Vicsek Flocking Dynamics and Phase Transition

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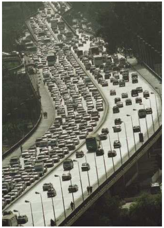
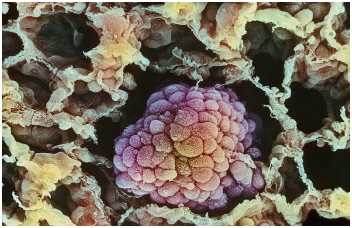
- Frouvelle and Liu, Dynamics in a kinetic model of oriented particles with phase transition, SIMA 2012
- Degond, Frouvelle, and Liu, Macroscopic limits and phase transition in a system of self-propelled particles
- Degond, Liu, Motsch, and Panferov, Existence theory for hydrodynamic models of self-alignment interactions
- Degong and Liu, Hydrodynamics of self-alignment interactions with precession and derivation of the Landau-Lifschitz-Gilbert equation, M3AS, 2012
- Xiuqing Chen and Liu, Global weak entropy solution to Doi-Saintillan-Shelley model for active and passive rod-like particle, Chen's talk this afternoon suspensions

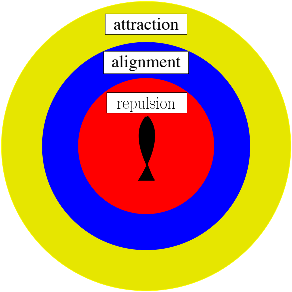
Emergence behavior of self-propelled agents

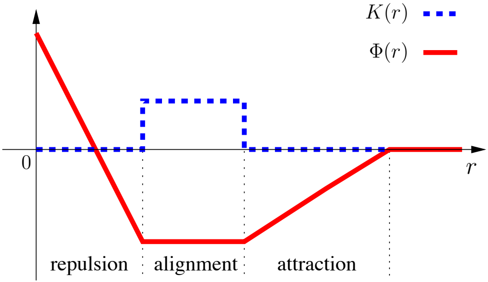
patterns, structures, correlations, synchronization, only local interactions, no leader






3 zones: repulsion, alignment, attraction
3 classes of models: agent based, kinetic, hydrodynamics





Aoki '82, Reynolds 86, Vicsek '95, Toner and Tu '98, Couzin '02, Topaz and Bertozzi '04, DOrsogna, Chuang, Bertozzi, Chayes 06', Cucker and Smale '07, Degond and Motsch '08, Ha and Tadmor '08, Ha and Liu '09, Carrillo, Fornasier, Rosado '10, etc

Agent based model of self-alignment with attraction-repulsion potential

$$\begin{aligned}
\frac{dx_k}{dt} &= v_k \\
dv_k &= P_{v_k^\perp}\left(\overline{v}_k\, dt + \sqrt{2\tau}\, dB_t^k\right), \quad P_{v^\perp} = \mathrm{Id} - v \otimes v.
\end{aligned}$$

where

$$\overline{v}_k = \nu \frac{j_k + r_k}{|j_k + r_k| + \delta},$$

$$j_k = \sum_j K(|x_j - x_k|) v_j, \quad r_k = -\nabla_x \Phi(x_k), \qquad \Phi(x) = \sum_j \phi(|x_j - x|),$$

Mean field kinetic equation

$$f_t + v \cdot \nabla_x f = -\nabla_v \cdot [(P_{v^\perp} v_f) f] + \tau \Delta_v f,$$

where

$$v_f = \nu \frac{j_f + r_f}{|j_f + r_f| + \delta}, \quad j_f = \int_{x',v'} K(|x' - x|) v' \, f(x', v', t) \, dx' dv',$$

$$r_f = -\nabla_x \int_{x',v'} \Phi(|x' - x|) \, f(x', v', t) \, dx' dv'.$$

c.f. Bolley, Caizo, & Carrillo 2012

Hydrodynamical equations

Hydrodynamical rescaling

$$f_t + v \cdot \nabla_x f = -\frac{1}{\epsilon} \left(\nabla_v \cdot [(P_{v^\perp} v_f) f] - \tau \Delta_v f \right),$$

- local aligment $K(|x_j - x_k|/\epsilon)$ gives the pressure term
- near local aligment $K(|x_j - x_k|/\sqrt{\epsilon})$ gives viscosity term
- near local van Der Waals potential $\Phi(x,u) = \iint_{R^n \times S_{n-1}} \phi(|x-y|/\sqrt{\epsilon})(f(y,v,t) - f(x,u,t)) dydv$ induced a capillary force

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho \Omega) = 0,$$
$$\rho(\partial_t \Omega + c_2 \Omega \cdot \nabla_x \Omega) + \tau P_{\Omega^\perp} \nabla_x \rho = c_3 P_{\Omega^\perp} \Delta(\rho \Omega) + c_4 P_{\Omega^\perp} \nabla_x \Delta \rho,$$

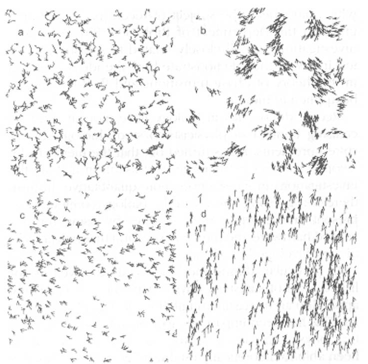
Symmetrization viscous hyperbolic system

In 2D, we set $\Omega = (\cos\varphi, \sin\varphi)$, $\hat{\rho} = a(\rho)$, $a'(\rho) = \frac{\sqrt{p'(\rho)}}{\rho}$,
$\lambda(\hat{\rho}) = a'(\rho)\rho$, $h(\hat{\rho}) = 2\ln\rho$.
Then the system recast as

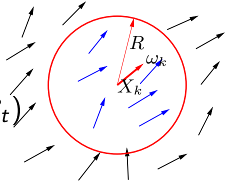
$$
\begin{aligned}
&(\partial_t + \Omega\cdot\nabla_x)\hat{\rho} + \lambda(\hat{\rho})\,(\Omega^\perp\cdot\nabla_x)\varphi = 0\\
&(\partial_t + c\Omega\cdot\nabla_x)\varphi + \lambda(\hat{\rho})\,(\Omega^\perp\cdot\nabla_x)\hat{\rho} = \mu\left(\Delta\varphi + \nabla_x h(\hat{\rho})\cdot\nabla_x\varphi\right).
\end{aligned}
$$

local classical solution

Flocking of self propelled particles, Vicsek et al PRL '95

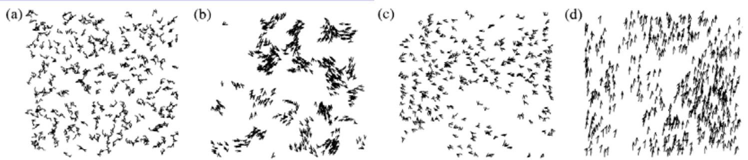


- time-discrete model
  - $t^n = n\Delta t$, k-th individual
  - $\mathbf{x}_k^n$: position at t^n
  - $\mathbf{v}_k^n$: velocity with $|\mathbf{v}_k^n| = 1$.
- $\mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \mathbf{v}_k^n \Delta t$
  alignment to local mean velocity
  $\bar{\mathbf{v}}_k^n = \sum_{j=1}^N \psi(|\mathbf{x}_j - \mathbf{x}_k|)\mathbf{v}_j^n + \text{noise}$
  $\mathbf{v}_k^{n+1} = \bar{\mathbf{v}}_k^n / |\bar{\mathbf{v}}_k^n|$
- Degond-Motsch '08
  $\dot{\mathbf{x}}_k = \mathbf{v}_k$
  $\bar{\mathbf{v}}_k = \sum_{j=1}^N \psi(|\mathbf{x}_j - \mathbf{x}_k|)\mathbf{v}_j$
  $\omega_k = \bar{\mathbf{v}}_k / |\bar{\mathbf{v}}_k|$
  $d\mathbf{v}_k = (\mathrm{Id} - \mathbf{v}_k \otimes \mathbf{v}_k)(\lambda(\rho_k)\omega_k dt + \sqrt{2\tau} dB_t)$
- We take $\lambda = \lambda_k = |\bar{\mathbf{v}}_k|$

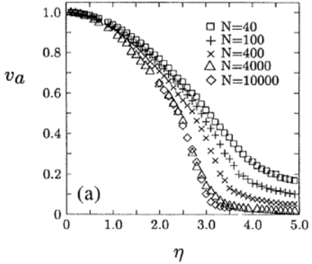


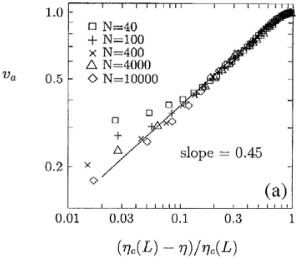
Vicsek model and phase transition

Order parameter/mean speed $|J(f)|$ and variance $1 - |J(f)|^2$

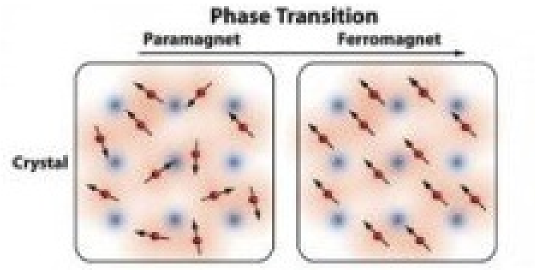


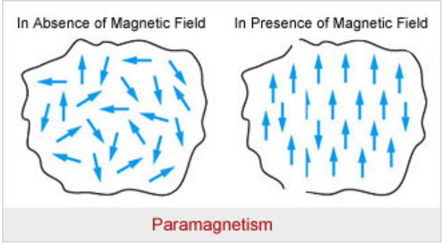
a) High noise, low density: particles moved independently
b) Low noise, low density: particles formed groups that were independent
c) High noise, high density: particles moved with some correlations
d) Low noise, high density: all particles moved in the same direction

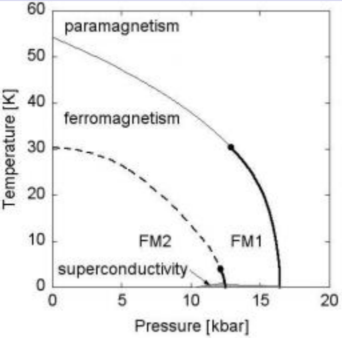




Paramagnetism to ferromagnetism phase transition near Curie temperature







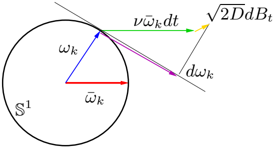
Dynamics of orientational alignment

- oriented particles $\{\omega_j\}_{j=1}^N \subset \mathbb{S}^{n-1}$, unit sphere in $\mathbb{R}^n$
- dynamics of orientational alignment, for $k = 1, \cdots, N$

$$\mathrm{d}\omega_k = (\mathsf{Id} - \omega_k \otimes \omega_k) J(t)\,\mathrm{d}t + \sqrt{2\tau}(\mathsf{Id} - \omega_k \otimes \omega_k) \circ \mathrm{d}B_t^k,$$

$J(t) = \frac{1}{N}\sum_{j=1}^N \omega_j(t)$.
Stochastic integral is in the Stratonovich sense.

- B_t Brownian motion in $\mathbb{R}^n$
- $(\mathsf{Id} - \omega \otimes \omega) \circ \mathrm{d}B_t = \mathrm{d}W_t$, Brownian motion on sphere



Dynamics of orientational alignment

- in 2D, $\omega_k(t) = e^{i\theta_k(t)}$,

$$\mathrm{d}\theta_k = \sin(\bar{\theta}(t) - \theta_k)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_t^k,$$

$$\sin(\bar{\theta}(t)) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j(t))$$

connected to Kuramoto nonlinear oscillator

$$\dot{\theta}_k = \Omega_k + \sin(\bar{\theta}(t) - \theta_k),$$

and Synchronization. S.-Y. Ha's talk on Monday

Standard deviation and Order Parameter $|\bar{\mathbf{v}}|$

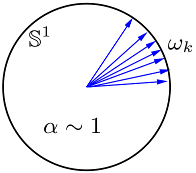
Standard deviation:

$$\sigma_v^2 = \frac{1}{N}\sum_{k=1}^{N} |\mathbf{v}_k - \bar{\mathbf{v}}|^2 = \frac{1}{N}\sum_{k=1}^{N} |\mathbf{v}_k|^2 - |\bar{\mathbf{v}}|^2, \quad \bar{\mathbf{v}} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{v}_k$$

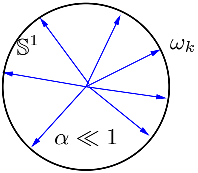
If $|\mathbf{v}_k| = 1$, then

$$\sigma_v^2 = 1 - |\bar{\mathbf{v}}|^2,$$

Measures alignment



$\alpha \sim 1$: ω aligned



$\alpha \ll 1$: ω random

Mean field equation, Fokker-Planck eq, Smoluchowski eq

- potential $\psi(\omega, t) = -\omega \cdot J(t)$, least potential if ω aligns to J
- identity: $(\mathrm{Id} - \omega \otimes \omega)J = -\nabla_\omega \psi$
- recast as steepest descent motion:

$$\mathrm{d}\omega_k = -\nabla_\omega \psi(\omega_k)\mathrm{d}t + \sqrt{2\tau}\mathrm{d}W_t^k$$

- probability density function, $f(\omega, t)$, $\omega \in \mathbb{S}^{n-1}$,
- mean field equation (Fokker-Planck eq, Smoluchowski eq), $\int_{\mathbb{S}^{n-1}} f(\omega, t) = 1 = \int_{\mathbb{S}^{n-1}} \mathrm{d}\omega$

$$\partial_t f = d\Delta_\omega f + \nabla_\omega (f \nabla_\omega \psi) := Q(f), \quad d = \frac{\tau}{\rho}$$

$$\psi(\omega, t) = -\omega \cdot J(t), \quad J(t) = \int_{\mathbb{S}^{n-1}} \omega f(\omega, t),$$

- interaction kernel

$$\psi(\omega, t) = \int_{\mathbb{S}^{n-1}} K(\omega, \omega') f(\omega', t), \quad K(\omega, \omega') = -\omega \cdot \omega'$$

Free energy-dissipation relation

- free energy:

$$\mathcal{F}(f) = d \int_{\mathbb{S}^{n-1}} f \ln f + \frac{1}{2} \int_{\mathbb{S}^{n-1}} \psi f.$$

- chemical potential ψ

$$\mu = \frac{\delta \mathcal{F}}{\delta f} = d \ln f + \psi, \quad \mathbf{v} = -\nabla_\omega \mu = -d \nabla_\omega \ln f - \nabla_\omega \psi,$$

- recast as the continuity equation (Doi 1981, Hess 1976):

$$f_t + \nabla_\omega \cdot (f \mathbf{v}) = 0,$$

- dissipation of free energy:

$$\partial_t \mathcal{F} + \mathcal{D} = 0, \quad \mathcal{D}(f) = \int_{\mathbb{S}^{n-1}} f |\nabla_\omega \mu|^2.$$

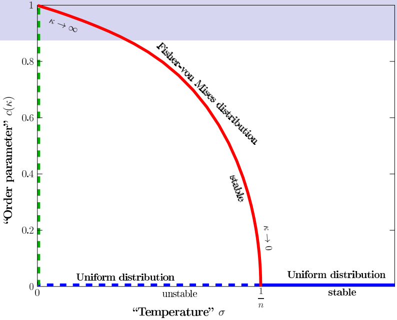
Onsager theory on orientational phase transition, 1949

- recall free energy:

$$\mathcal{F}(f) = d\int_{\mathbb{S}^{n-1}} f \ln f + \frac{1}{2}\int_{\mathbb{S}^{n-1}} \psi f, \quad \psi(\omega) = \int_{\mathbb{S}^{n-1}} K(\omega,\omega')f(\omega')$$

- $K(\omega,\omega') = |\omega \times \omega'|$, Onsager : excluded volume potential for rodlike polymers
- $K(\omega,\omega') = |\omega \times \omega'|^2 = 1 - (\omega\cdot\omega')^2$, Maier-Saupe kernel
- $K(\omega,\omega') = -\omega\cdot\omega'$, **Dipolar interaction kernel**.
- $K(\omega,\omega') = -\alpha\omega\cdot\omega' - \beta(\omega\cdot\omega')^2$, magnetic rod suspension
- **equilibria** f_{eq} are given by minimizing $\mathcal{F}(f)$ subject to $\int_{\mathbb{S}^{n-1}} f = 1$
- phase transition in equilibria probability density function near critical temperature/noise level or critical mass

Critical parameter $d_c = \frac{1}{n}$ and Linear stability



Around the constant state : $f = 1 + g$.

$$\partial_t g = d\Delta_\omega g + (n-1)(1+\cancel{g})\,\omega\cdot J[g] \cancel{- (\mathrm{Id}-\omega\otimes\omega)J[g]\cdot\nabla_\omega g},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J[g] = (n-1)\left(\frac{1}{n} - d\right)J[g] + \cancel{\int_{\mathbb{S}^{n-1}}(\mathrm{Id}-\omega\otimes\omega)J[g]g}$$

Linearly stable when $d > d_c$, unstable when $d < d_c$.

Characterization of equilibria

Proposition

The steady states to the Fokker- Planck eq. are probability measures f on $\mathbb{S}^{n-1}$ satisfy one of the following equivalent conditions

- *equilibrium:* $f \in C^2(\mathbb{S}^{n-1})$ *and* $Q(f) = 0$
- *no dissipation,* $f \in C^1(\mathbb{S}^{n-1})$ *and* $\mathcal{D}(f_{eq}) = 0$
- *critical states:* $f \in C^0(\mathbb{S}^{n-1})$ *and critical point of* $\mathcal{F}(f)$ *subject to* $\int_{\mathbb{S}^{n-1}} f = 1$.

$$\frac{\delta \mathcal{F}}{\delta f} = \mu \quad (\textit{Lagrange multiplier, a constant})$$

- *Gibbs/Boltzmann states:* f *positive, symmetric, analytic*

$$\mu = \frac{\delta \mathcal{F}}{\delta f} = d \ln f - J[f] \cdot \omega = \textit{constant} \quad (\textit{chemical potential})$$

Characterization of equilibria: Von Mises-Fisher distribution, 1953

recall equilibria (Gibbs/Boltzmann states) are given by $d \ln f - J[f] \cdot \omega = C$. Denote $J[f] = |J(f)|\Omega$, $\Omega \in \mathbb{S}^{n-1}$

$$f = Z^{-1} \exp(\kappa\, \omega \cdot \Omega), \quad \kappa = |J(f)|/d$$

Von Mises-Fisher distribution with concentration parameter $\kappa \geqslant 0$:

$$M_{\kappa\Omega}(\omega) = Z^{-1} \exp(\kappa\, \omega \cdot \Omega), \quad Z = \int_{\mathbb{S}^{n-1}} \exp(\kappa\, \omega \cdot \Omega)\,.$$

Compatibility equation for κ:

$$|J[M_{\kappa\Omega}]| = \kappa d = c(\kappa) := \frac{\int_0^\pi \cos\theta\, e^{\kappa\cos\theta} \sin^{n-2}\theta\, \mathrm{d}\theta}{\int_0^\pi e^{\kappa\cos\theta} \sin^{n-2}\theta\, \mathrm{d}\theta}.$$

$0 \leqslant c(\kappa) \leqslant 1$, order parameter/mean speed for equilibria

Characterization of equilibria ($d > 0$): two phases

Lemma

Let $\beta = c(\kappa)^2 + n\dfrac{c(\kappa)}{\kappa} - 1$. *For any* $\kappa > 0$, *we have* $\beta > 0$.

The function $\kappa \mapsto \dfrac{c(\kappa)}{\kappa}$ ($= d$ for the compatibility equation) is decreasing (its derivative is $-\frac{\beta}{\kappa}$), starting from $\frac{1}{n}$ at 0.

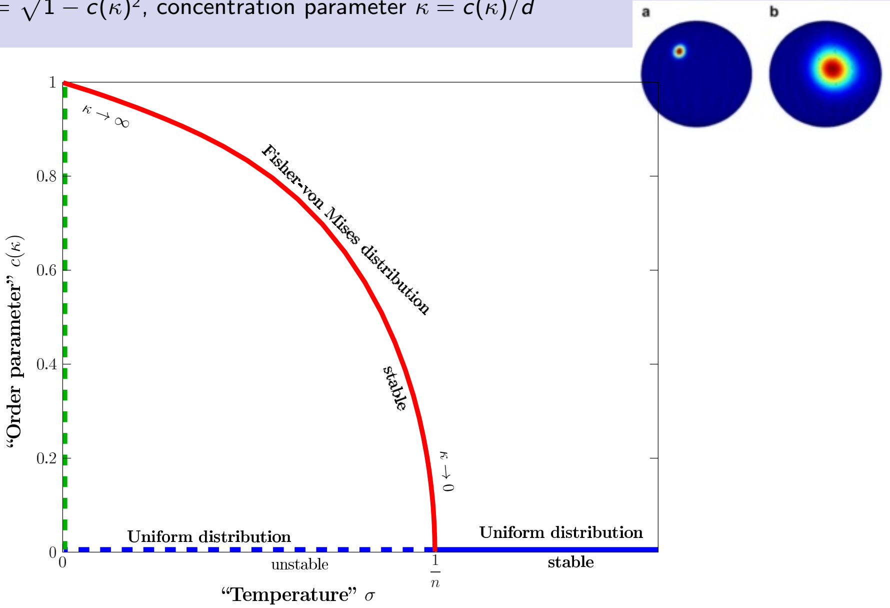
- If $d \geqslant d_c$, only one solution to compatibility equation: $\kappa = 0$. Equilibrium : constant state 1 (disordered phase).
- If $d < d_c$, either $\kappa = 0$ or κ is the unique positive solution of the compatibility equation (ordered phase $M_{\kappa\Omega}, \Omega \in \mathbb{S}^{n-1}$).
- Near the critical value of $d_c = 1/n$, $c(\kappa(d)) \sim \sqrt{(n+2)(d_c - d)}$.

Proposition

Any steady state to Fokker- Planck equation is of the above forms

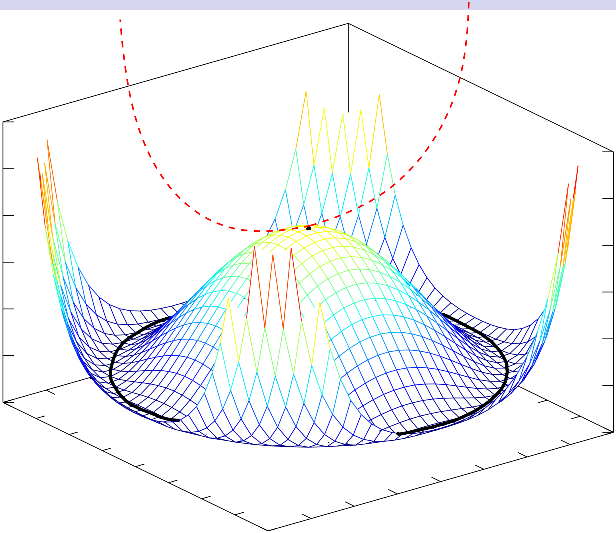
Order parameter $c(\kappa)$ vs d

s.d. $= \sqrt{1 - c(\kappa)^2}$, concentration parameter $\kappa = c(\kappa)/d$



Super-critical $d < 1/n$, cartoon shape of free energy

spontaneous symmetry breaking, spontaneous magnetization



Basic results (are also valid for other kernels)

- For $f_0 \in H^s(\mathbb{S}^{n-1})$, $f_0 \geq 0$, $d \geq 0$, $s \in \mathbb{R}$,
  $\exists!$ solution $f \in C^\infty\big(0,\infty; H^s(\mathbb{S}^{n-1})\big)$ (also analytic in a Gevrey class if $d > 0$) to Fokker- Planck equation.

- Instantaneous regularity, uniform boundedness :

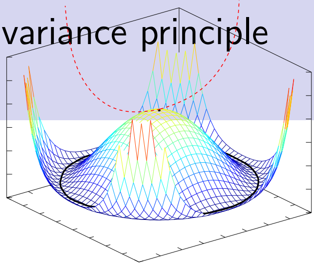
  $$\|f(\cdot,t)\|_{H^{s+m}(\mathbb{S}^{n-1})} \leqslant C\left(1+\frac{1}{t^m}\right)\|f_0\|_{H^s(\mathbb{S}^{n-1})}, \quad t > 0$$

  constant C depending only on $d > 0$, m, n, s.

- Strong maximum principle: if $f_0 \not\equiv 1$, regular, $d > 0$

  $$e^{-(n-1)\int_0^t |J(s)|ds} \min f_0(\omega) < f(\omega,t) < e^{(n-1)\int_0^t |J(s)|ds} \max f_0(\omega)$$

Characterization of equilibria: LaSalle invariance principle (also valid for other kernels)



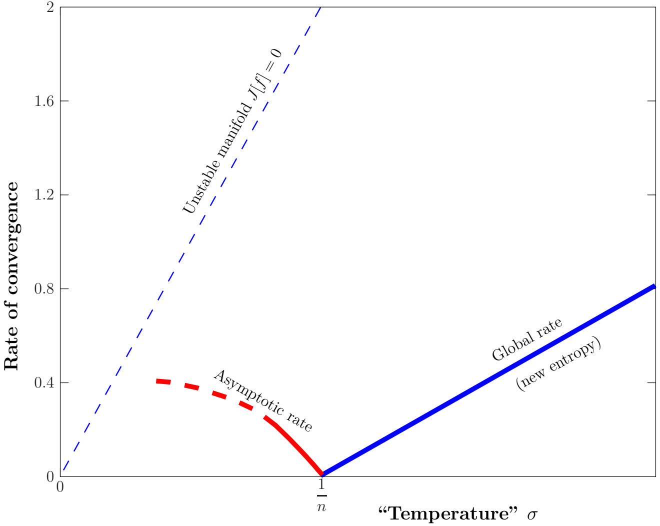
Proposition

Let f_0 be a probability measure on $\mathbb{S}^{n-1}$ and $f(t)$ be the solution with initial data f_0. Denote by $\mathcal{F}_\infty = \lim_{t\to\infty} \mathcal{F}(f(t))$. Then

- *the ω-limit set* $\mathcal{E}_\infty = \{f \in C^\infty(\mathbb{S}^{n-1}) : \mathcal{F}(f) = \mathcal{F}_\infty, \mathcal{D}(f_\infty) = 0\}$ *is not empty.*
- *$f(t)$ converges to $\mathcal{E}_\infty$ in $H^s(\mathbb{S}^{n-1})$ for any $s \in \mathbb{R}$:*

$$\lim_{t\to\infty} \inf_{g\in\mathcal{E}_\infty} \|f(\cdot,t) - g\|_{H^s(\mathbb{S}^{n-1})} = 0, \qquad \text{for any } s \in \mathbb{R}$$

We now show the rates of convergence:



Conformal Laplacian $\widetilde{\Delta}_{n-1}$ on sphere $\mathbb{S}^{n-1}$

- for n odd,

$$\widetilde{\Delta}_{n-1} = \prod_{\ell=0}^{(n-3)/2} \left(-\Delta_\omega + \ell(n-2-\ell)\right)$$

- for n even,

$$\widetilde{\Delta}_{n-1} = \left[-\Delta_\omega + \left(\frac{n-2}{2}\right)^2\right]^{1/2} \prod_{\ell=0}^{(n-4)/2} \left(-\Delta_\omega + \ell(n-2-\ell)\right)$$

- for any spherical harmonic of degree ℓ, Y_ℓ, one has

$$\widetilde{\Delta}_{n-1} Y_\ell = \ell(\ell+1)\ldots(\ell+n-2) Y_\ell,$$

Laplace-Belltrami: $\quad -\Delta_\omega Y_\ell = \ell(\ell+n-2) Y_\ell$

Some notations

- Subspace

$$\dot{H}^s(\mathbb{S}^{n-1}) = \Big\{g \in H^s(\mathbb{S}^{n-1}), \int_{\mathbb{S}^{n-1}} g = 0\Big\}, \quad s \in \mathbb{R}$$

$$\|g\|^2_{\dot{H}^s(\mathbb{S}^{n-1})} = \langle (-\Delta_\omega)^s g, g\rangle$$

- Define, for g a mean-zero function, the following norms:

$$\|g\|^2_{\widetilde{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} g \widetilde{\Delta}^{-1}_{n-1} g,$$
$$\|g\|^2_{\widetilde{H}^{-\frac{n-3}{2}}(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} -\Delta_\omega g \widetilde{\Delta}^{-1}_{n-1} g,$$

equivalent to the $\dot{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})$ and $\dot{H}^{-\frac{n-3}{2}}(\mathbb{S}^{n-1})$ Sobolev norms.

Lemma

1. *For any* $g \in \dot{H}^s(\mathbb{S}^{n-1})$, $h \in \dot{H}^{-s+1}(\mathbb{S}^{n-1})$,

$$\left| \int_{\mathbb{S}^{n-1}} g \nabla_\omega h \right| \leqslant C \|g\|_{\dot{H}^s(\mathbb{S}^{n-1})} \|h\|_{\dot{H}^{-s+1}(\mathbb{S}^{n-1})}$$

2. *For any* $g \in \dot{H}^{s+1}(\mathbb{S}^{n-1})$,

$$\left| \int_{\mathbb{S}^{n-1}} g \nabla_\omega (-\Delta_\omega)^s g \right| \leqslant C \|g\|^2_{\dot{H}^s(\mathbb{S}^{n-1})}$$

3. *For any* $g \in \dot{H}^{-\frac{n-3}{2}}(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} g \nabla_\omega \widetilde{\Delta}_{n-1}^{-1} g = 0$$

$$\widetilde{\Delta}_{n-1} \nabla_\omega g + (n-1)\omega \widetilde{\Delta}_{n-1} g = \nabla_\omega \widetilde{\Delta}_{n-1} g$$

A new entropy

Let $f = 1 + g$, $h = \widetilde{\Delta}_{n-1}^{-1} g$

$$\langle h, g_t \rangle = d\langle h, \Delta_\omega g\rangle + \langle \nabla_\omega h, \cancel{(\mathrm{Id} - \omega \otimes \omega)} J[g](1 + \cancel{g})\rangle$$

$$\langle h, g_t \rangle = -d\langle h, -\Delta_\omega g\rangle + \tfrac{n-1}{(n-1)!}|J(f)|^2$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})} &= -\,d\|f-1\|^2_{\widetilde{H}^{-\frac{n-3}{2}}(\mathbb{S}^{n-1})} + \tfrac{1}{(n-2)!}|J(f)|^2 \\ &\leqslant -\,(n-1)(d - \tfrac{1}{n})\|f-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})}.\end{aligned}$$

The above conservation laws only involves quadratic quantities, i.e., contribution only comes from linear terms!

Remarks on $H^{-\frac{n-1}{2}}(\mathbb{S}_{n-1})$, $M(\mathbb{S}_{n-1})$, and new entropy

- $L^1(\mathbb{S}^{n-1}) \hookrightarrow M(\mathbb{S}^{n-1}) \hookrightarrow H^{-\frac{n-1}{2}-\epsilon}(\mathbb{S}_{n-1})$
- $H^{-\frac{n-1}{2}}(\mathbb{S}_{n-1}) \not\subset M(\mathbb{S}_{n-1})$, $M(\mathbb{S}_{n-1}) \not\subset H^{-\frac{n-1}{2}}(\mathbb{S}_{n-1})$
- $L^1_+(\mathbb{S}_{n-1}) \hookrightarrow H_+^{-\frac{n-1}{2}}(\mathbb{S}_{n-1}) \subset M_+(\mathbb{S}_{n-1}) = \mathcal{D}'_+(\mathbb{S}_{n-1}) \subset M(\mathbb{S}_{n-1}) \hookrightarrow H^{-\frac{n-1}{2}-\epsilon}(\mathbb{S}_{n-1})$
- For $f_0 \in H^s(\mathbb{S}^{n-1})$, $s \in \mathbb{R}$, f_0 can be negative, $d > 0$, $\exists!$ solution $f \in C\big(0,\infty; H^s(\mathbb{S}^{n-1})\big)$ to FP equation
- For $f_0 \in H^s(\mathbb{S}^{n-1})$, $s \geq -(n-1)/2$, f_0 can be negative, $d = 0$, $\exists!$ solution $f \in C\big(0,\infty; H^s(\mathbb{S}^{n-1})\big)$ to FP equation
- For $d = 0$, $\mathbf{f}_0 \in H^{-(n-1)/2}(\mathbb{S}^{n-1})$

$$\|f_0-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}} + |J(0)|t \leq \|f(\cdot,t)-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}} \leq \|f_0-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}} + |J|_\infty t$$

Global exponential rate for under-critical case $d > 1/n$

Let $f = 1 + g$. By a high-low decomposition, for any $s \in \mathbb{R}$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2_{\dot{H}^s} + d\|g\|^2_{\dot{H}^{s+1}} \leq \frac{C}{(N+1)(N+n-1)}\|g\|^2_{\dot{H}^{s+1}} + C\|g^N\|^2_{\dot{H}^s}$$

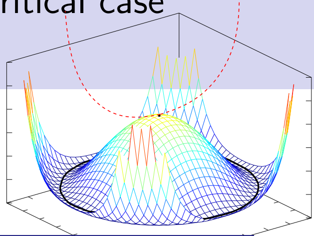
we can extend above result to:

Theorem

For $d > 1/n$, $f_0 \in H^s(\mathbb{S}^{n-1})$, $s \geq -(n-1)/2$, we have global exponential decay towards the uniform distribution with rate $(n-1)(d-\frac{1}{n})$

$$\|f-1\|_{H^s(\mathbb{S}^{n-1})} \leqslant C\|f_0-1\|_{H^s(\mathbb{S}^{n-1})} \exp\left(-(n-1)(d-\tfrac{1}{n})t\right).$$

Asymopotic exponential rate for super-critical case $d < 1/n$



Proposition

- *If* $J[f_0] = 0$ *then* $J[f(t)] = 0$ *for all* t *and* $\mathcal{E}_\infty = \{1\}$, $\mathcal{F}_\infty = 0$. *FP equation becomes heat equation, exponential decay with rate* $2nd$ *to the uniform distribution.*
- *If* $J[f_0] \neq 0$ *then* $J[f(t)] \neq 0$ *for all* t *and* $\mathcal{E}_\infty = \{M_{\kappa\Omega}, \Omega \in \mathbb{S}^{n-1}\}$, $\mathcal{F}_\infty < 0$. *Furthermore, for any* $s \in \mathbb{R}$

$$\lim_{t\to\infty} \|f(\cdot, t) - M_{\kappa\Omega(t)}\|_{H^s(\mathbb{S}^{n-1})} = 0$$

where $\Omega(t) = J[f(t)]/|J[f(t)]|$.

ODE: $\frac{\mathrm{d}}{\mathrm{d}t}J[f] = M(t)J[f]$

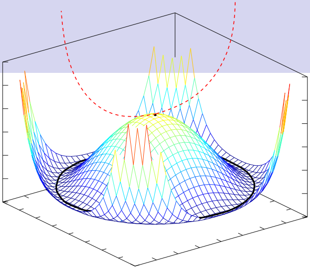
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J[f] &= -d(n-1)J[f] + \left(\int_{\mathbb{S}} (\mathrm{Id} - \omega\otimes\omega)\, f \,\mathrm{d}\omega\right) J[f] \\ &= \left((1-(n-1)d)\mathrm{Id} - \int_{\mathbb{S}} \omega\otimes\omega\, f\right) J[f] =: M(t)J[f],\end{aligned}$$

$M(t)$ is smooth, so we have a global unique solution.
If $J[f(0)] = 0$, then $J[f(t)] \equiv 0$, reduced to the heat equation.
If $J[f(0)] \neq 0$, then $J[f(t)] \neq 0$.
If $f(t) \to 1$, then $M(t) \to (n-1)(\frac{1}{n} - d)$. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|J[f]|^2 = J[f]\cdot M(t)J[f] \geqslant ((n-1)(\tfrac{1}{n}-d) - \varepsilon)|J[f]|^2.$$

$|J[f]| \to \infty$. This is a contradiction.

Asymptotic around $M_{\kappa\Omega(t)}$, $\Omega(t) = \frac{J[f]}{|J[f]|}$



- let $\cos\theta = \omega \cdot \Omega(t)$
- If $d < 1/n$, fix the corresponding $\kappa > 0$.

$$f = M_{\kappa\Omega}\left(1 + \alpha(\cos\theta - c(\kappa) + g\right), \quad \langle g\rangle_{M_{\kappa\Omega}} = 0,\ \langle g\omega\rangle_{M_{\kappa\Omega}} = 0$$

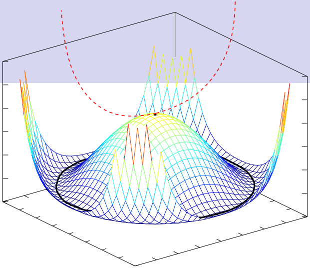
- for large t,

$$\mathcal{F}(f) - \mathcal{F}(M_{\kappa\Omega}) \leq \frac{1}{2}\Big((\beta+\epsilon)(d-\beta)\alpha^2 + (d+\epsilon)\langle g^2\rangle_{M_{\kappa\Omega}}\Big)$$

$$\mathcal{D}(f) \geq \lambda_\kappa\Big((\beta^2-\epsilon)\alpha^2 + (d^2-\epsilon)\langle g^2\rangle_{M_{\kappa\Omega}}\Big)$$

$\lambda_\kappa \geq (n-1)e^{-2\kappa}$, 1st positive eigenvalue $1/M_{\kappa\Omega}\nabla\cdot(M_{\kappa\Omega}\nabla)$

Asymptotic around $M_{\kappa\Omega(t)}$, $\Omega(t) = \frac{J[f]}{|J[f]|}$



- $$\mathcal{F}(f(t)) - \mathcal{F}(M_{\kappa\Omega(t)}) = \int_t^\infty \mathcal{D}(f)$$
- for any $r < \lambda_\kappa \beta$, $t > t_r$, denote $C = \frac{d}{\beta(d-\beta)}$
  $$\alpha(t) + C\langle g^2(t)\rangle_{M_{\kappa\Omega(t)}} \leq \left(\alpha(t_r) + C\langle g^2(t_r)\rangle_{M_{\kappa\Omega(t_r)}}\right) e^{-2r(t-t_r)}$$
- There is a $\Omega_\infty \in \mathbb{S}^{n-1}$ s.t. for large t
  $$|\Omega(t) - \Omega_\infty| \leq \int_t^\infty |\frac{d\Omega}{dt}| \leq Ce^{-rt}, \quad \|f(t) - M_{\kappa\Omega_\infty}\|_{L^2(\mathbb{S}^{n-1})} \leq Ce^{-rt}$$
- The asymptotic rate $r_\infty(d) \geq 2(n-1)(\frac{1}{n} - d) + O(\frac{1}{n} - d)^{3/2}$ in the neighborhood of $d = 1/n$.

Algebraic rate at critical case $d = 1/n$

$J[f_0] \neq 0$, we have $J[f(t)] \neq 0$ for all $t > 0$.

Set $\Omega(t) = \frac{J[f(t)]}{|J[f(t)]|}$. $\cos\theta = \omega \cdot \Omega$

Set $f = 1 + h$, $J[f] = \langle (1+h)\omega \rangle = \langle h\omega \rangle = \langle h \cos\theta \rangle \Omega$

Set $h = g + \alpha \cos\theta + \frac{1}{2}\alpha^2(\cos^2\theta - \frac{1}{n}) + \frac{1}{6}\alpha^3(\cos^3\theta - \frac{3}{n+2}\cos\theta)$,

where $\alpha = n\langle h \cos\theta \rangle$

$\langle g \rangle = \langle g \cos\theta \rangle = 0$

$$\mathcal{F}(f) = \int_t^\infty \mathcal{D}(f)$$

$$1 - \varepsilon \leqslant \frac{4n^3(n+2)}{2n^2(n+2)\langle g^2 \rangle + \alpha^4} \mathcal{F}(1+h) \leqslant 1 + \varepsilon$$

$$\mathcal{D}(f) \geqslant (1-\varepsilon)\frac{n-1}{n^2}\left(\langle g^2 \rangle + \frac{1}{n^3(n+2)^2}\alpha^6\right)$$

Algebraic rate at critical case $d = 1/n$

For any $r < \frac{8(n-1)}{n^2(n+2)}$

$$\frac{1}{2}(2n^2(n+2)\langle g^2\rangle + \alpha^4) \geqslant r\int_t^\infty (2n^2(n+2)\langle g^2\rangle + \alpha^4)^{\frac{3}{2}}$$

$$2n^2(n+2)\langle g^2\rangle + \alpha^4 \leqslant \frac{1}{r^2(t-t_0)^2}.$$

For $r' < \frac{8(n-1)}{n(n+2)}$, there is a $t_0 > 0$ such that for all $t > t_0$,

$$\|f-1\|_{L^2}^2 = \langle h^2\rangle \leqslant \frac{1}{\sqrt{r'}(t-t_0)}.$$

Summary

- We consider alignment model for interacting, self-propelled, oriented particles system $\{\omega_j\}_{j=1}^N \subset \mathbb{S}^{n-1}$, unit sphere in $\mathbb{R}^n$

$$\mathrm{d}\omega_k = (\mathsf{Id} - \omega_k \otimes \omega_k)(J(t)\,\mathrm{d}t + \sqrt{2\tau}\,\mathrm{d}B_t^k), \quad J(t) = \frac{1}{N}\sum_{j=1}^N \omega_j(t)$$

  motivated by Vicsek model of flocking of birds, alignment in ferromagnetism

- This model is well described by mean field equation (also known as nonlinear Fokker-Planck equation, Doi-Onsager equation, Smoluchowski equation, McKean-Vlasov equation)

$$\partial_t f = d\Delta_\omega f + \nabla_\omega (f \nabla_\omega \psi), \quad \psi(\omega, t) = \int_{\mathbb{S}^{n-1}} k(\omega, \omega') f(\omega', t),$$

  where $k(\omega, \omega') = -\omega \cdot \omega'$ is dipolar interaction kernel.

Summary

- There is a critical noise parameter $d = \frac{1}{n}$ (analog to Curie temperature). For $d \geq 1/n$, the only equilibrium is the uniform distribution; for $d < 1/n$, there is also a family of non-isotropic equilibria: Fisher-Von Mises distribution, $M_{\kappa\Omega}(\omega)$, $\Omega \in \mathbb{S}^{n-1}$ with concentration parameter $\kappa(d)$.
- For $d > \frac{1}{n}$, we discovered a new entropy

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})} &= -\,d\|f-1\|^2_{\widetilde{H}^{-\frac{n-3}{2}}(\mathbb{S}^{n-1})} + \tfrac{1}{(n-2)!}|J(f)|^2 \\ &\leqslant -\,(n-1)(d-\tfrac{1}{n})\|f-1\|^2_{\widetilde{H}^{-\frac{n-1}{2}}(\mathbb{S}^{n-1})}.\end{aligned}$$

  The norms above involve the conformal Laplacian. The above conservation laws only involves quadratic quantities, i.e., contribution only comes from linear terms!
- Rates of convergence to the equilibrium are given by following theorem.

Theorem (Rates of convergence to equilibrium)

Suppose $f_0 \geq 0$, $J[f_0] \neq 0$, and $f_0 \in H^s(\mathbb{S}^{n-1})$ for some $s \in \mathbb{R}$. Then there exists a unique global weak solution to the nonlinear Fokker-Planck equation, $f \in C^\infty((0,+\infty) \times \mathbb{S}^{n-1})$ and $f > 0$ all time $t > 0$;

- *For $d > \frac{1}{n}$, for all $t_0 > 0$, there is a constant C depending only on t_0, s, p, n, d, s.t.*

$$\|f(t) - 1\|_{H^p(\mathbb{S}^{n-1})} \leqslant C\|f_0\|_{H^s(\mathbb{S}^{n-1})}\, e^{-(n-1)(d-\frac{1}{n})t}, \quad t \geq t_0$$

- *For $d = \frac{1}{n}$, $r < \frac{2}{n(n-1)^{p-1}(n+2)}$, there existes t_0, s.t.*

$$\|f(t) - 1\|_{H^p(\mathbb{S}^{n-1})} \leqslant 1/\sqrt{rt}, \quad t > t_0$$

- *For $d < \frac{1}{n}$, $r < r_\infty(d)$, there existes t_0 and $\Omega \in \mathbb{S}^{n-1}$, s.t.*

$$\|f(t) - M_{\kappa\Omega}\|_{H^p(\mathbb{S}^{n-1})} \leqslant e^{-rt}, \quad t > t_0$$

  where rate function $r_\infty(d) \geqslant c(\frac{1}{n} - d)$ when d is close to $\frac{1}{n}$.

Thank You!