SimTech
Cluster of Excellence

Universität Stuttgart
Germany

Multi-Resolution Methods for Quantifying Uncertainties in Geophysical Applications

R. Bürger, I.Kröker, W. Nowak and C. Rohde

University of Stuttgart · Institute of Applied Analysis and Numerical Simulation

14. International Conference on Hyperbolic Problems: Theory, Numerics, Applications
Università di Padova, 25. June 2012

Outline

- Motivation
- Stochastic Multi-Resolution approach (SMR)
- Clarifier-thickener problem with random feed in one space dimension
- Random perturbed quarter five-spot problem in two space dimensions
- Conclusion and remarks

Governing Equation

Find the unknown $u : D \times [0,T) \times \Omega \to [0,1]$

$$\begin{cases} u_t + \operatorname{div} f(\mathrm{x},t,u,\omega) + q(\mathrm{x},\omega) = 0 & \text{in } D \times (0,T) \times \Omega, \\ u(\mathrm{x},0) = u_0(\mathrm{x}) & \text{in } D. \end{cases}$$

Here are $D \subset \mathbb{R}^d$, $d = 1, 2$, $T > 0$, (Ω, P) a probability space, f nonlinear flux function, q-source term.

Stochastic discretization methods

- Monte-Carlo (MC): to high computational effort
- Polynomial-Chaos: R. Abgrall, B. Despres.
- Stochastic-Elements: J. Troyen, A. Ern.

Stochastic Multi-Resolution Approach

Let $\xi = \xi(\omega)$ be a random variable. Assume $\xi \sim \mathcal{U}(0,1)$. Define

$$\phi_{i,l}^{N_\mathrm{r}}(\xi) = 2^{N_\mathrm{r}/2}\phi_i(2^{N_\mathrm{r}}\xi - l), \quad i = 0, \dots, N_\mathrm{o}, \quad l = 0, \dots, 2^{N_\mathrm{r}} - 1.$$

Here ϕ_i is the i-th Legendre polynomial.

Let $\xi = \xi(\omega)$ be a random variable. Assume $\xi \sim \mathcal{U}(0,1)$. Define

$$\phi_{i,l}^{N_\mathrm{r}}(\xi) = 2^{N_\mathrm{r}/2}\phi_i(2^{N_\mathrm{r}}\xi - l), \quad i = 0,\dots,N_\mathrm{o}, \quad l = 0,\dots,2^{N_\mathrm{r}} - 1.$$

Here ϕ_i is the i-th Legendre polynomial.

The polynomials $\phi_{0,0}^{N_\mathrm{r}},\dots,\phi_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}$ satisfy

$$\left\langle \phi_{i,k}^{N_\mathrm{r}},\, \phi_{j,l}^{N_\mathrm{r}} \right\rangle = \delta_{ij}\delta_{kl}.$$

and their support is $\mathrm{Supp}(\phi_{i,k}^{N_\mathrm{r}}) = [2^{-N_\mathrm{r}}k, 2^{-N_\mathrm{r}}(k+1)]$.

Let $\xi = \xi(\omega)$ be a random variable. Assume $\xi \sim \mathcal{U}(0,1)$. Define

$$\phi_{i,l}^{N_\mathrm{r}}(\xi) = 2^{N_\mathrm{r}/2}\phi_i(2^{N_\mathrm{r}}\xi - l), \quad i = 0,\ldots,N_\mathrm{o}, \quad l = 0,\ldots,2^{N_\mathrm{r}}-1.$$

Here ϕ_i is the i-th Legendre polynomial.

The polynomials $\phi_{0,0}^{N_\mathrm{r}},\ldots,\phi_{N_\mathrm{o},2^{N_\mathrm{r}}-1}^{N_\mathrm{r}}$ satisfy

$$\left\langle \phi_{i,k}^{N_\mathrm{r}},\, \phi_{j,l}^{N_\mathrm{r}} \right\rangle = \delta_{ij}\delta_{kl}.$$

and their support is $\mathrm{Supp}(\phi_{i,k}^{N_\mathrm{r}}) = [2^{-N_\mathrm{r}}k, 2^{-N_\mathrm{r}}(k+1)]$.

The projection of a random field $w(\mathrm{x},t,\cdot) \in L^2(\Omega)$ is defined by

$$\Pi^{N_\mathrm{o},N_\mathrm{r}}[w](\mathrm{x},t,\xi) := \sum_{l=0}^{2^{N_\mathrm{r}}-1}\sum_{i=0}^{N_\mathrm{o}} w_{i,l}^{N_\mathrm{r}}(\mathrm{x},t)\phi_{i,l}^{N_\mathrm{r}}(\xi),$$

$$w_{i,l}^{N_\mathrm{r}}(\mathrm{x},t) := \left\langle w(\mathrm{x},t,\cdot),\, \phi_{i,l}^{N_\mathrm{r}} \right\rangle.$$

Stochastic Galerkin approach

For $N_\mathrm{r}, N_\mathrm{o} \in \mathbb{N}$ find $u_0^{N_\mathrm{r}}, \dots, u_P^{N_\mathrm{r}} : D \times (0,T) \to \mathbb{R}$ such that

$$\int_\Omega \left(\partial_t \Pi^{N_\mathrm{o},N_\mathrm{r}}\left[u\right](\mathrm{x},t,\omega) + \mathrm{div}\, f\left(\mathrm{x},t,\Pi^{N_\mathrm{o},N_\mathrm{r}}\left[u\right],\omega\right)\right) \times \phi_\alpha^{N_\mathrm{r}}\, dP(\omega) = 0$$

hold for all $\alpha = 0, \dots, P$.
Here

- $\alpha = (i,l)$ for $i = 0, \dots, N_\mathrm{o}$, $l = 0, \dots, 2^{N_\mathrm{r}} - 1$.
- $P = (N_\mathrm{o}+1)2^{N_\mathrm{r}} - 1$.

The orthogonality of $\phi_\alpha^{N_\mathrm{r}}$ leads to the $(P+1)$-dimensional partially decoupled system

$$\partial_t u_0^{N_\mathrm{r}}(\mathrm{x},t) + \mathrm{div}\left\langle f\left(\mathrm{x},t,\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_0^{N_\mathrm{r}}\right\rangle = 0$$

$$\vdots$$

$$\partial_t u_P^{N_\mathrm{r}}(\mathrm{x},t) + \mathrm{div}\left\langle f\left(\mathrm{x},t,\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_P^{N_\mathrm{r}}\right\rangle = 0$$

Note:

- Remains (Weak-) hyperbolic
- Coupled in PC
- Decoupled in MR

The orthogonality of $\phi_\alpha^{N_\mathrm{r}}$ leads to the $(P+1)$-dimensional partially decoupled system

$$\begin{aligned}
\partial_t u_0^{N_\mathrm{r}}(\mathrm{x},t) + \mathrm{div}\left\langle f\left(\mathrm{x},t,\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_0^{N_\mathrm{r}}\right\rangle &= 0 \\
&\vdots \\
\partial_t u_P^{N_\mathrm{r}}(\mathrm{x},t) + \mathrm{div}\left\langle f\left(\mathrm{x},t,\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_P^{N_\mathrm{r}}\right\rangle &= 0
\end{aligned}$$

Note:

- Remains (Weak-) hyperbolic
- Coupled in PC
- Decoupled in MR

- Lower Polynomial Order
- Better for parallel computing

Numerical Experiments in One Space Dimension

Clarifier-Thickener Problem with random feed

Find the unknown concentration u with

$$\begin{aligned} u_t(x,t,\omega) + \big(g(x,t,u,\omega)\big)_x &= 0 && \text{in} \quad \mathbb{R} \times (0,T) \times \Omega. \\ u(x,0,\omega) &= u_0(x) && \text{in} \quad \mathbb{R}. \end{aligned} \qquad (CT)$$

The flux function g is determined for $t \in (0,T)$ and $\omega \in \Omega$ by

$$g(x,t,u,\omega) := \begin{cases} q_{\mathrm{L}}(t)(u - u_{\mathrm{F}}(t,\omega)) & \text{for } x < -1, \\ q_{\mathrm{L}}(t)(u - u_{\mathrm{F}}(t,\omega)) + b(u) & \text{for } -1 < x < 0, \\ q_{\mathrm{R}}(u - u_{\mathrm{F}}(t,\omega)) + b(u) & \text{for } 0 < x < 1, \\ q_{\mathrm{R}}(u - u_{\mathrm{F}}(t,\omega)) & \text{for } x > 1. \end{cases}$$

Here q_{L}, q_{R} and u_{F} satisfy

$$q_{\mathrm{L}}(.) \in C^1([0,T]),\ q_{\mathrm{L}}(.) < 0,\ q_{\mathrm{R}} > 0,\ 0 \le u_{\mathrm{F}}(\omega) \le 1.$$

Literature:

M. Bustos et al, Sedimentation and thickening

Second Formulation of the Model

The flux $g(x,t,u,\omega)$ has discontinuities at the points $x=-1,0,1$. We view the flux g as depending of two parameters

$$\gamma^1(x,t) := \begin{cases} q_{\mathrm{L}}(t) & \text{for } x<0, \\ q_{\mathrm{R}} & \text{for } x>0, \end{cases} \gamma^2(x,t) := \begin{cases} 1 & \text{for } x \in (-1,1), \\ 0 & \text{for } x \notin (-1,1). \end{cases}$$

With the flux

$$f(t,u,\gamma^1,\gamma^2,\omega) := \gamma^1(\cdot,\omega)(u-u_{\mathrm{F}}(t,\omega)) + \gamma^2 b(u)$$

we can understand the problem (CT) as a system of balance laws

$$\begin{aligned} u(x,t,\omega)_t + \big(f(t,u,\gamma^1,\gamma^2,\omega)\big)_x &= 0, \\ \gamma^1_t(x,t) = H(-x)q_{\mathrm{L},t}(t), \qquad \gamma^1_t(x,t) &= 0 \end{aligned} \qquad (CTF)$$

for the unknown vector $(u,\gamma^1,\gamma^2)^T \in [0,1]\times\mathbb{R}^2$.
The system (CTF) is weakly hyperbolic.

- 1D Finite–Volume Method

$$u_i^{\alpha,n+1} = u_i^{\alpha,n} - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^{\alpha,n} - F_{i-1/2}^{\alpha,n}\right) \quad (i \in \mathbb{Z},\ n \in \mathbb{N},\ \alpha = 0,\dots,P).$$

- 1D Finite–Volume Method

$$u_i^{\alpha,n+1} = u_i^{\alpha,n} - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^{\alpha,n} - F_{i-1/2}^{\alpha,n}\right) \quad (i \in \mathbb{Z},\ n \in \mathbb{N},\ \alpha = 0,\dots,P).$$

- We use Lax-Friedrichs numerical flux:

$$F_{i+1/2}^{\alpha,n} := \frac{1}{2}\Big(f^\alpha(t^n, u_i^{0,n},\dots,u_i^{\alpha,n},\gamma_i^{1\,n},\gamma_i^{2,n}) + f^\alpha(t^n, u_{i+1}^{0,n},\dots,u_{i+1}^{P,n},\gamma_{i+1}^{1\,n},\gamma_{i+1}^{2,n})\Big) + \frac{\Delta x}{2\Delta t^n}(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}).$$

- 1D Finite–Volume Method

$$u_i^{\alpha,n+1} = u_i^{\alpha,n} - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^{\alpha,n} - F_{i-1/2}^{\alpha,n}\right) \quad (i \in \mathbb{Z},\ n \in \mathbb{N},\ \alpha = 0, \dots, P).$$

- We use Lax-Friedrichs numerical flux:

$$F_{i+1/2}^{\alpha,n} := \frac{1}{2}\Big(f^\alpha(t^n, u_i^{0,n}, \dots, u_i^{\alpha,n}, \gamma_i^{1\,n}, \gamma_i^{2,n}) \\ + f^\alpha(t^n, u_{i+1}^{0,n}, \dots, u_{i+1}^{P,n}, \gamma_{i+1}^{1\,n}, \gamma_{i+1}^{2,n})\Big) + \frac{\Delta x}{2\Delta t^n}(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}).$$

- Flux function:

$$f^\alpha(t, u^0, \dots, u^P, \gamma^1, \gamma^2) = \\ \sum_{\alpha=0}^{P} \gamma^1 u_\alpha^{N_\mathrm{r}} - \sum_{\alpha=0}^{P} \gamma^1 u_\mathrm{F}^\alpha(t) + \gamma^2 \left\langle b\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_\alpha^{N_\mathrm{r}} \right\rangle .$$

- 1D Finite–Volume Method

$$u_i^{\alpha,n+1} = u_i^{\alpha,n} - \frac{\Delta t^n}{\Delta x}\left(F_{i+1/2}^{\alpha,n} - F_{i-1/2}^{\alpha,n}\right) \quad (i \in \mathbb{Z},\ n \in \mathbb{N},\ \alpha = 0,\dots,P).$$

- We use Lax-Friedrichs numerical flux:

$$F_{i+1/2}^{\alpha,n} := \frac{1}{2}\Big(f^\alpha(t^n, u_i^{0,n},\dots,u_i^{\alpha,n},\gamma_i^{1\,n},\gamma_i^{2,n}) \\ + f^\alpha(t^n, u_{i+1}^{0,n},\dots,u_{i+1}^{P,n},\gamma_{i+1}^{1\,n},\gamma_{i+1}^{2,n})\Big) + \frac{\Delta x}{2\Delta t^n}(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}).$$

- Flux function:

$$f^\alpha(t,u^0,\dots,u^P,\gamma^1,\gamma^2) = \\ \sum_{\alpha=0}^{P}\gamma^1 u_\alpha^{N_\mathrm{r}} - \sum_{\alpha=0}^{P}\gamma^1 u_\mathrm{F}^\alpha(t) + \gamma^2\left\langle b\left(\Pi^{N_\mathrm{o},N_\mathrm{r}}[u]\right), \phi_\alpha^{N_\mathrm{r}}\right\rangle.$$

- Initial values: $u_i^{0,0} = \dots = u_i^{P,0} = 0$

Numerical Experiments

- Batch flux function

$$b(u) := \begin{cases} 10^{-4}u((1-u)^5) & : 0 \leq u \leq u_*, \\ p_2(u) & : u_* < u \leq u_{Max}, \\ 0 & : u_{Max} < u. \end{cases}$$

p_2-second order polynomial.

- The solid volume feed fraction u_F satisfies

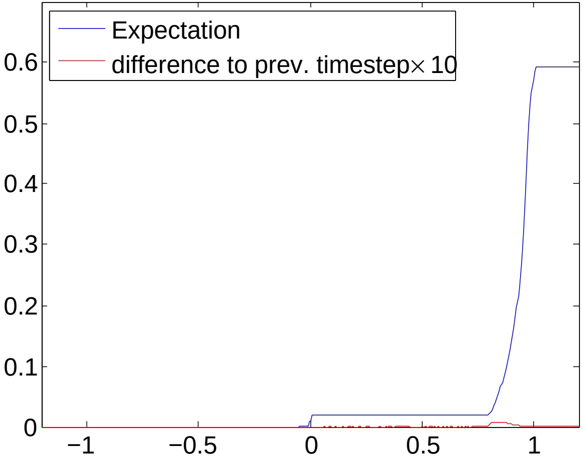
$$u_\mathrm{F}(t,\omega) := 0.15 + 0.05\xi(\omega),$$

such that ξ is uniformly distributed on $[0,1]$. Consequently

$$\mathbb{E}\left[u_\mathrm{F}\right] = 0.175$$

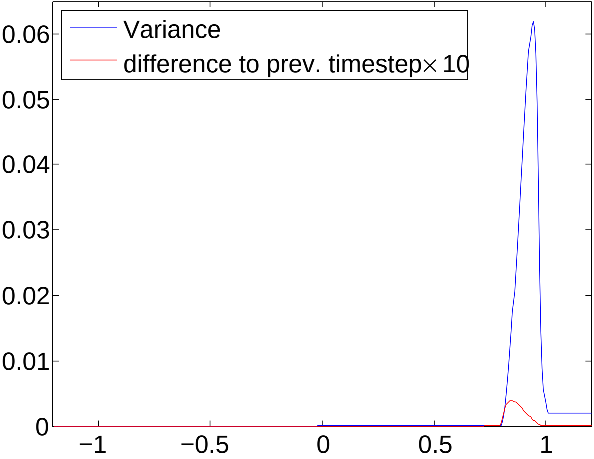
- $q_\mathrm{L} = -7.2 \cdot 10^{-6}$, $q_\mathrm{R} = 3.0 \cdot 10^{-6}$.

Expectation



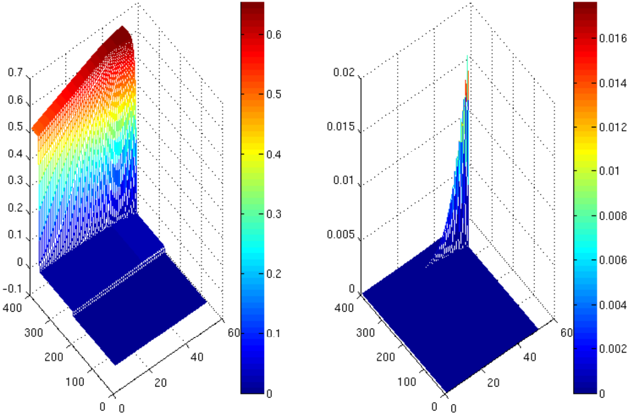
Expectation (blue line) and the difference to the previous time-step (red line) on $T = 10^6$ and $u_\mathrm{F} = 0.15 + 0.05\xi$ computed with $N_\mathrm{r} = 4$, $N_\mathrm{o} = 4$.

Variance



Variance (blue line) and the difference to the previous time-step (red line) $T = 10^6$ and $u_F = 0.15 + 0.05\xi$computed with $N_r = 4,\ N_o = 4$.

Distribution of the numerical solution



Distribution of the numerical solution on $T = 10^6$ with $u_{\mathrm{F}} = 0.15 + 0.05\xi$ and the difference to the previous time-step.

L^1-Error

Reference - Monte-Carlo solution with 500 000 samples.

N_o	$N_r = 3$	$N_r = 4$	$N_r = 5$
2	1.405881e-02	4.252602e-03	1.415577e-03
3	1.002693e-02	3.134362e-03	1.016522e-03
4	7.384129e-03	2.323872e-03	7.785858e-04
5	5.422152e-03	1.642281e-03	6.374276e-04

Table: L^1-Error of the SMR-approach.

Samples	50000	100000	200000	400000
L^1-error	5.614726e-03	1.392341e-03	1.132936e-03	6.651745e-04

Table: L^1-Error for the MC-approach.

CPU-Time

	Computed on 4 CPU's				Computed on 16 CPU's	
N_o	N_r=2	N_r=3	N_r=4	N_r=5	N_r=4	N_r=5
2	350 s.	1021 s.	1674 s.	3949 s.		
3	721 s.	2055 s.	3362 s.	7920 s.		
4	1332 s.	3767 s.	6032 s.	14427 s.	1753 s.	4188 s.
5	2277 s.	6069 s.	10195 s.	24346 s.	2701 s.	6351 s.

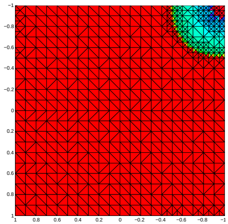
Table: Computation on Intel-Xeon E7-4830 (2.13 GHz). Time in sec.

Samples:	50000	100000	200000	400000	500000
Time (s.)	24009 s.	47651s.	96005 s.	192019 s.	353391 s.

Table: During of Monte-Carlo computations on 32 CPU's. Computed on AMD Opteron(tm) Processor 2376 (2.3GHz)

Numerical Experiments in two Space Dimensions

Randomly Perturbed Quarter Five-Spot Problem



Find the unknown saturation $S : D \times [0,T) \times \Omega \to [0,1]$

$$\begin{cases} S_t + \mathrm{div}\,(\mathbf{v}_s f(S)) + q(\mathrm{x}, S) = 0 & \text{in } D \times (0,T) \times \Omega, \\ S(x,0) = S_0(x) & \text{in } D. \end{cases}$$

Here

$\mathbf{v}_s$ is a given velocity field satisfies

$$\mathbf{v}_s = (v^1 + \xi(\omega), v^2)^t, \qquad \xi \sim \mathcal{U}(0,1)$$

and

$$\mathrm{div}\,(\mathbf{v}_s) = 0 \quad \text{in} \quad D \times \Omega.$$

f nonlinear fractional flow function.

q Source.

Finite Volume Method

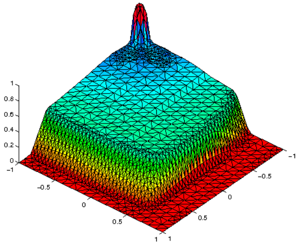
The semidiscrete Central-Upwind scheme (Kurganov, Petrova (2005)) for $\bar{S} = \left(\bar{S}^0, \ldots, \bar{S}^P\right)$ is given by

$$\frac{d}{dt}\bar{S}_j^\alpha := -\frac{1}{|T_j|}\sum_{k=1}^{3} h_{jk}\left(\frac{a_{jk}^{in}F^\alpha(\tilde{S}_{jk}, M_j(k), t) + a_{jk}^{out}F^\alpha(\tilde{S}_j, M_j(k))}{a_{jk}^{in} + a_{jk}^{out}}\right)\cdot \mathbf{n}_{jk}$$
$$+\frac{1}{|T_j|}\sum_{k=1}^{3} h_{jk}\frac{a_{jk}^{in}a_{jk}^{out}}{a_{jk}^{in} + a_{jk}^{out}}\left[\tilde{S}_{jk}^\alpha(M_j(k)) - \tilde{S}_j^\alpha(M_j(k))\right].$$

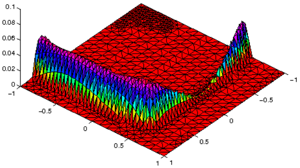
Here

$$F^\alpha(S, x, y, t) := \left\langle \mathbf{v}_s f\left(\Pi^{N_\mathrm{o}, N_\mathrm{r}}[S]\right), \phi_\alpha^{N_\mathrm{r}}\right\rangle, \quad \alpha = 0, \ldots, P,$$

$\bar{S}_j$	the cell average over the triangle T_j,
h_{jk}	is the length of the k-th side, $k = 1, 2, 3$,
$M_j(k)$	is the midpoint of the k-th side,
n_{jk}	is the outer normal on the k-th side,
a_{jk}^{in}, a_{jk}^{out}	directional local speeds on the k-th side,
$\tilde{S}_j(G)$	is a admissible reconstruction on the point G over the cell T_j.



(a)



(b)

Expectation (a) and Variance (b) of the numerical solution computed on $N_\mathrm{r} = 2$, $N_\mathrm{o} = 2$, (this implies 12-dimensional system) on 2154 Triangles.

Conclusion and Remarks

- The computational effort of Stochastic Multi-Resolution approach is significantly lower the the computational effort of Monte-Carlo approach.
- The Stochastic Multi-Resolution requires higher dimensional system than Polynomial Chaos, but this system is partially decoupled and we need lower polynomial order.
- Stochastic Multi-Resolution requires less synchronisation in parallel computation.

Thank you for your attention!

Literature

Rémi Abgrall.
A simple, flexible and generic deterministic apporach to uncertainty quantifications in non linear problems: application to fluid flow problems.
2007.

Bradley K. Alpert.
A class of bases in L^2 for the sparse representation of integral operators.
SIAM J. Math. Anal., 24(1):246–262, 1993.

R. Bürger, K. H. Karlsen, N. H. Risebro, and J. D. Towers.
Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units.
Numer. Math., 97(1):25–65, 2004.

María Cristina Bustos, Fernando Concha, Raimund Bürger, and Elmer M. Tory.
Sedimentation and thickening, volume 8 of *Mathematical Modelling: Theory and Applications*.
Kluwer Academic Publishers, Dordrecht, 1999.
Phenomenological foundation and mathematical theory.

Alexander Kurganov and Guergana Petrova.
Central-upwind schemes on triangular grids for hyperbolic systems of conservation laws.
Numerical Methods for Partial Differential Equations, 21(3):536–552, 2005.

O. P. Le Maître, H. N. Najm, R. G. Ghanem, and O. M. Knio.
Multi-resolution analysis of Wiener-type uncertainty propagation schemes.
J. Comput. Phys., 197(2):502–531, 2004.

Gaël Poëtte, Bruno Després, and Didier Lucor.
Uncertainty quantification for systems of conservation laws.
J. Comput. Phys., 228(7):2443–2467, 2009.

J. Tryoen, O. Le Maître, M. Ndjinga, and A. Ern.
Intrusive Galerkin methods with upwinding for uncertain nonlinear hyperbolic systems.
J. Comput. Phys., 229(18):6485–6511, 2010.