JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Adaptive large time step methods for geophysical flows

Combining the discontinuous Galerkin method with multi-d evolution operators

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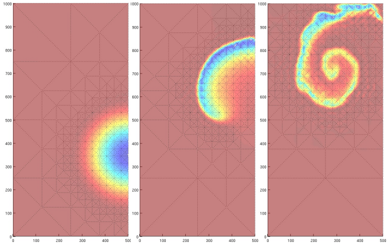
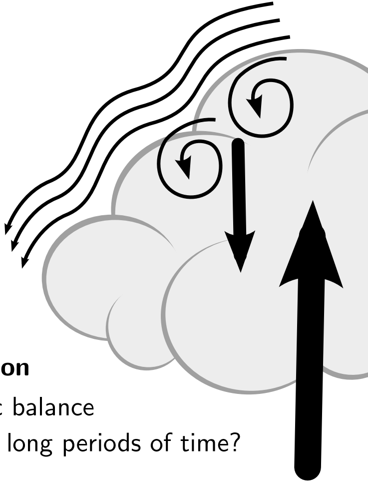
Application





Meteorology: Cloud Simulation

- Gravity induces hydrostatic balance
- How do clouds evolve over long periods of time?

Figure: Oceanographic flows

Multiscale phenomena of geophysical flows

-wave speeds differ by several orders: $\|\mathbf{u}\| << c \Rightarrow \mathbf{M}, \mathbf{Fr} := \dfrac{\|\mathbf{u}\|}{\mathbf{c}} << \mathbf{1}$
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$$\frac{\max(|u| + c, |v| + c)\Delta t}{\Delta x} \leq 1$$

$$\max\left(\left(1 + \frac{1}{\mathbf{Fr}}\right) \sqrt{u^2 + v^2}\right) \frac{\Delta t}{\Delta x} \leq 1$$

- number of time steps $\mathcal{O}(1/\mathbf{Fr})$

-low Mach / low Froude number problem

[Bijl & Wesseling ('98), Klein et al.('95, '01), Meister ('99,01), Munz &Park ('05), Degond et al. ('11) ...]

Cancelation problem

- Sesterhenn et al. ('99)
- h ... water depth in the shallow flow
- "pressure term" $\frac{1}{2\mathbf{Fr^2}}\nabla h^2 \Longrightarrow$
- $h_L, h_R = h_L + \delta h, \quad \delta h \approx \mathcal{O}(\mathbf{Fr^2})$

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- $h_L, h_R = h_L + \delta h, \quad \delta h \approx \mathcal{O}(\mathbf{Fr^2})$
- BUT round off errors can yield the cancelation effects

$$
\begin{aligned}
h_R^2 - h_L^2 &= ((h_L^2 + 2h_L\delta h + \delta h^2)(1+\epsilon_1) - h_L^2)(1+\epsilon_2) \\
h_R^2 - h_L^2 &= \delta h\left[(2h_L + \delta h_L) + \epsilon_1 \frac{(h_L+\delta h)^2}{\delta h} + h.o.t.\right]
\end{aligned}
$$

leading order error in the pressure term $\approx \frac{1}{\mathbf{Fr^2}}\epsilon_1 \mathcal{O}(\frac{1}{\mathbf{Fr^2}}) \approx \mathcal{O}(1)$!

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-Remedy: introduce background values and work with perturbations only

- **AIM:**
  - reduce adverse effect of $1 + 1/\mathbf{Fr}$
  - large time step scheme: Δt does not depends on $\mathbf{Fr}$
  - efficient scheme for advection effects
  - stability and accuracy of the scheme is independent on $\mathbf{Fr}$

Asymptotic preserving schemes

Goal: *Derive a scheme, which gives a consistent approximation of the limiting equations for* $\varepsilon = \mathbf{Fr} \to \mathbf{0}$

[S.Jin&Pareschi('01), Gosse&Toscani('02), Degond et al.('11), ...]

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$$
\begin{aligned}
&\partial_t z + \partial_x m + \partial_y n = 0 \\
&\partial_t m + \partial_x (m^2/(z+b)) + \partial_y (mn/(z+b)) + \frac{1}{2\mathbf{Fr}^2} \partial_x (z^2) = -\frac{1}{\mathbf{Fr}^2} b \partial_x z \\
&\partial_t n + \partial_x (mn/(z+b)) + \partial_y (n^2/(z+b)) + \frac{1}{2\mathbf{Fr}^2} \partial_y (z^2) = -\frac{1}{\mathbf{Fr}^2} b \partial_y z
\end{aligned}
$$

Asymptotic expansion

-rigorous analysis [Klainerman & Majda ('81)]
-formally: $(\varepsilon = \mathbf{Fr})$

$$z^{\varepsilon}(x,t;\varepsilon) = z^{(0)}(x,t) + \varepsilon z^{(1)}(x,t) + \varepsilon^2 z^{(2)}(x,t)$$
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plug into the SWE $\Longrightarrow$

$$z^{(0)} = z^{(0)}(t); \quad \partial_x(h^{(0)} - b) = 0$$
$$\partial_x h^{(1)} = 0$$
$$\partial_t z^{(0)} = \partial_x(h^{(0)} u^{(0)}) \equiv \partial_x m^{(0)}$$
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limiting system as $\varepsilon \to 0$ $(\partial_t b = 0)$

$$h^{(0)}(x) = b(x) + const. \tag{1}$$
$$\partial_t h^{(0)} = \partial_x m^{(0)}$$
$$\partial_t u^{(0)} + u^{(0)} \partial_x u^{(0)} + \partial_x z^{(2)} = 0$$

Does a numerical scheme give a consistent approximation of (1) ?

Time discretization

Key idea:

- semi-implicit time discretization: splitting into the linear and nonlinear part
- linear operator modells gravitational (acoustic) waves are treated implicitly
- rest nonlinear terms are treated explicitly

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$$\mathbf{w} = (z, m, n)^T, \quad z = h - b;\ b < 0$$

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$$\mathcal{L}(\mathbf{w}) := \begin{pmatrix} \partial_x(m) + \partial_y(n) \\ \frac{b}{\mathbf{Fr}^2}\partial_x z \\ \frac{b}{\mathbf{Fr}^2}\partial_y z \end{pmatrix}$$

[Restelli ('07), Giraldo & Restelli ('10)]

- $\mathcal{L}$: spatially varying linear system

$$\mathbf{w}_t + \mathbf{A}_1(b)\mathbf{w}_x + \mathbf{A}_2(b)\mathbf{w}_y = 0$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\mathbf{Fr}^2} b(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\mathbf{Fr}^2} b(x,y) & 0 & 0 \end{pmatrix} \quad \Longrightarrow E_\Delta^L$$

Multi-d evolution operator in [Arun, M.L., Kraft, Prasad (2009)]

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Multi-d evolution operator in [Arun, M.L., Kraft, Prasad (2009)]

- REST: nonlinear system $\mathcal{N}$

$$z_t = 0$$

$$m_t + (m^2/(z-b))_x + \frac{1}{2\mathbf{Fr}^2}(z^2)_x + (mn/(z-b)))_y = 0$$

$$n_t + (mn/(z-b))_x + (n^2/(z-b))_y + \frac{1}{2\mathbf{Fr}^2}(z^2)_y = 0 \quad \Longrightarrow E_\Delta^N$$

Semi-implicit time discretization

$$\mathbf{w}^{n+1} = \mathbf{w}^n + \frac{\Delta t}{2}\left[\mathcal{L}(\mathbf{w}^n) + \mathcal{L}(\mathbf{w}^{n+1})\right] + \Delta t \mathcal{N}(\mathbf{w}^{n+1/2})$$

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- spatial discretization: FV update using flux differences
+ EG-evolution operator to evaluate fluxes at interfaces (multi-d Riemann solver)

$$\mathcal{L}(\mathbf{w}^\ell) = \frac{1}{\Delta x_k}\sum_{k=1}^{2} \delta_{x_k}(\mathbf{F}_L(\mathbf{E}_0(\mathbf{w}^\ell))), \quad \ell = n, n+1$$

$$\mathcal{N}(\mathbf{w}^{n+1/2}) = \frac{1}{\Delta x_k}\sum_{k=1}^{2} \delta_{x_k}(\mathbf{F}_N(\mathbf{E}_{\Delta t/2}(\mathbf{w}^n)))$$

AP property for the semi-implicit time discretization scheme

semi-discrete scheme:

$$z^{n+1} = z^n - \frac{\Delta t}{2}\left[m_x^{n+1} + m_x^n\right] \tag{2}$$

$$m^{n+1} = m^n - \frac{\Delta t}{2}\left[\frac{b}{\varepsilon^2} z_x^{n+1} + \frac{b}{\varepsilon^2} z_x^n\right] - \Delta t \left[\frac{1}{2\varepsilon^2}(z_x^{n+1/2})^2 + (mu)_x^{n+1/2}\right] \tag{3}$$

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- we assume that $z^n, z^{n+1/2}, m^n, m^{n+1/2}$ approximate the limiting eqs. (1)

• Eq.(3) yields for ε^{-2}

$$\frac{b}{2}\left(z_x^{(0),n+1} + z_x^{(0),n}\right) + \frac{1}{2} z^{(0),n+1/2} z_x^{(0),n+1/2} = 0$$

$\Longrightarrow z^{(0),n+1}(x) = const.$

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- Eq.(2) yields for ε^0 consistent approx. of

$$\partial_t z^{(0)} = \partial_x m^{(0)}$$

- periodic, slip BC $\Longrightarrow z^{(0),n+1}(x) = z_x^{(0),n}(x)$
- $m^{(0),n+1}(x) = const.$

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- Eq.(3) yields for ε^0 terms :

$$\begin{aligned} m^{(0),n+1} &= m^{(0),n} - \frac{\Delta t}{2}\Big[b\Big(z_x^{(2),n+1} - z_x^{(2),n}\Big) \\ &\qquad + z^{(0),n+1/2} z_x^{(2),n+1/2} - (mu)_x^{(0),n+1/2}\Big] \\ &\approx m^{(0),n} - \Delta t\Big[h^{(0),n+1/2} z_x^{(2),n+1/2} - (hu^2)^{(0),n+1/2}\Big] \end{aligned}$$

- which is a consistent approx. of the momentum eq.

$$\partial_t u^{(0)} = u^{(0)}\partial_x u^{(0)} + \partial_x z^{(2)}$$

Application to atmospheric flow

Compressible Euler equations

$$\begin{aligned} \partial_t \rho' &+ \nabla \cdot (\rho \mathbf{u}) &&= 0 \\ \partial_t (\rho \mathbf{u}) &+ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p' \, \mathrm{Id}) &&= -\rho' g \mathbf{k} \\ \partial_t (\rho\theta)' &+ \nabla \cdot (\rho\theta \mathbf{u}) &&= 0 \end{aligned}$$

with background state $\bar{p}$, $\bar{\rho}$, $\bar{\theta}$ in **hydrostatic balance**

$$\boxed{\partial_y \bar{p} = -\bar{\rho} g}$$

State variables: $\mathbf{w} = [\rho', \rho u, \rho v, (\rho\theta)']^\mathsf{T}$

- Potential temperature $\theta := T/\pi$
- *Exner-pressure* $\pi(y) := 1 - \dfrac{gy}{c_p \theta}$

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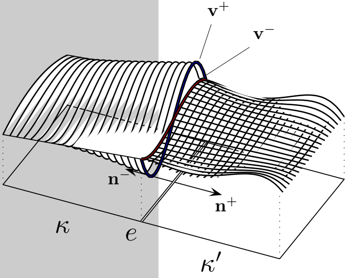
In short:

$$\partial_t \mathbf{w} + \nabla \cdot \underbrace{F}_{\text{Flux}} (\mathbf{w}) = \underbrace{s}_{\text{Source term}} (\mathbf{w})$$

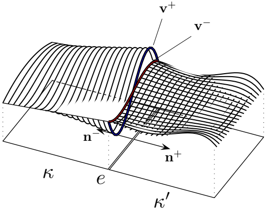
- Goal:
  - approximate the Euler eqs. using the above splitting into linearized and nonlinear waves and semi-implicite time discretization
  - space discretization using the discontinuous Galerkin method and P1, P2 - elements
  - use multi-d evolution in order to approximate fluxes along cell interfaces ... **verify AP !**



1 Key Ingredients of the Discretization

Discontinuous Galerkin FEM

DG-FEM are finite element methods based on completely discontinuous finite element spaces.



Ingredients:

Triangulation $\mathcal{T}_h = \{\kappa\}$
approximate solutions by dividing the domain Ω into finite subregions

Parametric function space $\mathbf{V}_h^p$
Piecewise pth-order polynomials in each element

Averages and jumps on interior edges
Approximation possibly discontinuous across interelement boundaries

$$\{v\}_e = \frac{1}{2}(v_\kappa^+ + v_\kappa^-)\ ,\ [\![v]\!]_e = v_\kappa^+\mathbf{n}^+ + v_\kappa^-\mathbf{n}^-$$

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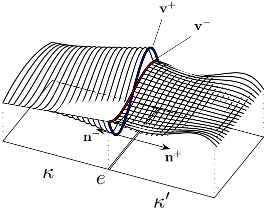

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Local variational formulation

$$\begin{cases} \text{Find } u_h \in \mathbf{V}_h^p,\ \text{s. t.} \\ B(u_h, v_h) = \ell(v_h) \quad \forall v_h \in \mathbf{V}_h^p \end{cases}$$

Discrete system & basis functions

Variational Formulation

Multiply

$$\partial_t \mathbf{w} + \nabla \cdot F(\mathbf{w}) = \partial_t \mathbf{w} + \sum_{s=1}^{d} \mathbf{f}_s(\mathbf{w}) = s(\mathbf{w})$$

with a test function $\mathbf{v}$ and perform integration by parts:

$$\sum_{\kappa \in \mathcal{T}_h} \Big[\int_\kappa \partial_t \mathbf{w} \cdot \mathbf{v} \;\, d\mathbf{x} - \sum_{s=1}^{d} \Big(\int_\kappa \mathbf{f}_s(\mathbf{w}) \cdot \partial_s \mathbf{v} \; d\mathbf{x} + \int_{\partial\kappa} \mathbf{f}_s\,(\mathbf{w}) \cdot \mathbf{v}\, n_s \;\, ds \;\Big) \Big] = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa s(\mathbf{w}) \cdot \mathbf{v} \;\, d\mathbf{x}.$$

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Another integration by parts yields

$$\sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \left[\partial_t \mathbf{w} - \sum_{s=1}^{d} \partial_s \mathbf{f}_s(\mathbf{w}) - s(\mathbf{w}) \right] \cdot \mathbf{v} \; d\mathbf{x} = \sum_{\kappa \in \mathcal{T}_h} \sum_{s=1}^{d} \int_{\partial\kappa} [\mathbf{f}_s(\mathbf{w}) - \mathbf{f}_s^*(\mathbf{w})] \cdot \mathbf{v} \, n_s \; ds$$

- choose $\mathbf{w}, \mathbf{v} \in \left[\mathbf{V}_h^p\right]^4$, insert quadrature rules
- numerical flux function $\mathbf{f}_s^*(\mathbf{w})$ required

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`FVEG::eg_flux_strong`

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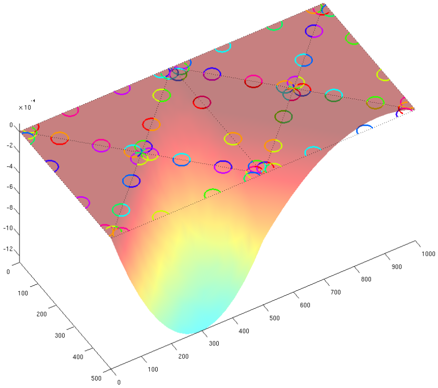
$\mathbf{f}_s^*(\mathbf{w})$ should approximate the flux of $\mathbf{w}$ through interior edges

- Finite Volume Method: one-dimensional approach **Rusanov flux**

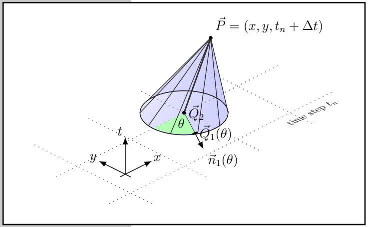
$$\mathbf{f}_s^*(\mathbf{w}) := \frac{1}{2}\left[\mathbf{f}_s(\mathbf{w}^+) + \mathbf{f}_s(\mathbf{w}^-) - \lambda(\mathbf{w}^- - \mathbf{w}^+)\right]$$

λ - max. wave speed

Truly multi-dimensional approach: **evolution operator (EG)**



- implemented in the CloudFlash code: `flash/FVEG.F90`



2

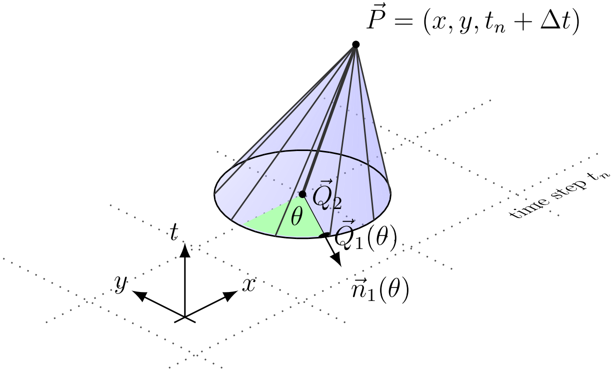
Evolution Galerkin Scheme

Replacing a one-dimensional numerical flux

Wave propagation for the Euler equations

Information travels along **bicharacteristic curves**

Integration along each curve + averaging over the cone mantle yields integral representation for the solution at the pick of the cone



M. Lukáčová-Medvid'ová, K.W. Morton, and Gerald Warnecke.
Finite volume evolution Galerkin methods for hyperbolic systems.
J. Sci. Comp. 2004.

Short derivation of integral representation

Step 1: Formulation as a quasilinear system

$$\partial_t \mathbf{w} + \underline{A}_1(\mathbf{w})\,\partial_x \mathbf{w} + \underline{A}_2(\mathbf{w})\,\partial_y \mathbf{w} = \mathbf{s}(\mathbf{w})$$

with matrices $\underline{A}_1(\mathbf{w})$, $\underline{A}_2(\mathbf{w})$ and source term $\mathbf{s}(\mathbf{w})$.
Freeze Jacobians $\underline{A}_1$, $\underline{A}_2$ if they depend on $\mathbf{w}$

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Step 2: Quasi-diagonalization

Let $\underline{R}$ denote the right eigenvectors of $\underline{P} = \underline{A}_1 n_x + \underline{A}_2 n_y$.
Change of variables $\mathbf{v} = \underline{R}^{-1}\mathbf{w}$ yields a quasi-diagonal system

$$\boxed{\partial_t \mathbf{v} + \mathrm{diag}(\underline{B}_1)\partial_x \mathbf{v} + \mathrm{diag}(\underline{B}_2)\partial_y \mathbf{v} = \mathbf{S} + \mathbf{r}}$$

where $\underline{B}_{1/2} := \underline{R}^{-1}\underline{A}_{1/2}\underline{R}$

$\mathbf{r} := \underline{R}^{-1}\mathbf{s}(\mathbf{w})$, $\quad \mathbf{S} := -(\underline{B}_1 - \mathrm{diag}(\underline{B}_1))\partial_x \mathbf{v} - (\underline{B}_2 - \mathrm{diag}(\underline{B}_2))\partial_y \mathbf{v}$.

Short derivation of integral representation (cont'd)

Step 3: Averaging over the cone mantle
For every direction $[n_x, n_y] = [\cos(\theta), \sin(\theta)]$ with $\theta \in [0, 2\pi]$:

The system

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Temporal integration over $[t_n, t_n + \tau]$ and averaging over the *wave-front*, i. e. $\theta \in [0, 2\pi]$, yields an integral representation for $[x, y, t_n + \tau]$.

Short derivation of integral representation (cont'd)

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Step 4: Back transform to primitive variables
Change of variables $\mathbf{w} = \underline{R}\mathbf{v}$.

Exact evolution operator for the linear subsystem

linear part for the Euler system

$$\partial_t \mathbf{w} + \mathcal{L}(\mathbf{w}) = 0$$

$$\mathbf{w} := \begin{pmatrix} \rho' \\ \rho u \\ \rho v \\ (\rho\theta)' \end{pmatrix} \qquad \mathcal{L}(\mathbf{w}) := \begin{pmatrix} \mathrm{div}(\rho \mathbf{u}) \\ \partial p'/\partial x \\ \partial p'/\partial y + g\rho' \\ \mathrm{div}(\overline{\theta}\rho \mathbf{u}) \end{pmatrix}$$

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- linearized version of p': $\dfrac{\partial p'}{\partial x} = \dfrac{c_p \overline{p}}{c_v \overline{\rho}\overline{\theta}} \dfrac{\partial(\rho\theta)'}{\partial x} = \tilde{\gamma}\dfrac{\partial(\rho\theta)'}{\partial x}$, where $\tilde{\gamma} = \gamma R$

Exact evolution operator for the linear subsystem

$$\partial \mathbf{w} + \mathbf{A}_1 \mathbf{w}_x + \mathbf{A}_2 \mathbf{w}_y = S(\mathbf{w})$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \bar{\theta} & 0 & 0 \\ \tilde{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & \bar{\theta} & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\bar{\theta} = \bar{\theta}(y)$

eigenstructure: $\lambda_1 = -a,\ \lambda_{2,3} = 0, \lambda_4 = a, \quad a := \sqrt{\tilde{\gamma}\bar{\theta}}$

Note: in the non-dimensional form $\tilde{\gamma} = \dfrac{\gamma R}{\mathbf{M}^2}$

Exact integral representation

$$
\begin{aligned}
\rho'(\mathbf{P}) = \frac{1}{2\pi a} \int_0^{2\pi} & \left[-\cos(\omega)\, u(\mathbf{Q}_1(\omega) - \sin(\omega)\, v(\mathbf{Q}_1(\omega)) + \frac{\tilde{\gamma}}{a^2} (\rho\theta)'(\mathbf{Q}_1(\omega)) \right] d \\
& + \rho'(\mathbf{Q}_2) - \frac{\tilde{\gamma}(\rho\theta)'(\mathbf{Q}_2)}{a^2} \\
& - \frac{1}{2\pi a} \int_0^{2\pi} \int_{t_n}^{t_n+\tau} \frac{1}{\tau - t} \left(\cos(\omega)\, u(\mathbf{x}_1(t,\omega)) + \sin(\omega)\, v(\mathbf{x}_1(t,\omega))\right. \\
& - \tfrac{1}{2\pi a} \int_0^{2\pi} \int_{t_n}^{t_n+\tau} \sin(\theta) g \rho'(\mathbf{x}_1(t,\omega))\, dt\, d\omega
\end{aligned}
$$

Expression intractable for a numerical scheme – Simplify!

Approximate

- temporal integrals by the rectangle rule
- to avoid large sonic circles use local evolution for $\tau \to 0$ [Sun & Ren ('09)]
- in practice τ fixed, $\frac{a\tau}{\Delta x} \approx 0.1$

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$$+ \rho'(\mathbf{Q}_2) - \frac{\tilde{\gamma}(\rho\theta)'(\mathbf{Q}_2)}{a^2}$$

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$$+ \rho'(\mathbf{Q}_2) - \frac{\tilde{\gamma}(\rho\theta)'(\mathbf{Q}_2)}{a^2}$$

- The resulting operator is a predictor for the cell-interface values of fluxes in the DG-update
- **The operator is asymptotic preserving !**
- It can be shown to be of order $\mathcal{O}(\tau^2)$.

Time integration

- Second order linear semi-implicit method: BDF2

$$\frac{\partial \mathbf{w}}{\partial t} = \mathcal{N}(\mathbf{w}) = \mathcal{R}(\mathbf{w}) + \mathcal{L}(\mathbf{w}),$$

$\mathcal{L}$ - linearised operator: acoustic waves / gravity waves
-implicit approx. in time

$\mathcal{R} := \mathcal{N} - \mathcal{L}$ -nonlinear part : explicit approx. in time

Time integration; cont. ...

- BDF time discretization:

$$\frac{\partial \mathbf{w}}{\partial t} = \{\mathcal{N}(\mathbf{w}) - \mathcal{L}(\mathbf{w})\} + \mathcal{L}(\mathbf{w})$$

$\Rightarrow$

$$\sum_{m=-1}^{1} \alpha_m \mathbf{w}^{n-m} = \sum_{m=0}^{1} \beta_m \left[\mathcal{N}(\mathbf{w}^{\mathbf{n-m}}) - \mathcal{L}(\mathbf{w}^{\mathbf{n-m}})\right] + \mathcal{L}(\mathbf{w}^{n+1})$$

Time integration; cont. ...

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implicit corrector

$$[1 - \gamma \Delta t \mathcal{L}]\, \mathbf{w}^{n+1} = \mathbf{w}^{ex} - \gamma \Delta t \sum_{m=0}^{1} \beta_m \mathcal{L}(\mathbf{w}^{n-m})$$

with the explicit predictor step

$$\mathbf{w}^{ex} := \sum_{m=0}^{1} \alpha_m \mathbf{w}^{n-m} + \gamma \Delta t \sum_{m=0}^{1} \beta_m \mathcal{N}(\mathbf{w}^{n-m})$$

$\alpha_0 = 4/3,\ \alpha_1 = -1/3, \gamma = 2/3, \beta_0 = 2, \beta_1 = -1$

Numerical Experiments

- Euler equations with hydrostatic assumption: hydrostatic balance $\partial_z p = -g\rho$
- DG space discretization (with Rusanov flux and EG operator)
- SSP Runge Kutta 2 or 3; BDF 2 time discretization
- adaptive mesh refinement, triangular grid - mesh refinement criterium:

$$|\theta'| \geq \max_{\mathbf{x}}(|\theta'(\mathbf{x}, t = 0)|)/10$$

Test 1: rising warm air bubble

- bubble with a cosine profile in $\theta = \overline{\theta} + \theta'$:

$$\theta' = \begin{cases} 0 & r > r_C,\ r = \|\mathbf{x} - \mathbf{x}_C\| \\ 0.25[1 + \cos(\pi_c r / r_C)] & r \leq r_C \end{cases}$$

$$\mathbf{x}_C = (500, 350),\ r_C = 250m,\ \overline{\theta} = 300K,$$

$$\mathbf{x} \in [0, 1000]^2,\ t \in [0, 700]$$

• in the momentum and energy eqs. regularized viscous terms with a small viscosity μ are added
$\mu = 0.1 m^2/s$

Error Analysis

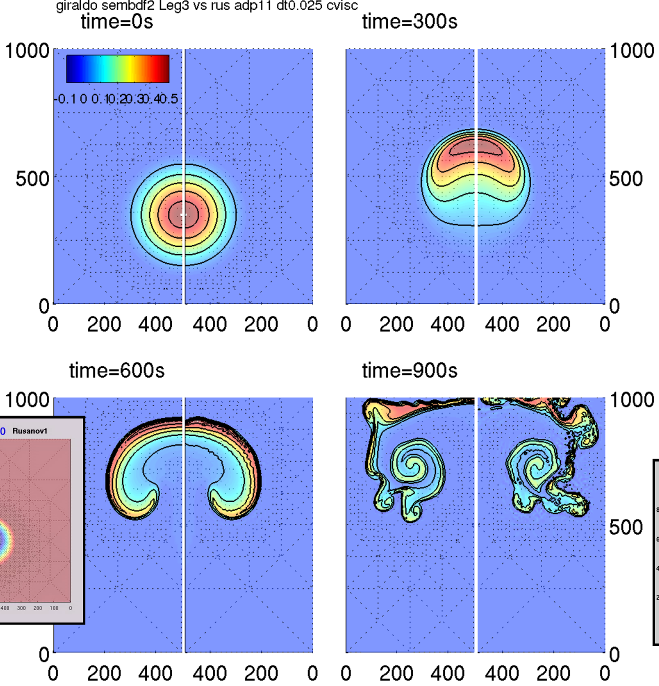
- comparison of the multi-D EG-flux and the 1-D Rusanov flux
- semi-implict: BFD2, quadratic elements, T = 150

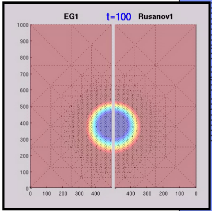
EG-flux

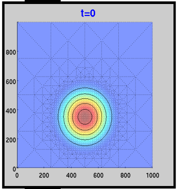
N = gridlevel	$\|u_N - u_{N+2}\|$	$\|u_{N+2} - u_{N+4}\|$	EOC
3	0.0375	0.0071	2.40
4	0.0226	0.0038	2.56
5	0.0071	0.0014	2.33
6	0.0038	0.0005	3.03
7	0.0014	0.0002	3.05

Rusanov flux

N = gridlevel	$\|u_N - u_{N+2}\|$	$\|u_{N+2} - u_{N+4}\|$	EOC
3	0.0815	0.0115	2.82
4	0.0354	0.0060	2.57
5	0.0115	0.0027	2.07
6	0.0060	0.0012	2.33
7	0.0027	0.0005	2.26


giraldo sembdf2 Leg3 vs rus adp11 dt0.025 cvisc
time=0s
time=300s
time=600s
time=900s
-0.1 0 0.1 0.2 0.3 0.4 0.5
1000
500
0
0 200 400 400 200 0

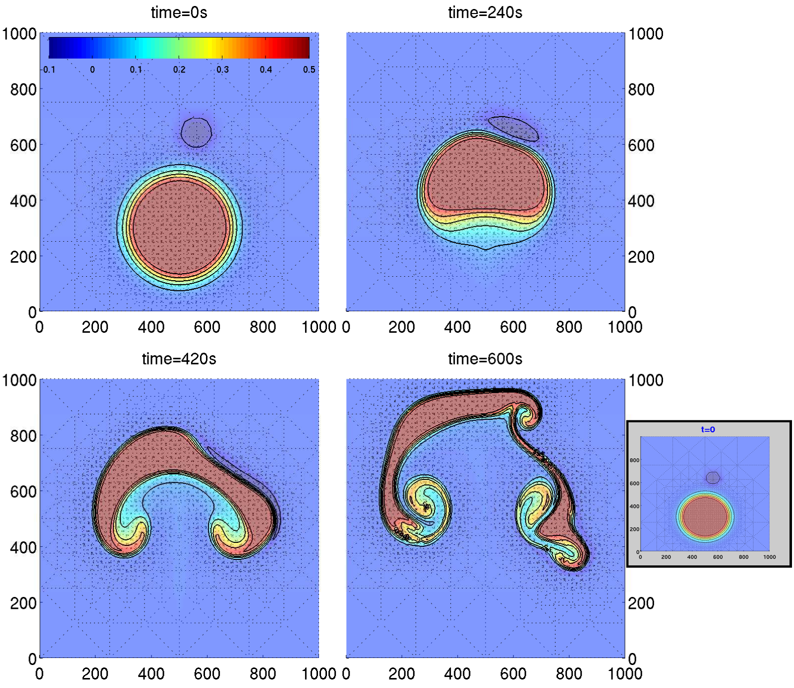

EG1
t=100
Rusanov1


t=0

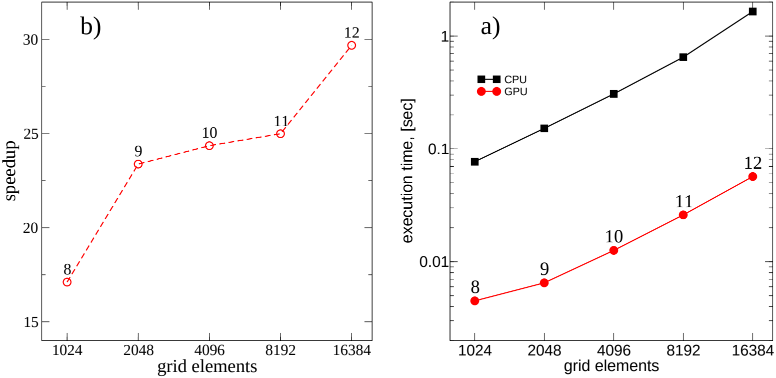
Test 2: small cold bubble on the top of large warm bubble

- **Robert test** (1993)

- both bubbles: a Gaussian profile
- warm air bubble: amplitude of 0.5 K
- cold air bubble: amplitude 0.17 K
- $\mu = 0.1 m^2/s$


time=0s
time=240s
time=420s
time=600s
-0.1
0
0.1
0.2
0.3
0.4
0.5
t=0

GPU parallelization



Speed up for the GPU implementation of the EG operator