# Relative entropy in diffusive relaxation

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14th International Conference on Hyperbolic Problems:
Theory, Numerics, Applications
June 25 – 29, 2012, Padova, Italy

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# Outline

# Main motivation for relaxation limits

Hydrodynamic limit for the Boltzmann equation:

$$\nu f_t + \xi \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \tag{1}$$

$\nu$: Mach number and $\varepsilon$: Knudsen number
if $\nu = \varepsilon$

$$(1) \longrightarrow \text{Navier–Stokes equations as } \varepsilon \downarrow 0$$

Simplest discrete velocity model: Carleman's equations

$$\begin{cases} \partial_t f_1 + \frac{1}{\varepsilon} \partial_x f_1 = \frac{1}{\varepsilon^2}(f_2^2 - f_1^2) \\ \partial_t f_2 - \frac{1}{\varepsilon} \partial_x f_2 = \frac{1}{\varepsilon^2}(f_1^2 - f_2^2) \end{cases}$$

$\rho = f_1 + f_2$ as $\varepsilon \downarrow 0$ satisfies $\rho_t = \frac{1}{2}(\log(\rho))_{xx}$

# Toy model (linear Cattaneo)

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = 0 \\ v_t^\varepsilon + c^2 u_x^\varepsilon = -\dfrac{1}{\varepsilon} v^\varepsilon \end{cases}$$

Time scaling: $\partial_t \longrightarrow \varepsilon \partial_t$

$$\begin{cases} u_t^\varepsilon + \dfrac{1}{\varepsilon} v_x^\varepsilon = 0 \\ v_t^\varepsilon + \dfrac{c^2}{\varepsilon} u_x^\varepsilon = -\dfrac{1}{\varepsilon^2} v^\varepsilon \end{cases}$$

Flux scaling: $v^\varepsilon \longrightarrow \varepsilon v^\varepsilon$

$$\begin{cases} u_t^\varepsilon + v_x^\varepsilon = 0 \\ \varepsilon^2 v_t^\varepsilon + c^2 u_x^\varepsilon = -v^\varepsilon \end{cases}$$

Formal limit: $u_t - c^2 u_{xx} = 0$

# References for diffusive relaxation

Kurtz, *Trans. AMS* ‘73; McKean, *Israel J. Math.* ‘75
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# Relative entropy[3]

$U$ (weak, entropy) solution and $\bar{U}$ *smooth solution* of systems of conservation (or balance) laws and $(\eta, q_\alpha)$ a convex entropy–entropy flux pair

Compute $\eta(U|\bar{U})_t + \sum_\alpha \partial_\alpha q_\alpha(U|\bar{U})$ for

$$\eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (U - \bar{U})$$
$$q_\alpha(U|\bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (F_\alpha(U) - F_\alpha(\bar{U}))$$

This shall lead to a *stability estimate*, used in many different contexts. Recently:

- Hyperbolic relaxation: L., Tzavaras *ARMA* '06; Tzavaras *Commun. Math. Sci.* '05; Berthelin, Vasseur *SIMA* '05; Berthelin, Tzavaras, Vasseur *J. Stat. Physics* '09;
- Weak–strong uniqueness: Demoulini, Stuart, Tzavaras *ARMA*; Feireisl, Novotny *ARMA* '12

[3] Dafermos, *ARMA '79*; DiPerna, *Indiana U. Math. J.* '79

# The model

Isentropic gas dynamics in three space dimensions with a damping term:

$$\begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m, \end{cases} \tag{2}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^3$, density $\rho \geq 0$ and momentum flux $m \in \mathbb{R}^3$. The pressure $p(\rho)$ satisfies $p'(\rho) > 0$ which makes the system hyperbolic. $\gamma$–law: $p(\rho) = k\rho^\gamma$ with $\gamma \geq 1$ and $k > 0$. In the diffusive relaxation limit $\varepsilon \downarrow 0$, solutions of (2) formally converge to those of the porous medium equation

$$\bar{\rho}_t - \triangle_x p(\bar{\rho}) = 0$$

# Hilbert's expansion/1

We now use the standard Hilbert's expansion

$$\rho = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots ,$$
$$m = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots ,$$

into (2) and into

$$\eta(\rho, m)_t + \frac{1}{\varepsilon} \operatorname{div}_x q(\rho, m) = -\frac{1}{\varepsilon^2} \nabla_m \eta(\rho, m) \cdot m = -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} \leq 0$$

for the mechanical energy $\eta(\rho, m) = \frac{1}{2} \frac{|m|^2}{\rho} + h(\rho)$

and the associated flux $q(\rho, m) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + m h'(\rho)$

$h''(\rho) = \frac{p'(\rho)}{\rho}$; $\rho h'(\rho) = p(\rho) + h(\rho)$

# Hilbert's expansion/2

From the equations we get:

$$O(\varepsilon^{-1}) \ \operatorname{div}_x m_0 = 0$$
$$O(1) \ \partial_t \rho_0 + \operatorname{div}_x m_1 = 0$$
$$O(\varepsilon) \ \partial_t \rho_1 + \operatorname{div}_x m_2 = 0$$

$$O(\varepsilon^{-2}) \ m_0 = 0$$
$$O(\varepsilon^{-1}) \ -m_1 = \nabla_x p(\rho_0)$$
$$O(1) \ -m_2 = \nabla_x (p'(\rho_0)\rho_1)$$
$$O(\varepsilon) \ -m_3 = \partial_t m_1 + \operatorname{div}_x \left( \frac{m_1 \otimes m_1}{\rho_0} \right) + \nabla_x \Big( p'(\rho_0)\rho_2 + \frac{1}{2} p''(\rho_0)\rho_1^2 \Big)$$

In particular, we recover the equilibrium relation $m_0 = 0$ for the state variables, the Darcy's law $m_1 = -\nabla_x p(\rho_0)$, and that $\rho_0$ satisfies porous medium equation

# Hilbert's expansion/3

From the entropy we get:

$$O(1)\ h(\rho_0)_t + \mathrm{div}_x \left( m_1 h'(\rho_0) \right) = -\frac{|m_1|^2}{\rho_0}$$

$$O(\varepsilon)\ \partial_t \left( h'(\rho_0)\rho_1 \right) + \mathrm{div}_x \left( m_2 h'(\rho_0) + m_1 h''(\rho_0)\rho_1 \right) = |m_1|^2 \frac{\rho_1}{\rho_0^2} - 2\frac{m_1 \cdot m_2}{\rho_0}$$

Thus we recover the entropy dissipation relation associated to the porous medium equation for $\rho_0$

$$h(\rho)_t - \mathrm{div}_x \left( h'(\rho) \nabla_x p(\rho) \right) = -\frac{|\nabla_x p(\rho)|^2}{\rho}$$

# Reformulation of the limiting equation

We rewrite $\bar{\rho}_t - \triangle_x p(\bar{\rho}) = 0$ as follows

$$\begin{aligned} \bar{\rho}_t + \frac{1}{\varepsilon}\partial_{x_i}\bar{m}_i &= 0 \\ \bar{m}_t + \frac{1}{\varepsilon}\partial_{x_i} f_i(\bar{\rho}, \bar{m}) &= -\frac{1}{\varepsilon^2}\bar{m} + e(\bar{\rho}, \bar{m}) \end{aligned} \tag{3}$$

for $(\bar{\rho}, \bar{m} = -\varepsilon \nabla_x p(\bar{\rho}))$ and the *error term*

$$\begin{aligned} \bar{e} := e(\bar{\rho}, \bar{m}) &= \frac{1}{\varepsilon}\,\mathrm{div}_x\left(\frac{\bar{m}\otimes\bar{m}}{\bar{\rho}}\right) - \varepsilon\partial_t\nabla_x p(\bar{\rho}) \\ &= \varepsilon\,\mathrm{div}_x\left(\frac{\nabla_x p(\bar{\rho})\otimes\nabla_x p(\bar{\rho})}{\bar{\rho}}\right) - \varepsilon\nabla_x\left(p'(\bar{\rho})\triangle_x p(\bar{\rho})\right) \\ &= O(\varepsilon) \end{aligned}$$

# Relative entropy/1

The relative entropy is of the form

$$
\begin{aligned}
\eta(\rho, m \,|\, \bar\rho, \bar m) &:= \eta(\rho, m) - \eta(\bar\rho, \bar m) - \eta_\rho(\bar\rho, \bar m)(\rho - \bar\rho) \\
&\qquad - \nabla_m \eta(\bar\rho, \bar m) \cdot (m - \bar m) \\
&= \frac{1}{2}\rho \left| \frac{m}{\rho} - \frac{\bar m}{\bar\rho} \right|^2 + h(\rho \,|\, \bar\rho)
\end{aligned}
$$

while the corresponding relative entropy-flux reads

$$
\begin{aligned}
q_i(\rho, m \,|\, \bar\rho, \bar m) &:= q_i(\rho, m) - q_i(\bar\rho, \bar m) - \eta_\rho(\bar\rho, \bar m)(m_i - \bar m_i) \\
&\qquad - \nabla_m \eta(\bar\rho, \bar m) \cdot (f_i(\rho, m) - f_i(\bar\rho, \bar m)) \\
&= \frac{1}{2} m_i \left| \frac{m}{\rho} - \frac{\bar m}{\bar\rho} \right|^2 + \rho (h'(\rho) - h'(\bar\rho)) \left( \frac{m_i}{\rho} - \frac{\bar m_i}{\bar\rho} \right) + \frac{\bar m_i}{\bar\rho} h(\rho \,|\, \bar\rho)
\end{aligned}
$$

# Relative entropy/2

## Proposition

*Let $(\rho, m)$ be a weak entropy solution of (2) and let $(\bar\rho, \bar m)$ be a smooth solution of (3). Then,*

$$\eta(\rho, m\,|\,\bar\rho, \bar m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho, m\,|\,\bar\rho, \bar m) \le -\frac{1}{\varepsilon^2}R(\rho, m\,|\,\bar\rho, \bar m) - Q - E,$$

*where*

$$Q = \frac{1}{\varepsilon}\nabla^2_{(\rho,m)}\eta(\bar\rho, \bar m)\begin{pmatrix}\bar\rho_{x_i}\\ \bar m_{x_i}\end{pmatrix}\cdot\begin{pmatrix}0\\ f_i(\rho, m|\bar\rho, \bar m)\end{pmatrix}$$
$$= -\sum_{i,j}\partial_{x_i x_j}h'(\bar\rho)f_{ij}(\rho, m|\bar\rho, \bar m)$$

$$R(\rho, m\,|\,\bar\rho, \bar m) = \rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 \qquad E = e(\bar\rho, \bar m)\cdot\frac{\rho}{\bar\rho}\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)$$

# Control of the quadratic term $Q$

## Lemma

*If* $p(\rho)$ *satisfies* $p''(\rho) \leq A\frac{p'(\rho)}{\rho}\ \forall\ \rho > 0$ *for some* $A > 0$*, then* $h(\rho)$ *verifies*

$$p(\rho\,|\bar{\rho}) \leq ch(\rho\,|\bar{\rho}) \qquad \forall\ \rho\,,\ \bar{\rho} > 0$$

*for a given constant* $c > 0$*. Moreover, there exists a* $C > 0$ *such that for any fixed* $i$

$$|f_i(\rho, m\,|\bar{\rho}, \bar{m})| \leq C\eta(\rho, m\,|\bar{\rho}, \bar{m})$$

$$f_{ij}(\rho, m\,|\bar{\rho}, \bar{m}) = \rho\left(\frac{m_i}{\rho} - \frac{\bar{m}_i}{\bar{\rho}}\right)\left(\frac{m_j}{\rho} - \frac{\bar{m}_j}{\bar{\rho}}\right) + p(\rho\,|\bar{\rho})\delta_{ij}$$

Remark: For a $\gamma$–law gases:
$\gamma > 1,\ h(\rho) = \frac{1}{\gamma-1}p(\rho);\ \gamma = 1,\ p(\rho\,|\bar{\rho}) = 0$

# Possible framework

We assume

**(H₁)** $\bar{\rho}$ is a smooth, positive solution of the multidimensional porous medium equation

$(\rho, m)$ be a weak solution of (2) such that $\rho \geq 0$, satisfying the entropy inequality, $\rho - \bar{\rho} \in L^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \eta(\rho, m \,|\, \bar{\rho}, \bar{m}) \Big|_{t=0} dx < +\infty$$

and

$$q(\rho, m \,|\, \bar{\rho}, \bar{m}) \to 0, \text{ as } |x| \to +\infty$$

the pressure $p(\rho)$ satisfies $p''(\rho) \leq A \frac{p'(\rho)}{\rho} \ \forall \ \rho > 0$ ; for instance, $p(\rho) = \rho^\gamma$, $\gamma \geq 1$

# Stability and convergence/1

We denote by

$$\varphi(t) = \int_{\mathbb{R}^3} \eta(\rho, m \,|\, \bar{\rho}, \bar{m})dx$$

## Theorem

*Let $T > 0$ be fixed. Under hypothesis $(\mathbf{H_1})$, the following stability estimate holds:*

$$\varphi(t) \leq C(\varphi(0) + \varepsilon^4), \quad t \in [0, T]\,,$$

*where $C$ is a positive constant depending only on $\|\rho_0 - \bar{\rho}_0\|_{L^1(\mathbb{R}^3)}$, $T$, $\bar{\rho}$ and its derivatives.*
*Moreover, if $\varphi(0) \to 0$ as $\varepsilon \downarrow 0$, then*

$$\sup_{t\in[0,T]} \varphi(t) \to 0, \ \ as \ \varepsilon \downarrow 0$$

# Stability and convergence/2

Proof.

$$\varphi(t)+\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^3} R(\rho, m\,|\,\bar{\rho}, \bar{m})\,d\tau dx \le \varphi(0)+\int_0^t\int_{\mathbb{R}^3}(|Q|+|E|)\,d\tau dx$$

$$\int_0^t\int_{\mathbb{R}^3}|Q|\,d\tau dx \le C_0\int_0^t \varphi(\tau)\,d\tau$$

$$\int_0^t\int_{\mathbb{R}^3}|E|\,d\tau dx \le \frac{\varepsilon^2}{2}\int_0^t\int_{\mathbb{R}^3}\left|\frac{e(\bar{\rho},\bar{m})}{\bar{\rho}}\right|^2\rho\,d\tau dx + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{R}^3}\rho\left|\frac{m}{\rho}-\frac{\bar{m}}{\bar{\rho}}\right|^2 d\tau dx$$

$$\le \varepsilon^4 t\big(C_1\|\rho-\bar{\rho}\|_{L^1(\mathbb{R}^3)} + C_2\|\bar{\rho}\|_\infty\big) + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{R}^3} R(\rho, m\,|\,\bar{\rho}, \bar{m})\,d\tau dx$$

# The model

$$\begin{cases} u_t - \frac{1}{\varepsilon} v_x = 0 \\ \\ v_t - \frac{1}{\varepsilon}\tau(u)_x = -\frac{1}{\varepsilon^2} v, \end{cases}$$

where $\tau$ satisfies the usual conditions $\tau'(u) > 0$ to guarantee strict hyperbolicity. For gas dynamics applications, $u$ denotes the specific volume, $v$ the velocity and

$$\tau\left(\frac{1}{\rho}\right) = -p(\rho),$$

where $p$ stands for the pressure of the gas and $\rho$ for its density.
Formal limit

$$u_t - \tau(u)_{xx} = 0,$$

thus with the relation (Darcy's law) at the limit $v = \tau(u)_x$.

# Entropy (in)equalities

$$\mathcal{E}(u,v) = \frac{1}{2}v^2 + \int^u \tau(s)ds = \frac{1}{2}v^2 + W(u),$$

with entropy flux given by

$$\mathcal{F}(u,v) = -v\tau(u)$$

and corresponding entropy inequality

$$\mathcal{E}(u,v)_t + \frac{1}{\varepsilon}\mathcal{F}(u,v)_x \leq -\frac{1}{\varepsilon^2}v^2 \leq 0$$

$\mathcal{E}(u,0) = W(u)$ entropy for the limiting equation:

$$\mathcal{E}(u,0)_t + \mathcal{F}(u,\tau(u)_x)_x = -\big(\tau(u)_x\big)^2$$

# Relative entropy estimate

$(\bar u, \bar v) = (\bar u, \varepsilon\tau(\bar u)_x)$ solves

$$\begin{cases} \bar u_t - \dfrac{1}{\varepsilon}\bar v_x = 0 \\ \bar v_t - \dfrac{1}{\varepsilon}\tau(\bar u)_x = -\dfrac{1}{\varepsilon^2}\bar v + \varepsilon\tau(\bar u)_{xt} \end{cases}$$

Relative entropy:

$$\mathcal{E}(u, v \,|\, \bar u, \bar v) = \frac{1}{2}(v - \bar v)^2 + W(u \,|\, \bar u)$$

$$\mathcal{E}(u, v \,|\, \bar u, \bar v)_t + \frac{1}{\varepsilon}\mathcal{F}(u, v \,|\, \bar u, \bar v)_x \le -\frac{1}{\varepsilon^2}(v - \bar v)^2 + \tau(\bar u)_{xx}\tau(u \,|\, \bar u) - \varepsilon\tau(\bar u)_{xt}(v - \bar v)$$

# The model

$$\begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m + \frac{1}{\varepsilon} \rho \nabla_x c \\ -\triangle_x c + c = \rho, \end{cases}$$

where $\rho \geq 0$, $c \in \mathbb{R}$, $m \in \mathbb{R}^3$ and the pressure $p(\rho)$ satisfies $p'(\rho) \geq 0$. Easiest case $p(\rho) = \rho^2$.
Formal limit:

$$\begin{cases} \rho_t + \operatorname{div}_x \left( \rho \nabla_x c - \nabla_x p(\rho) \right) = 0 \\ -\triangle_x c + c = \rho \end{cases}$$

# Entropy (in)equalities

Modified entropy–entropy flux pair, based on the mechanical energy of the system:

$$\mathcal{H}(\rho, m, c) = \eta(\rho, m) - \rho c$$
$$\mathcal{Q}(\rho, m, c) = q(\rho, m) - mc.$$

Then the entropy inequality becomes

$$\mathcal{H}(\rho, m, c)_t + \frac{1}{\varepsilon} \operatorname{div}_x \mathcal{Q}(\rho, m, c) \leq -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} - \rho c_t$$

From the elliptic equation:

$$\rho c_t = \frac{1}{2}\left(c^2 + |\nabla_x c|^2\right)_t - \operatorname{div}_x \left(c_t \nabla_x c\right)$$

Final relation:

$$\left(\mathcal{H}(\rho, m, c) + \frac{1}{2}\left(c^2 + |\nabla_x c|^2\right)\right)_t + \frac{1}{\varepsilon} \operatorname{div}_x \left(\mathcal{Q}(\rho, m, c) - \varepsilon c_t \nabla_x c\right) \leq -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho}$$

# Relative entropy estimate

$(\bar\rho, \bar m, \bar c) = \big(\bar\rho, -\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho) - \bar c\big), \bar c\big)$ solves

$$\begin{cases} \bar\rho_t + \frac{1}{\varepsilon}\operatorname{div}_x \bar m = 0 \\ \bar m_t + \frac{1}{\varepsilon}\operatorname{div}_x \frac{\bar m \otimes \bar m}{\bar\rho} + \frac{1}{\varepsilon}\nabla_x p(\bar\rho) = -\frac{1}{\varepsilon^2}\bar m + \frac{1}{\varepsilon}\bar\rho\nabla_x\bar c + e(\bar\rho, \bar m) \\ -\triangle_x \bar c + \bar c = \bar\rho \end{cases}$$

Relative entropy estimate:

$$\Big(\mathcal{H}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c) + \frac{1}{2}\big((c - \bar c)^2 + |\nabla_x(c - \bar c)|^2\big)\Big)_t$$
$$+ \frac{1}{\varepsilon}\operatorname{div}_x\big(\mathcal{Q}(\rho, m, c \,|\, \bar\rho, \bar m, \bar c) - \varepsilon(c - \bar c)_t\nabla_x(c - \bar c)\big)$$
$$\leq -\frac{1}{\varepsilon^2}R(\rho, m \,|\, \bar\rho, \bar m) - Q - P - E,$$

where $R$, $Q$ and $E$ are as before and $P = \frac{1}{\varepsilon}\frac{\bar m}{\bar\rho}(\rho - \bar\rho)\cdot\nabla_x(c - \bar c)$

# The model

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x - \frac{1}{\varepsilon} z_x = 0 \\ z_t - \frac{\mu}{\varepsilon} v_x = -\frac{1}{\varepsilon^2} z, \end{cases}$$

where $\mu > 0$ and the elastic stress function $\sigma$ satisfies the usual condition $\sigma'(u) > 0$ which guarantees strict hyperbolicity.

Formal limit:

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx}, \end{cases}$$

for which the viscoelastic response is given by $z = \sigma(u) + \mu v_x$

# Entropy (in)equalities

$$\mathbb{E}(u,v,z) = \int^{u} \sigma(s)ds + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2 = \Sigma(u) + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2,$$

with entropy flux given by

$$\mathbb{F}_{\varepsilon}(u,v,z) = -(\varepsilon\sigma(u)v + vz)$$

and corresponding entropy inequality

$$\mathbb{E}(u,v,z)_t + \frac{1}{\varepsilon}\mathbb{F}_{\varepsilon}(u,v,z)_x \leq -\frac{1}{\mu\varepsilon^2}z^2 \leq 0$$

$\mathbb{E}(u,v,0) = \Sigma(u) + \frac{1}{2}v^2$ entropy for the limiting system:

$$\mathbb{E}(u,v,0)_t + \mathbb{F}_1(u,v,\sigma(u)_x)_x = -\mu(v_x)^2$$

for

$$\mathbb{F}_1(u,v,\sigma(u)_x) = -(\sigma(u)v + \mu v v_x)$$

# Relative entropy estimate

$(\bar{u}, \bar{v}, \bar{z}) = (\bar{u}, \bar{v}, \varepsilon\mu\bar{v}_x)$ solves

$$\begin{cases} \bar{u}_t - \bar{v}_x = 0 \\ \bar{v}_t - \sigma(\bar{u})_x - \dfrac{1}{\varepsilon}\bar{z}_x = 0 \\ \bar{z}_t - \dfrac{\mu}{\varepsilon}\bar{v}_x = -\dfrac{1}{\varepsilon^2}\bar{z} + \varepsilon\mu\bar{v}_{xt} \end{cases}$$

Relative entropy:

$$\mathbb{E}(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z}) = \Sigma(u \,|\, \bar{u}) + \tfrac{1}{2}(v - \bar{v})^2 + \tfrac{1}{2\mu}(z - \bar{z})^2$$

$$\mathbb{E}(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z})_t + \frac{1}{\varepsilon}\mathbb{F}_\varepsilon(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z})_x \leq -\frac{1}{\mu\varepsilon^2}(z - \bar{z})^2 + \bar{v}_x\sigma(u \,|\, \bar{u}) - \varepsilon\bar{v}_{xt}(z - \bar{z})$$