High-frequency limit of the Maxwell-Landau-Lifshitz system in the diffractive optics regime

HYP2012, June 2012

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June 28, 2012

The equations

Maxwell-Landau-Lifshitz: $t \in \mathbb{R}$, $X = (x, y, z) \in \mathbb{R}^3$

$$(MLL) \begin{cases} \partial_t E - \nabla \times H = 0, \\ \partial_t H + \nabla \times E = -\partial_t M, \\ \partial_t M = -M \times H. \end{cases}$$

Constant solutions

$$(E, H, M) = (0, \alpha M_0, M_0), \ \alpha > 0.$$

Slow variable perturbations

$$\begin{cases} E(t, X) = \varepsilon \widetilde{E}(\varepsilon t, \varepsilon X), \\ H(t, X) = \alpha M_0 + \varepsilon \widetilde{H}(\varepsilon t, \varepsilon X), \\ M(t, X) = M_0 + \varepsilon \widetilde{M}(\varepsilon t, \varepsilon X). \end{cases}$$

A symmetric hyperbolic system

Let

$$u = (\widetilde{E}, \widetilde{H}, \alpha^{\frac{1}{2}} \widetilde{M}).$$

Then

$$\partial_t u + A(\nabla_X) u + \frac{1}{\varepsilon} L_0 u = B(u, u),$$

where

$$A(\nabla_X) = \begin{pmatrix} 0 & -\nabla_X \times & 0 \\ \nabla_X \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -M_0 \times & \alpha^{\frac{1}{2}} M_0 \times \\ 0 & \alpha^{\frac{1}{2}} M_0 \times & -\alpha M_0 \times \end{pmatrix},$$

$$B(u, v) = \frac{1}{2} \begin{pmatrix} 0 \\ \alpha^{-\frac{1}{2}} (u^3 \times v^2 + v^3 \times u^2) \\ -(u^3 \times v^2 + v^3 \times u^2) \end{pmatrix}.$$

Cauchy problem and long time stability

One dimensional setting:

$u(t,x,y,z) = v(t,x), x \in \mathbb{R}^1$. Then

$$\partial_t v + A(e_1)\partial_x v + \frac{1}{\varepsilon} L_0 v = B(v,v), \quad (1)$$

where $e_1 = (1,0,0)$.

Highly oscillatory initial data:

$$v(0,y) = a_0(x)e^{ikx/\varepsilon} + \overline{a_0(x)}e^{-ikx/\varepsilon} + \varepsilon a_1(x, kx/\varepsilon) + \varepsilon^2 a_2(x, kx/\varepsilon). \quad (2)$$

where $a_1(x,\theta)$ and $a_2(x,\theta)$ are real-valued and 2π-periodic in θ,
$a_0 \in H^s_x$, $a_1(x,\theta), a_2(x,\theta) \in H^1_\theta(H^s_x)$, $s > 1/2 + 3$.

The theorem

We study the solution on large time intervals $O(1/\varepsilon)$. Under appropriate assumptions on a_0 and a_1, we prove

Theorem

The Cauchy problem (1)-(2) admits a unique solution v on large time $[0, T/\varepsilon]$ where $T > 0$ is independent of ε. Moreover, for such times, the solution is well approximated by a WKB approximate solution v^a of which the leading terms satisfy cubic Schrödinger equations, and there holds:

$$|v - v^a|_{L^\infty([0,T/\varepsilon]\times\mathbb{R})} \leq C|(v - v^a)(0)|_{L^\infty(\mathbb{R})}$$

The first issue: Oscillatory initial data

Fast oscillations

Sobolev norm for $s > 0$:

$$|v(0)|_{H^s} = |a(x)e^{ikx/\varepsilon} + \ldots|_{H^s} = O(\varepsilon^{-s}) \to \infty.$$

Profile

Solution in the form

$$v(t,x) = V(t,x,\theta)|_{\theta=\frac{kx-\omega t}{\varepsilon}},$$

New variable $\theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Then

$$\begin{cases} \partial_t V + A(e_1)\partial_x V + \frac{1}{\varepsilon}\{-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0\}V = B(V,V), \\ V(0,x,\theta) = a(x)e^{i\theta} + \overline{a(x)}e^{-i\theta} + \varepsilon a_1(x,\theta) + \varepsilon^2 a_2(x,\theta). \end{cases} \tag{3}$$

The second issue

H^s estimates

For (3), the classical H^s estimate for semilinear symmetric hyperbolic operators yields

$$|V(t)|_{H^s} \leq |V(0)|_{H^s} + C \int_0^t |V(t')|_{H^s} dt',$$

where the constant C depend on $|V|_{L^\infty}$. Then, with Gronwall's lemma, the bound

$$|V(t)|_{H^s} \leq |V(0)|_{H^s} e^{Ct}.$$

The nature existence time for (3) is $O(1)$. In times $O(1/\varepsilon)$, the upper bound goes to infinity as ε goes to zero.

Main idea

Three steps:

1, Construct an approximate solution over times $O(1/\varepsilon)$,

2, Show existence of the exact solution over times $O(1/\varepsilon)$,

3, Give the error estimates.

Step 1—Approximate solution: Ansatz

We look for an approximate solution V^a of the form:

$$V^a(t,x,\theta) := W(\tau,t,x,\theta)|_{\tau=\varepsilon t}.$$

Where

$$W = \sum_{j=0}^{2} \varepsilon^j W_j, \quad W_j(\tau,t,x,\theta) = \sum_{p\in\mathbb{Z}} e^{ip\theta} W_{jp}(\tau,t,x). \tag{4}$$

Then

$$\partial_\tau W + \frac{1}{\varepsilon}\left(\partial_t + A(e_1)\partial_x\right)W + \frac{1}{\varepsilon^2}\{-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0\}W = \frac{1}{\varepsilon}B(W,W). \tag{5}$$

We plug (4) into (5) and cancel all the terms of orders $O(\varepsilon^j)$ for $j = -2,-1,0$.

Step 1—Approximate solution: WKB expansion

$O(\varepsilon^{-2})$: $L(\beta\partial_\theta)W_0 := (-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0)W_0 = 0, \quad \beta := (k, \omega).$

By (4), it amounts to solving the following equations for all $p \in \mathbb{Z}$:

$$L(ip\beta)W_{0p} := (-ip\omega + A(e_1)ipk + L_0)W_{0p} = 0.$$

$$\iff W_{0p} \in \ker L(ip\beta) \iff W_{0p} = \pi_p W_{0p}.$$

Where, π_p are projections onto $\ker L(ip\beta)$. There always holds

$$\pi_p L(ip\beta) = L(ip\beta)\pi_p = 0.$$

The dispersion relation: (k, ω) is chosen such that

$$\det L(i\beta) = 0.$$

Step 1—Approximate solution: WKB expansion

$O(\varepsilon^{-1})$: $(\partial_t + A(e_1)\partial_x)W_{0p} + L(ip\beta)W_{1p} := \sum_{p_1+p_2=p} B(W_{0p_1}, W_{0p_2})$.

Apply π_p to above equation:

$$\pi_p(\partial_t + A(e_1)\partial_x)\pi_p W_{0p} = \pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, W_{0p_2}).$$

Fact 1 (transport):

$$\pi_p(\partial_t + A(e_1)\partial_x)\pi_p = (\partial_t + \rho\partial_x)\pi_p, \quad \rho = \rho(k, \omega, p) \in \mathbb{R}.$$

Fact 2 (compatibility conditions)

$$\pi_p \sum_{p_1+p_2=p} B(\pi_{0p_1}, \pi_{0p_2}) = 0.$$

This gives $(\partial_t + \rho\partial_x)W_{0p} = 0.$

Step 1—Approximate solution: WKB expansion

$O(\varepsilon^{-1})$: $\left(\partial_t + A(e_1)\partial_x\right) W_{0p} + L(ip\beta) W_{1p} := \sum_{p_1+p_2=p} B(W_{0p_1}, W_{0p_2})$.

Applying $L(ip\beta)^{-1}$:

$$(1-\pi_p)W_{1p} = -L(ip\beta)^{-1}A(e_1)\partial_x(\pi_p W_{0p}) + L(ip\beta)^{-1} \sum_{p_1+p_2=p} B(W_{0p_1}, W_{0p_2}).$$

Step 1—Approximate solution: WKB expansion

$O(\varepsilon^0)$:

$$\partial_\tau W_{0p} + \big(\partial_t + A(e_1)\partial_x\big) W_{1p} + L(ip\beta) W_{2p} := 2 \sum_{p_1+p_2=p} B(W_{0p_1}, W_{1p_2}).$$

Apply π_p to above equation:

$$\partial_\tau W_{0p} + \pi_p\big(\partial_t + A(e_1)\partial_x\big) W_{1p} = 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, W_{1p_2}).$$

The decomposition $W_{1p} = \pi_p W_{1p} + (1-\pi_p) W_{1p}$. Then

$$\partial_\tau W_{0p} + \pi_p A(e_1)\partial_x (1-\pi_p) W_{1p} + \pi_p\big(\partial_t + A(e_1)\partial_x\big)\pi_p W_{1p}$$
$$= 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, \pi_{p2} W_{1p_2}) + 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, (1-\pi_{p2}) W_{1p_2}).$$

Step 1—Approximate solution: WKB expansion

$$\partial_\tau W_{0p} + \pi_p A(e_1)\partial_x(1-\pi_p)W_{1p} + \pi_p(\partial_t + A(e_1)\partial_x)\pi_p W_{1p}$$
$$= 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, \pi_{p2} W_{1p_2}) + 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, (1-\pi_{p2}) W_{1p_2}).$$

Recall the fact:

The transport operator

$$\pi_p(\partial_t + A(e_1)\partial_x)\pi_p W_{1p} = (\partial_t + \rho\partial_x)\pi_p W_{1p}.$$

The compatibility conditions

$$\pi_p \sum_{p_1+p_2=p} B(\pi_2 W_{0p_1}, \pi_{0p_2} W_{1p_2}) = 0.$$

Step 1—Approximate solution: WKB expansion

$$\partial_\tau W_{0p} + \pi_p A(e_1)\partial_x(1-\pi_p)W_{1p} + (\partial_t + \rho\partial_x)\pi_p W_{1p} = 2\pi_p \sum_{p_1+p_2=p} B(W_{0p_1}, (1-\pi_{p2})W_{1p_2}).$$

Fact 3 (Schrödinger)

$$\begin{aligned}\pi_p A(e_1)\partial_x(1-\pi_p)W_{1p} &= -\pi_p A(e_1)\partial_x L(ip\beta)^{-1}A(e_1)\partial_x \pi_p W_{0p} + \cdots \\ &= \frac{i\tilde{\rho}}{2}\partial_x^2 W_{0p} + \cdots, \quad \tilde{\rho} = \tilde{\rho}(k,\omega,p) \in \mathbb{R}.\end{aligned}$$

$$\partial_\tau W_{0p}+\pi_p A(e_1)\partial_x(1-\pi_p)W_{1p}+(\partial_t+\rho\partial_x)\pi_p W_{1p}=2\pi_p\sum_{p_1+p_2=p}B(W_{0p_1},(1-\pi_{p2})W_{1p_2}).$$

The equation becomes:

$$\partial_\tau W_{0p}+\frac{i\tilde{\rho}}{2}\partial_x^2 W_{0p}+(\partial_t+\rho\partial_x)\pi_p W_{1p}=F(W_{0p}).$$

Decomposed into two equations:

$$\partial_\tau W_{0p}+\frac{i\tilde{\rho}}{2}\partial_x^2 W_{0p}=F(W_{0p})\quad \text{(Schrödinger)},$$
$$(\partial_t+\rho\partial_x)\pi_p W_{1p}=0\quad \text{(transport)}.$$

Step 1—Approximate solution: conclusion

$\partial_t V + A(e_1)\partial_x V + \frac{1}{\varepsilon}\{-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0\}V = B(V,V)$

Precisely, we construct an approximate solution to (3) in the form

$$V^a(t,x,\theta) = \left[(W_{01}e^{i\theta} + c.c.) + \varepsilon W_1 + \varepsilon^2 W_2\right](\varepsilon t, t, x, \theta),$$

where W_{01} solves cubic Schrödinger equation, such that

$$\partial_t V^a + A(e_1)\partial_x V^a + \frac{1}{\varepsilon}\{-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0\}V^a = B(V^a, V^a) + \varepsilon^2 R.$$

Step 2—Long time $O(1/\varepsilon)$ existence: spectral

Spectral decomposition: $A(e_1)\xi + \frac{L_0}{i} = \sum_{j=1}^{6} \lambda_j \Pi_j + \sum_{j'=7}^{9} \lambda_{j'} \Pi_{j'}$,

where actually $\lambda_{j'} = 0, j' = 7, 8, 9$.

Semiclassical Fourier multiplier

$$\Pi_0 = \sum_{j=1}^{6} \Pi_j(\varepsilon \mathcal{D}_x + k\mathcal{D}_\theta), \quad \Pi_s = \sum_{j'=7}^{9} \Pi_{j'}(\varepsilon \mathcal{D}_x + k\mathcal{D}_\theta).$$

Where

$$\sigma(\varepsilon D_x + k\mathcal{D}_\theta + kp)\Big(\sum_{q\in\mathbb{Z}} e^{iq\theta} f_q(x)\Big) := \sum_{q\in\mathbb{Z}} e^{iq\theta} \sigma(\varepsilon \mathcal{D}_x + kq + kp) f_q(x),$$

$$\sigma(\varepsilon \mathcal{D}_x + kq + kp) f_q(x) := \int e^{ix\xi} \sigma(\varepsilon\xi + kq + kp) \hat{f}_q(\xi) d\,\xi.$$

Step 2—Long time $O(1/\varepsilon)$ existence: Change of variable

Define

$$V_0 = \Pi_0 V, \quad V_s = \Pi_s V.$$

Then

$$\begin{cases} \partial_t V_0 + \frac{i}{\varepsilon}\mathcal{A}_0 V_0 - \frac{\omega\partial_\theta}{\varepsilon} V_0 = \Pi_0 B(V_0 + V_s, V_0 + V_s), \\ \partial_t V_s - \frac{\omega\partial_\theta}{\varepsilon} V_s = \Pi_s B(V_0 + V_s, V_0 + V_s), \end{cases}$$

where

$$\mathcal{A}_0 := \sum_{j=1}^{6} \lambda_j(\varepsilon\mathcal{D}_x + k\mathcal{D}_\theta)\Pi_j(\varepsilon\mathcal{D}_y + k\mathcal{D}_\theta).$$

Step 2:—Long time $O(1/\varepsilon)$ existence: New equations

Direct calculation:

$$\Pi_0 B(\Pi_0, \Pi_0) = \Pi_0 B(\Pi_s, \Pi_s) = \Pi_s B(\Pi_s, \Pi_s) = \Pi_s B(\Pi_0, \Pi_s) = 0.$$

Then

$$\begin{cases} \partial_t V_0 + \dfrac{i}{\varepsilon}\mathcal{A}_0 V_0 - \dfrac{\omega\partial_\theta}{\varepsilon} V_0 = 2\Pi_0 B(V_0, V_s), \\ \partial_t V_s - \dfrac{\omega\partial_\theta}{\varepsilon} V_s = \Pi_s B(V_0, V_0). \end{cases}$$

Step 2—Long time $O(1/\varepsilon)$ existence: Normal form method

First, changing variable $V_s = \varepsilon W_s$ gives

$$\begin{cases} \partial_t V_0 + \frac{i}{\varepsilon}\mathcal{A}_0 V_0 - \frac{\omega\partial_\theta}{\varepsilon} V_0 = 2\varepsilon\Pi_0 B(V_0, W_s), \\ \partial_t W_s - \frac{\omega\partial_\theta}{\varepsilon} W_s = \frac{1}{\varepsilon}\Pi_s B(V_0, V_0). \end{cases}$$

Then introduce the nonlinear change of variable

$$N := W_s - J(V_0, V_0),$$

where the symmetric bilinear form J has the form

$$J(\sum_{p\in\mathbb{Z}} u_p e^{ip\theta}, \sum_{q\in\mathbb{Z}} v_q e^{iq\theta}) := \sum_{p,q} J_{pq}(u_p, v_q) e^{i(p+q)\theta},$$

with J_{pq} to be determined below.

Step 2—Long time $O(1/\varepsilon)$ existence: Normal form method

The equation in N is

$$\partial_t N - \frac{\omega\partial_\theta}{\varepsilon} N = \frac{1}{\varepsilon}\Pi_s B(V_0, V_0) + \frac{i}{\varepsilon} J\Big(\sum_{j=1}^{6} \lambda_j(\varepsilon\mathcal{D}_x + k\mathcal{D}_\theta)\Pi_j(\varepsilon\mathcal{D}_x + k\mathcal{D}_\theta)$$

$$+ \frac{i}{\varepsilon} J\Big(V_0, \sum_{j=1}^{6} \lambda_j(\varepsilon\mathcal{D}_x + k\mathcal{D}_\theta)\Pi_j(\varepsilon\mathcal{D}_x + k\mathcal{D}_\theta)V_0\Big) \; - 2\varepsilon J\Big(\Pi_0 B(V_0, V$$

Let

$$J_{pq}\big(\Pi_j(\xi)a, \Pi_{j'}(\eta)b\big) := i\sum_{j=1}^{6}\sum_{j'=1}^{6} \frac{\Pi_s B(\Pi_j(\xi)a, \Pi_{j'}(\eta)b)}{\lambda_j(\xi) + \lambda_{j'}(\eta)}, \text{ for all } a, b \in \mathbb{C}^9.$$

The three singular terms disappear.

Step 2—Long time $O(1/\varepsilon)$ existence: Compatibility conditions

$$J_{pq}(\Pi_j(\xi)a, \Pi_{j'}(\eta)b) := i\sum_{j=1}^{6}\sum_{j'=1}^{6}\frac{\Pi_s B(\Pi_j(\xi)a, \Pi_{j'}(\eta)b)}{\lambda_j(\xi)+\lambda_{j'}(\eta)}, \text{ for all } a, b \in \mathbb{C}^9.$$

Compatibility conditions

J_{pq} well defined, that means it is a bounded bilinear form, iff there exists $C > 0$, such that for all ξ, η, j, j':

$$\Pi_s B(\Pi_j(\xi)a, \Pi_{j'}(\eta)b) \le C|\lambda_j(\xi)+\lambda_{j'}(\eta)||a||b|.$$

This is satisfied here.

Step 2—Long time $O(1/\varepsilon)$ existence: Normal form method

The system in (V_0, N) becomes

$$\begin{cases} \partial_t V_0 + \dfrac{i}{\varepsilon}\mathcal{A}_0 V_0 - \dfrac{\omega\partial_\theta}{\varepsilon} V_0 = \varepsilon F_0(V_0, N), \\ \partial_t N - \dfrac{\omega\partial_\theta}{\varepsilon} N = \varepsilon F_1(V_0, N). \end{cases}$$

Rescale the time $(\mathcal{V}_0, \mathcal{N})(\tau, y, \theta) = (V_0, N)(\tau/\varepsilon, y, \theta)$. Then

$$\begin{cases} \partial_\tau \mathcal{V}_0 + \dfrac{i}{\varepsilon^2}\mathcal{A}_0 \mathcal{V}_0 - \dfrac{\omega\partial_\theta}{\varepsilon^2} \mathcal{V}_0 = F_0(V_0, N), \\ \partial_\tau \mathcal{N} - \dfrac{\omega\partial_\theta}{\varepsilon^2} \mathcal{N} = F_1(V_0, N). \end{cases}$$

Classical theory gives the local well-posedness for time τ on the interval $[0, T]$. Back to t, on $[0, T/\varepsilon]$.

Step 3—Error estimates: Changes of variables for the approximate solution

For the WKB solution V^a, by the changes of variable:

$$(V_0^a, V_s^a) := (\Pi_0 V^a, \Pi_s V^a), \quad W_s^a := V_s^a/\varepsilon,$$
$$N^a := W_s^a - J(V_0^a, V_0^a), \quad (\mathcal{V}_0^a, \mathcal{N}^a)(\tau) := (V_0^a, N^a)(\tau/\varepsilon).$$

Then

$$\begin{cases} \partial_\tau \mathcal{V}_0^a + \frac{i}{\varepsilon^2}\mathcal{A}_0 \mathcal{V}_0^a - \frac{\omega\partial_\theta}{\varepsilon^2} V_0^a = 2\Pi_0 B(\mathcal{V}_0^a, \mathcal{N}^a + J(\mathcal{V}_0^a, \mathcal{V}_0^a)) + \varepsilon \Pi_0 R, \\ \partial_\tau \mathcal{N}^a - \frac{\omega\partial_\theta}{\varepsilon^2}\mathcal{N}^a = -2J(\Pi_0 B(\mathcal{V}_0^a, \mathcal{N}^a + J(\mathcal{V}_0^a, \mathcal{V}_0^a)), \mathcal{V}_0^a) - 2\varepsilon J(\mathcal{V}_0^a, \Pi_0 R) + \end{cases}$$

Error estimates—The error

Define

$$\Phi(\tau, y, \theta) := (\mathcal{V}_0 - \mathcal{V}_0^a)(\tau, y, \theta), \quad \Psi(\tau, y, \theta) := (\mathcal{N} - \mathcal{N}^a)(\tau, y, \theta).$$

Then

$$\begin{cases} \partial_\tau \Phi + \frac{i}{\varepsilon^2}\mathcal{A}_0\Phi - \frac{\omega\partial_\theta}{\varepsilon^2}\Phi = 2\Pi_0 B(\mathcal{V}_0^a, \Psi) + H_0(\mathcal{V}_0^a, \mathcal{N}^a, \Phi, \Psi) - \varepsilon\Pi_0 R, \\ \partial_\tau \Psi - \frac{\omega\partial_\theta}{\varepsilon^2}\Psi = H_s(\mathcal{V}_0^a, \mathcal{N}, \Phi, \Psi) + 2\varepsilon J(\mathcal{V}_0^a, \Pi_0 R) - \Pi_s R. \end{cases}$$

Initial data:

$$\begin{cases} \Phi(0) = \varepsilon\Pi_0 b - \varepsilon^2\Pi_0 b_1, \\ \Psi(0) = -\Pi_s b + \varepsilon b_2, \end{cases}$$

where $\varepsilon b + \varepsilon^2 b_1 = v(0) - v^a(0)$.

Error estimates—Prepared data

Choose suitable initial data such that $\Pi_s b = O(\varepsilon)$ which is equivalent to

$$|\Pi_0(v - v^a)(0)|_{L^\infty} = O(\varepsilon), \quad |\Pi_s(v - v^a)(0)|_{L^\infty} = O(\varepsilon^2).$$

Then

$$\begin{cases} \Phi(0) = \varepsilon\Pi_0 b - \varepsilon^2\Pi_0 b_1 = O(\varepsilon), \\ \Psi(0) = -\Pi_s b + \varepsilon b_2 = O(\varepsilon). \end{cases}$$

Rescale the solution

Define

$$\Phi_1 = \Phi/\varepsilon, \qquad \Psi_1 = \Psi/\varepsilon.$$

Then

$$\begin{cases} \partial_\tau \Phi_1 + \frac{i}{\varepsilon^2}\mathcal{A}_0\Phi_1 - \frac{\omega\partial_\theta}{\varepsilon^2}\Phi_1 = 2\Pi_0 B(\mathcal{V}_0^a, \Psi_1) + \frac{1}{\varepsilon}H_0(\mathcal{V}_0^a, \mathcal{N}^a, \varepsilon\Phi_1, \varepsilon\Psi_1) - \Gamma \\ \partial_\tau \Psi_1 - \frac{\omega\partial_\theta}{\varepsilon^2}\Psi_1 = \frac{1}{\varepsilon}H_s(\mathcal{V}_0^a, \mathcal{N}, \varepsilon\Phi_1, \varepsilon\Psi_1) + 2J(\mathcal{V}_0^a, \Pi_0 R) - R_s. \end{cases} \tag{6}$$

The initial datum

$$\Phi_1(0) = -\Pi_0 b - \varepsilon\Pi_0 b_1 = O(1), \quad \Psi_1(0) = \frac{\Pi_s b}{\varepsilon} + b_2 = O(1). \tag{7}$$

Error estimates

Classical theory gives

Proposition

The Cauchy problem (6)-(7) *admits a unique solution* (Φ_1, Ψ_1) *on* $[0, T^*[$ *with* $T^* > 0$ *independent of* ε, *and for any* $T < T_3^*$*:*

$$\|\Phi_1\|_{L^\infty([0,T],H^1(\mathbb{T}_\theta,H_x^{s-2}))} \leq C, \quad \|\Psi_1\|_{L^\infty([0,T],H^1(\mathbb{T}_\theta,H_x^{s-2}))} \leq C.$$

Back to the original time and variables, by Sobolev embedding:

Corollary

There holds for all $T < T^*$*:*

$$\|\Pi_0(v - v^a)\|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}_y)} \leq C\varepsilon, \quad \|\Pi_s(v - v^a)\|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}_y)} \leq C\varepsilon^2.$$

Thank you for your attention!