Low Mach number limit for the compressible viscous MHD equations

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The goal of Low Mach number limit:

derive incompressible (slightly compressible) models from compressible models when the Mach number goes to zero.

Mach number: $Ma = \dfrac{v}{c}$

v : relative velocity of the source to the medium
c : speed of sound in the medium

The isentropic compressible Euler equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P = 0. \end{cases}$$

$$P = a\rho^\gamma, \quad \gamma > 1.$$

Denote ϵ the Mach number, we introduce

$$\rho(x,t) = \rho^\epsilon(x, \epsilon t), \quad u(x,t) = \epsilon u^\epsilon(x, \epsilon t).$$

The original Euler equations become

$$\begin{cases} \partial_t \rho^\epsilon + \mathrm{div}(\rho^\epsilon u^\epsilon) = 0, \\ \partial_t(\rho^\epsilon u^\epsilon) + \mathrm{div}(\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \dfrac{a\nabla(\rho^\epsilon)^\gamma}{\epsilon^2} = 0. \end{cases}$$

Let $\epsilon \to 0^+$, we formally obtain $\rho^\epsilon(x,t) \to \rho^0(t)$.
The initial datum $\rho^\epsilon(x) \to \bar{\rho}_0 \Rightarrow \rho^\epsilon(t) \to \bar{\rho}_0$.
Taking $\bar{\rho}_0 \equiv 1 \Rightarrow \mathrm{div} v = 0$
(assume that $u^\epsilon \to v$ as $\epsilon \to 0^+$).

The limiting equations (incompressible Euler) read

$$\partial_t v + v \cdot \nabla v + \nabla \pi = 0, \qquad \mathrm{div} v = 0.$$

For well-prepared initial data

$$\left\| \left(\frac{\rho_0^\epsilon - \bar{\rho}_0}{\epsilon}, u_0^\epsilon - v_0(x) \right) \right\|_{H^s(\Omega)} = O(\epsilon)$$

$$\Longrightarrow \left\| \left(\frac{\rho^\epsilon - \bar{\rho}_0}{\epsilon} - \frac{\epsilon\pi}{\psi_0}, u^\epsilon - v \right)(t) \right\|_{H^s(\Omega)} \leq K\epsilon, \quad 0 \leq t < T^*$$

$$\Omega = \mathbb{T}^d \ \text{ or } \ \mathbb{R}^d, \quad \psi_0 = \sqrt{P'(\bar{\rho}_0)}, \quad s > \frac{d}{2} + 1.$$

T^*: the maximal existing time of the smooth solutions to the incompressible Euler equations.

In this talk:

The low Mach number limit to the full compressible magnetohydrodynamics equations

Model Magnetohydrodynamics (MHD) studies the dynamics of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields.

The applications of MHD cover a very wide range of physical objects: liquid metals, astrophysics, geophysics, plasma physics, cosmic plasmas, et. al.

Full MHD equations:

$$
\begin{aligned}
&\rho_t + \mathrm{div}(\rho \mathbf{u}) = 0,\\
&(\rho \mathbf{u})_t + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathrm{div}\Psi,\\
&\mathcal{E}_t + \mathrm{div}\big(\mathbf{u}(\mathcal{E}' + P)\big)\\
&\qquad\qquad = \mathrm{div}\big((\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + \nu \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{u}\Psi + \kappa \nabla \theta\big),\\
&\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \qquad \mathrm{div}\mathbf{H} = 0.
\end{aligned}
$$

$\rho \geq 0$: the density

$\mathbf{u} \in \mathbb{R}^3$: the velocity

$\mathbf{H} \in \mathbb{R}^3$: the magnetic field

θ: the temperature

Ψ: the viscous stress tensor given by

$$\Psi = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) + \lambda \,\mathrm{div}\mathbf{u}\, \mathbf{I}_d$$

$\mathcal{E}$: the total energy given by

$$\mathcal{E} = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{H}|^2 \text{ and } \mathcal{E}' = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right)$$

e: the internal energy

$\frac{1}{2}\rho|\mathbf{u}|^2$: the kinetic energy

$\frac{1}{2}|\mathbf{H}|^2$: the magnetic energy

$P = P(\rho, \theta)$, $e = e(\rho, \theta)$ satisfy the equations of state

$$P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}$$

$\mathbf{I}$: the 3×3 identity matrix

$\nabla \mathbf{u}^{\mathrm{T}}$: the transpose of the matrix $\nabla \mathbf{u}$

λ, μ: the viscosity coefficients of the flow satisfying

$$2\mu + 3\lambda > 0, \quad \mu > 0$$

$\nu > 0$: the magnetic diffusivity

$\kappa > 0$: the heat conductivity

The compressible MHD equations can be derived from the complete equations describing an electromagnetic dynamics

[compressible Navier–Stokes system coupled with Maxwell system]

as the dielectric constant tends to zero.

This is the so-called magnetohydrodynamic approximation.

Remark: Although the electric field $\mathbf{E}$ does not appear in the MHD equations, it is indeed induced according to the following relation

$$\mathbf{E} = \nu \nabla \times \mathbf{H} - \mathbf{u} \times \mathbf{H}$$

by the moving conductive flow in the magnetic field.

Two categories on studying the low Mach number limit to the full compressible MHD equations

1. Small variations on density and temperature

2. Large variations on density and temperature

Here we study the low Mach number limit to the full compressible MHD equations in the framework of local smooth solutions and consider the three-dimensional case only.

Remark:

1. For the low Mach number limit to the full compressible MHD equations in the framework of weak solutions, see :

P. Kukučka, J. Math. Fluid Mech. (2011); A. Novotny, et. al., M3AS (2011); Y.-S. Kwon, K. Trivisa, JDE (2011).

2. For low Mach number limit to the isentropic MHD equations, see:

Hu-Wang (SIAM JMA 2009), Jiang-Ju-L (SIAM JMA 2010, CMP 2010).

CASE I: Small variations on density and temperature

We shall focus our efforts on the ionized fluid obeying the perfect gas relations

$$P = \mathfrak{R}\rho\theta, \quad e = c_V\theta, \tag{1}$$

$\mathfrak{R} > 0$: the gas constant

$c_V > 0$: the heat capacity at constant volume

We rewrite the full MHD equations as follows

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{2}$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \frac{\nabla(\rho\theta)}{\epsilon^2} = (\nabla \times \mathbf{H}) \times \mathbf{H} + \mathrm{div}\Psi, \tag{3}$$

$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \quad \mathrm{div}\mathbf{H} = 0, \tag{4}$$

$$\begin{aligned} &\rho(\partial_t \theta + \mathbf{u} \cdot \nabla \theta) + (\gamma - 1)\rho\theta \mathrm{div}\mathbf{u} \\ &\qquad = \epsilon^2 \nu |\nabla \times \mathbf{H}|^2 + \epsilon^2 \Psi : \nabla \mathbf{u} + \kappa \Delta \theta, \end{aligned} \tag{5}$$

ϵ: the Mach number

$\mu, \lambda, \nu, \kappa$: the scaled parameters

$\gamma = 1 + \mathfrak{R}/c_V$: the ratio of specific heats

We further restrict ourselves to the small density and temperature variations, i.e.

$$\rho = 1 + \epsilon q^\epsilon, \quad \theta = 1 + \epsilon \phi^\epsilon, \quad \mathbf{u} = \mathbf{u}^\epsilon, \quad \mathbf{H} = \mathbf{H}^\epsilon. \tag{6}$$

Putting (6) and (1) into the system (2)–(5), and using the identities

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = \nabla \operatorname{div}\mathbf{H} - \Delta\mathbf{H},$$
$$\nabla(|\mathbf{H}|^2) = 2\mathbf{H}\cdot\nabla\mathbf{H} + 2\mathbf{H}\times\operatorname{curl}\mathbf{H},$$
$$\operatorname{curl}(\mathbf{u}\times\mathbf{H}) = \mathbf{u}(\operatorname{div}\mathbf{H}) - \mathbf{H}(\operatorname{div}\mathbf{u}) + \mathbf{H}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{H},$$

we can rewrite (2)–(5) as

$$\partial_t q^\epsilon + \mathbf{u}^\epsilon \cdot \nabla q^\epsilon + \frac{1}{\epsilon}(1+\epsilon q^\epsilon)\mathrm{div}\mathbf{u}^\epsilon = 0, \tag{7}$$

$$(1+\epsilon q^\epsilon)(\partial_t \mathbf{u}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon) + \frac{1}{\epsilon}\big[(1+\epsilon q^\epsilon)\nabla\phi^\epsilon + (1+\epsilon\phi^\epsilon)\nabla q^\epsilon\big]$$
$$- \mathbf{H}^\epsilon \cdot \nabla \mathbf{H}^\epsilon + \frac{1}{2}\nabla(|\mathbf{H}^\epsilon|^2) = 2\mu \mathrm{div}(\mathbb{D}(\mathbf{u}^\epsilon)) + \lambda\nabla(\mathrm{tr}\mathbb{D}(\mathbf{u}^\epsilon)), \tag{8}$$

$$(1+\epsilon q^\epsilon)(\partial_t \phi^\epsilon + \mathbf{u}^\epsilon \cdot \nabla\phi^\epsilon) + \frac{\gamma-1}{\epsilon}(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)\mathrm{div}\mathbf{u}^\epsilon$$
$$= \kappa\Delta\phi^\epsilon + \epsilon\{2\mu|\mathbb{D}(\mathbf{u}^\epsilon)|^2 + \lambda(\mathrm{tr}\mathbb{D}(\mathbf{u}^\epsilon))^2\} + \epsilon\nu|\nabla \times \mathbf{H}^\epsilon|^2, \tag{9}$$

$$\partial_t \mathbf{H}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{H}^\epsilon + \mathrm{div}\mathbf{u}^\epsilon \mathbf{H}^\epsilon - \mathbf{H}^\epsilon \cdot \nabla \mathbf{u}^\epsilon = \nu\Delta\mathbf{H}^\epsilon, \quad \mathrm{div}\mathbf{H}^\epsilon = 0. \tag{10}$$

Therefore, the formal limit as $\epsilon \to 0^+$ of (7)–(10) is the following incompressible MHD equations

(suppose that the limits $\mathbf{u}^\epsilon \to \mathbf{w}$ and $\mathbf{H}^\epsilon \to \mathbf{B}$ exist.)

$$\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \pi + \frac{1}{2} \nabla (|\mathbf{B}|^2) - \mathbf{B} \cdot \nabla \mathbf{B} = \mu \Delta \mathbf{w}, \tag{11}$$

$$\partial_t \mathbf{B} + \mathbf{w} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{w} = \nu \Delta \mathbf{B}, \tag{12}$$

$$\mathrm{div} \mathbf{w} = 0, \quad \mathrm{div} \mathbf{B} = 0. \tag{13}$$

Consider the system (7)-(10) in the Torus $\mathbb{T}^3$ or the whole space $\mathbb{R}^3$.

Proposition (Local existence of the limiting system, Duvaut-Lions (1972), Sermange-Temam (1983))

Let $s > 3/2 + 2$*. The initial data* $(\mathbf{w}, \mathbf{B})|_{t=0} = (\mathbf{w}_0, \mathbf{B}_0)$ *satisfy*

$$\mathbf{w}_0 \in H^s, \mathbf{B}_0 \in H^s, \operatorname{div} \mathbf{w}_0 = 0, \operatorname{div} \mathbf{B}_0 = 0.$$

Then, there exist a $\hat{T}^* \in (0, \infty]$ *and a unique solution* $(\mathbf{w}, \mathbf{B}) \in L^\infty(0, \hat{T}^*; H^s)$ *to* (11)–(13) *satisfying*

$$\operatorname{div} \mathbf{w} = 0 \text{ and } \operatorname{div} \mathbf{B} = 0, \text{ and for any } 0 < T < \hat{T}^*,$$

$$\sup_{0 \le t \le T} \left\{ ||(\mathbf{w}, \mathbf{B})(t)||_{H^s} + ||(\partial_t \mathbf{w}, \partial_t \mathbf{B})(t)||_{H^{s-2}} + ||\nabla \pi(t)||_{H^{s-2}} \right\} \le C.$$

Theorem (Jiang-Ju-L, Nonlinearity(2012))

Let $s > 3/2 + 2$. *Suppose that* $(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{H}_0^\epsilon(x), \phi_0^\epsilon(x)))$ *satisfy*

$$\|q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x) - \mathbf{w}_0(x), \mathbf{H}_0^\epsilon(x) - \mathbf{B}_0(x), \phi_0^\epsilon(x)\|_{H^s} = O(\epsilon).$$

Let $(\mathbf{w}, \mathbf{B}, \pi)$ *be a smooth solution to* (11)–(13) *obtained in the above proposition satisfying*

$(\mathbf{w}, \pi) \in C([0, T^*], H^{s+2}) \cap C^1([0, T^*], H^s)$, $T^* > 0$ *finite.*

Then $\exists\ \epsilon_0 > 0$, *for all* $\epsilon \leq \epsilon_0$, *the system* (7)-(10) *with initial data* $(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{H}_0^\epsilon(x), \phi_0^\epsilon(x)))$ *has a unique smooth solution* $(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \phi^\epsilon) \in C([0, T^*], H^s)$.

Moreover, $\exists\ K > 0$, *independent of* ϵ, *for all* $\epsilon \leq \epsilon_0$,

$$\sup_{t\in[0,T^*]} \left\| \left\{ (q^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \phi^\epsilon) - \left(\frac{\epsilon}{2}\pi, \mathbf{w}, \mathbf{B}, \frac{\epsilon}{2}\pi\right) \right\}(t) \right\|_{H^s} \leq K\epsilon.$$

Remark: From Theorem above, we know that for sufficiently small ϵ and well-prepared initial data, the full MHD equations (2)–(5) admits a unique smooth solution on the same time interval where the smooth solution of the incompressible MHD equations exists.

Remark: The KEY points in the proof:

energy estimates + compact arguments + convergence-stability lemma.

Remark: The approach is still valid for the ideal non-isentropic compressible MHD equations.

CASE II: Large variations on density and temperature

Full MHD equations:

$$
\begin{aligned}
&\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0,\\
&(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = (\nabla \times \mathbf{H}) \times \mathbf{H} + \operatorname{div}\Psi,\\
&\mathcal{E}_t + \operatorname{div}\big(\mathbf{u}(\mathcal{E}' + p)\big)\\
&\qquad\qquad = \operatorname{div}\big((\mathbf{u} \times \mathbf{H}) \times \mathbf{H} + \nu \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{u}\Psi + \kappa \nabla\theta\big),\\
&\mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nu \nabla \times \mathbf{H}), \qquad \operatorname{div}\mathbf{H} = 0.
\end{aligned}
$$

with

$$\Psi = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) + \lambda \operatorname{div}\mathbf{u}\, \mathbf{I}_3,$$

$$\mathcal{E} = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) + \frac{1}{2}|\mathbf{H}|^2 \text{ and } \mathcal{E}' = \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right).$$

As before, we shall focus our efforts on the ionized fluid obeying the perfect gas relations

$$p = \mathfrak{R}\rho\theta, \quad e = c_V\theta, \tag{14}$$

$\mathfrak{R} > 0$: the gas constant

$c_V > 0$: the heat capacity at constant volume

Let ϵ be the Mach number.

Consider the full MHD system in the physical regime:

$$P \sim P_0 + O(\epsilon), \quad \mathbf{u} \sim O(\epsilon), \quad \mathbf{H} \sim O(\epsilon), \quad \nabla\theta \sim O(1),$$

where $P_0 > 0$ is a certain given constant which is normalized to be $P_0 = 1$.

Consider the case the pressure P is a small perturbation of the given constant state 1 while the temperature θ has a finite variation.

We introduce the following transformation to ensure the positivity of P and θ

$$P(x,t) = e^{\epsilon p^{\epsilon}(x,\epsilon t)}, \quad \theta(x,t) = e^{\theta^{\epsilon}(x,\epsilon t)}. \tag{15}$$

Note that (14) and (15) imply that

$$\rho(x,t) = e^{\epsilon p^{\epsilon}(x,\epsilon t) - \theta^{\epsilon}(x,\epsilon t)}$$

since we take $\mathfrak{R} \equiv c_V \equiv 1$.

Set

$$\mathbf{H}(x,t) = \epsilon \mathbf{H}^{\epsilon}(x, \epsilon t), \quad \mathbf{u}(x,t) = \epsilon \mathbf{u}^{\epsilon}(x, \epsilon t), \tag{16}$$

and

$$\mu = \epsilon \mu^{\epsilon}, \quad \lambda = \epsilon \lambda^{\epsilon}, \quad \nu = \epsilon \nu^{\epsilon}, \quad \kappa = \epsilon \kappa^{\epsilon}. \tag{17}$$

The MHD system with (14) takes the following equivalent form:

$$
\begin{aligned}
&\partial_t p^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) p^\epsilon + \frac{1}{\epsilon} \mathrm{div}(2\mathbf{u}^\epsilon - \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla \theta^\epsilon) \\
&\quad = \epsilon e^{-\epsilon p^\epsilon} [\nu^\epsilon |\mathrm{curl}\, \mathbf{H}^\epsilon|^2 + \Psi(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon + \theta^\epsilon} \nabla p^\epsilon \cdot \nabla \theta^\epsilon, \quad (18) \\
&e^{-\theta^\epsilon} [\partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon] + \frac{\nabla p^\epsilon}{\epsilon} \\
&\quad = e^{-\epsilon p^\epsilon} [(\mathrm{curl}\, \mathbf{H}^\epsilon) \times \mathbf{H}^\epsilon + \mathrm{div} \Psi^\epsilon (\mathbf{u}^\epsilon)], \quad (19) \\
&\partial_t \mathbf{H}^\epsilon - \mathrm{curl}\, (\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) - \nu^\epsilon \Delta \mathbf{H}^\epsilon = 0, \quad \mathrm{div} \mathbf{H}^\epsilon = 0, \quad (20) \\
&\partial_t \theta^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \theta^\epsilon + \mathrm{div} \mathbf{u}^\epsilon \\
&\quad = \epsilon^2 e^{-\epsilon p^\epsilon} [\nu^\epsilon |\mathrm{curl}\, \mathbf{H}^\epsilon|^2 + \Psi^\epsilon(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon] + \kappa^\epsilon e^{-\epsilon p^\epsilon} \mathrm{div}(e^{\theta^\epsilon} \nabla \theta^\epsilon), \quad (21)
\end{aligned}
$$

with

$$
\Psi^\epsilon(\mathbf{u}^\epsilon) = 2\mu^\epsilon \mathbb{D}(\mathbf{u}^\epsilon) + \lambda^\epsilon \mathrm{div} \mathbf{u}^\epsilon\, \mathbf{I}_3,
$$

$$
\Psi(\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon = 2\mu^\epsilon |\mathbb{D}(\mathbf{u}^\epsilon)|^2 + \lambda^\epsilon |\mathrm{tr} \mathbb{D}(\mathbf{u}^\epsilon)|^2.
$$

Formally, as ϵ goes to zero, suppose that

$$(\mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon) \to (\mathbf{w}, \mathbf{B}, \vartheta)$$

in some sense, and

$$(\mu^\epsilon, \lambda^\epsilon, \nu^\epsilon, \kappa^\epsilon) \to (\bar{\mu}, \bar{\lambda}, \bar{\nu}, \bar{\kappa}),$$

then taking the limit to (18)–(21), we have

$$\operatorname{div}(2\mathbf{w} - \bar{\kappa}\, e^{\vartheta} \nabla\vartheta) = 0, \tag{22}$$

$$e^{-\vartheta}[\partial_t \mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w}] + \nabla\pi = (\operatorname{curl}\mathbf{B}) \times \mathbf{B} + \operatorname{div}\Phi(\mathbf{w}), \tag{23}$$

$$\partial_t \mathbf{B} - \operatorname{curl}(\mathbf{w} \times \mathbf{B}) - \bar{\nu}\Delta\mathbf{B} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{24}$$

$$\partial_t \vartheta + (\mathbf{w}\cdot\nabla)\vartheta + \operatorname{div}\mathbf{w} = \bar{\kappa}\operatorname{div}(e^{\vartheta}\nabla\vartheta), \tag{25}$$

with some function π, where $\Phi(\mathbf{w}) = 2\bar{\mu}\mathbb{D}(\mathbf{w}) + \bar{\lambda}\operatorname{div}\mathbf{w}\,\mathbf{I}_3$.

We supplement the system (18)–(21) with the following initial conditions

$$(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)|_{t=0} = (p^\epsilon_{\rm in}(x), \mathbf{u}^\epsilon_{\rm in}(x), \mathbf{H}^\epsilon_{\rm in}(x), \theta^\epsilon_{\rm in}(x)), \quad x \in \mathbb{R}^3. \quad (26)$$

For simplicity, we also assume that

$$\mu^\epsilon \equiv \bar{\mu} > 0, \quad \nu^\epsilon \equiv \bar{\nu} > 0, \quad \kappa^\epsilon \equiv \bar{\kappa} > 0, \quad \lambda^\epsilon \equiv \bar{\lambda}.$$

We denote

$$\|v\|_{H^\sigma_\eta} := \|v\|_{H^{\sigma-1}} + \eta\|v\|_{H^\sigma}$$

for any $\sigma \in \mathbb{R}$ and $\eta \geq 0$.

For each $\epsilon > 0$, $t \geq 0$ and $s \geq 0$, we will also use the following norm:

$$\begin{aligned}
&\|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(t)\|_{s,\epsilon} \\
&\quad := \sup_{\tau\in[0,t]} \left\{ \|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)(\tau)\|_{H^s} + \|(\epsilon p^\epsilon, \epsilon\mathbf{u}^\epsilon, \epsilon\mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(\tau)\|_{H^{s+2}_\epsilon} \right\} \\
&\qquad + \left\{ \int_0^t [\|\nabla(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon)\|^2_{H^s} + \|\nabla(\epsilon\mathbf{u}^\epsilon, \epsilon H^\epsilon, \theta^\epsilon)\|^2_{H^{s+2}_\epsilon}](\tau)d\tau \right\}^{1/2}.
\end{aligned}$$

Theorem (Uniform solutions, (Jiang-Ju-L-Xin,2011))

Let $s \geq 4$. Assume that the initial data $(p^\epsilon_{\rm in}, \mathbf{u}^\epsilon_{\rm in}, \mathbf{H}^\epsilon_{\rm in}, \theta^\epsilon_{\rm in})$ satisfy

$$\|(p^\epsilon_{\rm in}, \mathbf{u}^\epsilon_{\rm in}, \mathbf{H}^\epsilon_{\rm in}, \theta^\epsilon_{\rm in} - \bar{\theta})(t)\|_{s,\epsilon} \leq L_0 \tag{27}$$

for all $\epsilon \in (0,1]$ and two given positive constants $\bar{\theta}$ and L_0.

Then there exist positive constants T_0 and $\epsilon_0 < 1$, depending only on L_0 and $\bar{\theta}$, such that the Cauchy problem (18)–(21), (26) *has a unique solution $(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon)$ satisfying*

$$\|(p^\epsilon, \mathbf{u}^\epsilon, \mathbf{H}^\epsilon, \theta^\epsilon - \bar{\theta})(t)\|_{s,\epsilon} \leq L, \qquad \forall\, t \in [0, T_0],\ \forall\, \epsilon \in (0, \epsilon_0], \tag{28}$$

where L depends only on L_0, $\bar{\theta}$ and T_0.

Theorem (Convergence results,(Jiang-Ju-L-Xin,2011))

Assume further that the initial data satisfy the following conditions

$$|\theta_0^\epsilon(x)-\bar{\theta}|\le N_0|x|^{-1-\zeta},\quad |\nabla\theta_0^\epsilon(x)|\le N_0|x|^{-2-\zeta},\quad \forall\epsilon\in(0,1], \tag{29}$$

$$\big(p_{\rm in}^\epsilon, {\rm curl}\,(e^{-\theta_{\rm in}^\epsilon}\mathbf{u}_{\rm in}^\epsilon), \mathbf{H}_{\rm in}^\epsilon, \theta_{\rm in}^\epsilon\big)\to(0,\mathbf{w}_0,\mathbf{B}_0,\vartheta_0) \tag{30}$$

in $H^s(\mathbb{R}^3)$ *as* $\epsilon\to 0^+$, *where* N_0 *and* ζ *are fixed positive constants.*

Then the solution sequence $(p^\epsilon,\mathbf{u}^\epsilon,\mathbf{H}^\epsilon,\,\theta^\epsilon)$

converges weakly in $L^\infty(0,T_0;H^s(\mathbb{R}^3))$

and

strongly in $L^2(0,T_0;\,H^{s_2}_{\rm loc}(\mathbb{R}^3))$ *for all* $0\le s_2<s$

to the limit $(0,\mathbf{w},\mathbf{B},\vartheta)$,

where $(\mathbf{w},\mathbf{B},\vartheta)$ *satisfies the system* (22)–(25) *with initial data* $(\mathbf{w},\mathbf{B},\vartheta)|_{t=0}=(\mathbf{w}_0,\mathbf{B}_0,\vartheta_0)$.

Two parts in the proofs:

1. Uniformly bounded estimates with respect to the parameter ϵ of the original system;
   - H^s estimates on $(\mathbf{H}^\epsilon, \theta^\epsilon)$ and $(\epsilon p^\epsilon, \epsilon \mathbf{u}^\epsilon)$
   - H^{s+1} estimates on $(\epsilon \mathbf{u}^\epsilon, \epsilon p^\epsilon, \epsilon \mathbf{H}^\epsilon, \theta^\epsilon)$
   - H^{s-1} estimates on $(\mathrm{div} \mathbf{u}^\epsilon, \nabla p^\epsilon)$
   - H^{s-1} estimate on $\mathrm{curl}\, \mathbf{u}^\epsilon$
2. Derivation of the limiting equations form the obtained estimates and the description of the oscillations in ill-prepared initial data case. (applying a result of G. Métivier $\&$ S. Schochet,(ARMA,2001))

Main features:

- The propagation of oscillations is described by the wave equations with unknown variable coefficients
- Strong coupling of the hydrodynamic motion and the magnetic fields

Key points of the proofs:

- The effect of fluid and magnetic diffusions, and heat conductivity
- Div-Curl decomposition of the velocity
- Refined energy estimates
- Weak compact arguments
- Detailed analysis of the oscillation equations

Here we use some ideas from: Métivier-Schochet (ARMA,2001), Alazard(ARMA,2006), Levermore-Sun-Trivisa (SIAM JMA, 2012).

Remark: The fluid diffusion term, heat conductivity term, and magnetic diffusivity term in the system (18)-(21) play a crucial role in our uniformly bounded estimates.

(in order to control some undesirable higher-order terms)

Remark: For the two cases–

- non-isentropic MHD equations with zero magnetic diffusivity
- non-isentropic MHD equations with zero fluid diffusion and heat conductivity coefficients

Some new ideas are needed to deal with the low Mach number limit, see Jiang-Ju-L (arXiv:1111.2926v1).

Summary

Low Mach number limit to the full MHD equations in two cases:

1. Small variations on density and temperature: well-prepared initial data in the torus or the whole space
2. Large variations on density and temperature: ill-prepared initial data in the whole space

Thank You !