) + G''(0)U^2/2 + \cdots$$

$$\lambda^{1/3}U^n(\lambda^{-2/3}t, \lambda^{-1}x) = \lambda^{(1-n)/3}U_\lambda^n(t,x) \quad \rightarrow \quad \lambda^{(1-n)/3}w^n(t,x)$$

$$\lambda^{1/3}u(\lambda^{-2/3}t, \lambda^{-1}x) \quad \rightarrow \quad w(t,x)/f''(0)$$

Similar computations for scaling of the flux function

Application to incompressible Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} = -i\mathbf{P}\int d^3p\, \mathbf{k}\cdot\mathbf{v}(\mathbf{p})\mathbf{v}(\mathbf{k}-\mathbf{p}), \quad \mathbf{k}\cdot\mathbf{v} = 0$$

(in Fourier space)

Renormalized variables:

$$\mathbf{v}(k) = ik^{-4}\exp\left[\int_0^{\tau} A(\tau')d\tau'\right]\mathbf{w}(\mathbf{k}), \quad t = t_c - \int_{\tau}^{\infty}\exp\left[-\int_0^{\tau'} A(\tau'')d\tau''\right]d\tau'$$

Renormalized equations:

$$\frac{\partial \mathbf{w}}{\partial \tau} = N[\mathbf{w}] - A\mathbf{w}, \quad N[\mathbf{w}] = \mathbf{P}\int d^2\mathbf{o}_p d\eta\, \frac{k^5}{pq^4}[\mathbf{w}(\mathbf{p})\mathbf{o}_k]\mathbf{w}(\mathbf{k}-\mathbf{p})$$

$$\mathbf{k} = e^{\xi}\mathbf{o}_k, \quad A = \int d^2\mathbf{o}_k d\xi\, \mathbf{w}N[\mathbf{w}] \Big/ \int d^2\mathbf{o}_k d\xi\, \|\mathbf{w}\|^2$$

Translational invariance and conservation law:

$$\xi \to \xi + \Delta\xi, \quad \int d^2\mathbf{o}_k d\xi\, \|\mathbf{w}\|^2 = const$$

No solitary wave solutions are found.

Extension of the method to the case of multiple scales is necessary (AM, 2012).

Conclusion

Relation between blowup in shell models of turbulence and hyperbolic conservation laws is found.

Blowup in conservation laws has universal properties under scaling in time, space and state. The universal structure is independent both of initial conditions and of the flux function.

Potential for application of the method for the blowup problem in incompressible Euler equations.

Universal structure of blowup for systems of conservation laws and conservation laws in several space dimensions?

Reference: AM, Phys. Rev. E 85, 066317 (2012)