Bounded vorticity, bounded velocity (Serfati) solutions to the incompressible 2D Euler equations

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Introduction

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These are the Euler equations – they should be supplemented by initial data – $u(t = 0) = u_0$ and – if there is a boundary – by slip boundary conditions $\rightarrow$ $u \cdot \hat{n} = 0$ on the finite boundary, together with conditions at infinity.

Vorticity is defined by $\omega \equiv \partial_{x_1} u^2 - \partial_{x_2} u^1 \equiv \nabla^\perp \cdot u$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. It is the only component of curl $(u^1, u^2, 0) \equiv (0, 0, \omega)$.

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Now, $K \in L^p_{loc}(\mathbb{R}^2)$, $1 \leq p < 2$ and K is q-th power integrable at infinity, with $q > 2$. Hence, to calculate $K * \omega$ need $\omega \in L^{p'} \cap L^{q'}$ with $p' > 2$ and $q' < 2$, e.g. $\omega \in L^{\infty} \cap L^1$.

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- $\omega_0 \in \mathcal{BM}_{+,c} \cap H^{-1}$ then $\exists$ (Delort 1991).

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This is a report of work in progress.

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Fundamental questions: what is a ‘solution’? What happens to the Biot-Savart law? Where do we get ‘uniqueness’? What perturbations are allowed?

Related results

- Taniuchi (2004) gives complete, and very different proof of $\exists$, including Serfati flows, in full plane, allowing slightly unbounded initial vorticity (as Yudovich 1995 did in a bounded domain). Uses Littlewood-Paley decomposition and Bony's paradifferential calculus. Does not generalize to exterior domains.

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- Taniuchi, Tashiro, and Yoneda (2010) are concerned with *almost periodic* flows *in the full plane*. They prove $\exists$ and ! assuming $u_0 \in L^\infty$ and $\omega_0 \in Y^\theta_{ul}$, for $\theta = \log(e+q)$; Y^θ_{ul} means "uniformly local" L^p-norms grow like $\theta(p)$ – includes Serfati initial data. Again, proof relies on Littlewood-Paley theory and Bony's paradifferential calculus; highly non-local proof.

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- Brunelli (2010) $\rightarrow$ studies full plane flows with velocity *growing* at infinity. He assumes $\omega_0 \in L^\infty$ and

$$\int \frac{1}{|x-y|} |\omega_0(y)| \, dy < \infty$$

for some $x \in \mathbb{R}^2$ and gets $\exists$ and ! of (u, ω, Φ_t) such that $|u|$ grows at most like $\sqrt{|x|}$ at infinity. (Φ_t is the Lagrangian map.) The hypothesis excludes periodic flows.

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- Local versus non-local;
- Need new idea to substitute Biot-Savart law;
- Broader potential applications in Serfati's key idea (new representation formula).

Serfati's representation formula

Key new idea: consider a smooth cutoff a_ε of the origin ($a_\varepsilon(z) = 1$ if $z \in \mathbb{R}^2$ and $|z|$ small, and vanishes if $|z|$ large) and establish a formula like

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$$u(t,x) = u_0(x) + \int a_\varepsilon(x-y)K(x-y)(\omega(t,y) - \omega_0(y))\,dy +$$

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Indeed: $a_\varepsilon K \in L^1$ and $\omega \in L^\infty$ so the first integral converges; $\nabla_y\nabla_y[(1-a_\varepsilon)K] \in L^1$ because of the extra decay at infinity coming from taking two derivatives, hence, if $u \in L^\infty$, then the second integral also converges.

Where does this formula come from?

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Hence,

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$$= \partial_t \int a_\varepsilon(x-y)K(x-y)\omega(t,y)\,dy + \int (1-a_\varepsilon(x-y))K(x-y)\partial_t\omega\,dy.$$

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Therefore, integrating in time yields

$$u(t,x) = u_0(x) + \int a_\varepsilon(x-y)K(x-y)[\omega(t,y)-\omega_0(y)]\,dy$$

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4. Define limit vorticity as $\omega(t,x) \equiv \omega_0(X^{-1}(t,x))$. Show $\omega_N \to \omega$ in $L^\infty_{loc}(dt, L^p_{loc}(dx))$, any $p < \infty$. Show $u_N \to u$ uniformly in compacts in space and time.

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5. Conclude u, ω satisfy incompressible 2D Euler in distributions, ω constant on particle paths and, also, representation formula remains valid. Also, (limit) u is log-Lipschitz.

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The desired L^∞-estimate for u follows, hence, by Gronwall's lemma.

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Second, (hindsight from previous exterior domain work) know

$$K_\Omega(x,y) + K(x) \sim K(x-y)$$

$(K(x) = x^\perp/(2\pi|x|^2))$, not $K_\Omega \sim K(x-y)$.

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$$\begin{aligned}
&u(t,x)\\
&\quad = u_0(x) + \int_\Omega a_\varepsilon(x-y)J(x,y)(\omega(t,y)-\omega_0(y))\,dy\\
&\qquad - \int_0^t \int_\Omega (u(s,y)\cdot\nabla_y)\,\nabla_y^\perp\,[(1-a_\varepsilon(x-y))J(x,y)]\cdot u(s,y)\,dy\,ds\\
&\qquad + \frac{K(x)}{2}\int_0^t\int_\Gamma |u(s,y)|^2\,\nabla a_\varepsilon(x-y)\cdot d\sigma(y)\,ds.
\end{aligned}$$

Uniqueness

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Theorem

Let Ω be a smooth domain exterior to a connected and bounded set. Let $u_0 \in S$. Then there exists one and at most one weak solution of Euler in Ω with initial velocity u_0.

Continuous dependence

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Theorem

Suppose initial velocities, u_1^0 and u_2^0, with vorticities, ω_1^0 and ω_2^0, such that $u_1^0 - u_2^0$ lies in

$$S_p := \{u \in (L^\infty(\Omega))^2 : \; div\, u = 0, u \cdot \mathbf{n} = 0, \omega \in L^p(\Omega)\}$$

for some p in $(2, \infty]$, with $\|\cdot\|_{S^p} = \|\cdot\|_{L^\infty} + \|\omega(\cdot)\|_{L^p}$. Then, for all sufficiently small $s_0 = \|u_1^0 - u_2^0\|_{S^p}$,

$$\|u_1(t) - u_2(t)\|_{L^\infty} \leq s_0 e^{Ct} + C_t (s_0 t)^{e^{-Ct(2+t)}} [\log C_t + s_0 t e^{-Ct(2+t)}] [C(2+t)e^{Ct} + 1],$$

where C and C_t depend on the initial data and on p, with C_t a continuous function of time.

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2. Should try to improve continuous dependence result so as to have same norm comparison, not velocity stable in L^∞ in terms of initial perturbation in S. Recall Taniuchi et alli, continuous dependence in $B^0_{\infty,1}$. But only for full plane and only by Littlewood-Paley...

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$$u = u(x) = \begin{cases} (1,0) & \text{if } x_2 > 3, \\ (x_2 - 2, 0) & \text{if } 2 < x_2 < 3, \\ (0,0) & \text{if } x_2 < 2, \end{cases}$$

is Serfati in the exterior of the unit disk.

Thank you!