Threshold phenomena for critical wave equations.

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Linear waves I

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- Maxwell's equations imply that in vacuo, the electric and magnetic fields obey the same type of equations. Hence universal significance of the linear wave equation.
- Wave equation first studied by d'Alembert, who explicitly solves initial value problem in $1-d$:
  If $\Box u = -\partial_t^2 u + c^2 \partial_x^2 u = 0$ and $u(0,\cdot) = f$, $u_t(0,\cdot) = g$, then

$$u(t,x) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

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- In higher dimensions, $n \geq 2$, the solutions of $\Box u = -u_{tt} + \triangle u = 0$ (from now on $c = 1$) do decay for sufficiently nice data. Comes from dispersion, i. e. the fact that the solution decouples into traveling waves moving in *infinitely many different directions*.

$$\|u(t,\cdot)\|_{L^\infty} \lesssim t^{-\frac{n-1}{2}}, \ (t,x) \in \mathbf{R}^{n+1}$$

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- This is of utmost importance for the nonlinear waves we'll consider soon. The soliton phenomena there are a result of a delicate balancing between dispersion and the nonlinear effects.

Nonlinear waves

- Abstract approach : free wave equation is a simple Lagrangian field theory :

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- Critical 'points' u for this functional are characterized by Euler-Lagrange equations, which in this case are given exactly by

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- Natural generalizations of (1) lead to important nonlinear wave equations :

Wave Maps

- First, one can replace the scalar-valued function u by one which takes values in a Riemannian manifold. Then formally, the Lagrangian stays the same :

$$\mathcal{L}(u, \nabla_{t,x} u) = \int_{\mathbf{R}^{n+1}} (-|u_t|_g^2 + |\nabla_x u|_g^2)\, dxdt \qquad (2)$$

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- The resulting Euler-Lagrange equations are now nonlinear, and their solutions are 'Wave Maps' :

$$\Box u + \Gamma^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0$$

  For $M = S^k$, get $\Box u = u(-|u_t|^2 + |\nabla_x u|^2)$.

Another example : NLW

- Another way to change $\mathcal{L}$ is to add extra terms, for example a polynomial term :

$$\mathcal{L} = \int_{\mathbf{R}^{n+1}} (-u_t^2 + |\nabla_x u|^2 + \lambda |u|^{p+1})\, dxdt, \; p > 1$$

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- By scaling, one may reduce to $\lambda = \pm 1$, corresponding to the *defocussing* ($\lambda = +1$) as well as the *focussing* ($\lambda = -1$) cases. These display radically different behavior. Although it is not apparent at first, there is a somewhat similar classification into foc./defoc. for Wave Maps, depending on target.

Basic questions for the nonlinear problems

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- *Blow up dynamics.* If the solution breaks down in finite time, give description of the singularity formation. We'll see that in some cases soliton like solutions play an important role here.
- *Behavior at Infinity/Scattering*. If a solution exists for all $t > 0$, describe the behavior at $t = +\infty$. Does solution approach a free wave (in suitable sense) ?

The criticality distinction I

- Due to their underlying Lagrangian, both (WM) as well as (NLW) have Hamiltonians giving a preserved energy :

$$\int_{\mathbf{R}^n} \frac{1}{2}(|u_t|^2 + |\nabla_x u|^2)\, dx\, (WM),$$

$$\int_{\mathbf{R}^n} [\frac{1}{2}(|u_t|^2 + |\nabla_x u|^2) + \frac{\lambda}{p+1}|u|^{p+1}]\, dx\, (NLW),$$

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- If energy left invariant under scaling, model is *energy critical*. (WM) : $n = 2$, (NLW) : $p = \frac{n+2}{n-2}$, e. g. $p = 5$ for $n = 3$.

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- Basis philosophy : the sub-critical case is easier for local and global existence questions as well as for classification of blow ups. It is harder for questions relating to the behavior at infinity, such as scattering. The supercritical case is harder for global existence and blow up classification. The critical case is borderline.

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- Large data supercritical problems up to now untouched, except perturbatively or explicit blow up solutions.

Case in point : quick review of Wave Maps

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- In the supercritical case $n \geq 3$, self-similar singular solutions of the form $u(t, x) = v(\frac{x}{t})$ have been known since work by Shatah in 1988 for suitable targets, such as S^3. These have recently been shown to be stable under suitable small perturbations by R. Donninger ('11).

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- *Focussing* : $-u_{tt} + \triangle u = -|u|^{p-1}u$. Here one has finite-time blow-up solutions for any $p > 1$, by using simple ODE-type solutions :

$$u(t,x) = \frac{C}{(T-t)^{\frac{2}{p-1}}}$$

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- Conjectured to be true for all $p < 5$.

NLW for $n = 3$, critical focussing case

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- **Key new feature** in critical focussing case : existence of **static solutions** (balancing of dispersion/nonlinear growth).

$$W(x) = \frac{1}{(1 + \frac{|x|^2}{3})^{\frac{1}{2}}},\ W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x)$$

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  $W(x)$ is called the *ground state*.
- The ground states play a pivotal role in the global dynamics of the solutions for the critical focussing NLW.

The role of static solutions for critical NLW, $n = 3$.

- A celebrated result of Kenig-Merle(2006) states that solutions u with $E(u) < E(W)$ are governed by simple dichotomy :
  (i) : If $\|\nabla_x u(x,0)\|_{L^2_x} < \|\nabla_x W\|_{L^2_x}$, then solutions exist globally and scatter like free waves at infinity.
  (ii) $\|\nabla_x u(x,0)\|_{L^2_x} > \|\nabla_x W\|_{L^2_x}$, then finite time blow-up both for $t >< 0$.
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  The blow up is most likely of the ODE type, but this is not proved yet.
- Key question arises : what happens for solutions whose energy is strictly above that of W.
- Recent work has demonstrated the existence of a number of new types of dynamics with energies arbitrarily close to but strictly above that of W. From now on **all solutions radial**.

Dynamics with $E(u) > E(W)$ I.

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Theorem

(K.-Schlag '04) There exists a co-dimension 1 manifold ('stable manifold') of initial data passing through $(W, 0)$ *within a small neighborhood of* W *(with respect to sufficiently strong topology) resulting in solutions which decouple into dynamically rescaled* W *and an error scattering to zero like free wave :*

$$u(t,x) = W_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t) \to \lambda_\infty > 0$$

Thus solution scatters to re-scaled ground state.

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- Are other types of 'bubbling off' dynamics possible, e. g. more violent dynamics for $\lambda(t)$?

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$$u(t,x) = W_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t) = t^{-1-\nu}$$

on some interval $[t_0, 0)$ for t_0 sufficiently small. Hence we have continuum of blow-up rates ! Energy may be arbitrarily close to that of W.

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- These solutions are type II, which means

$$\limsup_{t\to 0} \|u(t,\cdot)\|_{H^1} < \infty$$

Not the case for ODE-type blow up solutions.

Dynamics with $E(u) > E(W)$ III.

- The continuum of blow up rates is related to the fact that these solutions are not of C^∞-class, but indeed only of $H^{1+\nu-}$-class. The data experience a small 'kink' across the boundary of light cone (for high enough derivatives).

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- Current work in progress (Donninger-K.) establishes existence of an infinite set of quantized blow up rates $\nu = 2k + 1$ and sufficiently large) corresponding to type II blow up solutions of C^∞-class.

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- Current work in progress (Donninger-K.) establishes existence of an infinite set of quantized blow up rates $\nu = 2k + 1$ and sufficiently large) corresponding to type II blow up solutions of C^∞-class.
- The previous examples may lead one to believe that all type II solutions either blow up in finite time of the form

$$u(t,x) = W_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t)(T-t) \to \infty$$

or else exist globally (e. g. toward $t = +\infty$) and scatter toward rescaled ground state or zero : **strong soliton resolution**.

Dynamics with $E(u) > E(W)$ IV.

- This is not true, however :

Theorem

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- Expected that the above solutions are C^∞ (not proved yet). Hence no quantization of blow-up at infinity.
- Recent work by Duyckaerts-Kenig-Merle shows : for type II solutions, either one has finite time bubbling-off blow-up or else they decouple as

$$u(t,x) = \sum_i \mu_i W_{\lambda_i(t)}(x) + \epsilon(t,x),\ \lambda_i(t)t \to \infty$$

Contrast to subcritical models

- The blow up at infinity phenomenon is a threshold phenomenon for critical problems which does not seem to occur for subcritical situations. For example, a recent result of Nakanishi-Schlag('10) for the subcritical nonlinear Klein-Gordon equation

$$-u_{tt} + \triangle u - u = u^3$$

on $\mathbf{R}^{3+1}$ shows that the strong resolution conjecture (i. e. trichotmoy between finite time blow up, infinite time scattering to zero or infinite time convergence to ground state) is correct for solutions of energy sufficiently close to the ground state.

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- The same seems true for the analogous Schrodinger equation

$$iu_t + \triangle u = -|u|^2 u$$

on $\mathbf{R}^{3+1}$ according to suggestive work by T. Tao ('04).

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- Intuition behind this : data which are not in the co-dimension one manifold constructed by K.-Schlag in '04 will leave a 'small tube' around the one-parameter family $\{W_\lambda\}_{\lambda>0}$ of ground states. This is due to one negative unstable eigen mode in the linearization around W.

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- A recent ('10) result by K.-Nakanishi-Schlag shows that upon leaving this 'tube', the solution either blows up in finite time (certainly an ODE-type blow up) or else scatters to zero like a free wave.

Dynamics with $E(u) > E(W)$ VI.

- One may wonder how the exotic type II blow up solutions (finite time and at $t = \pm\infty$) fit into this framework. The key is that they are far away from the 'tube of re-scaled ground states' with respect to the topology in which one can construct the stable manifold which divides scattering from ODE-like blow up (conjecturally at this time).

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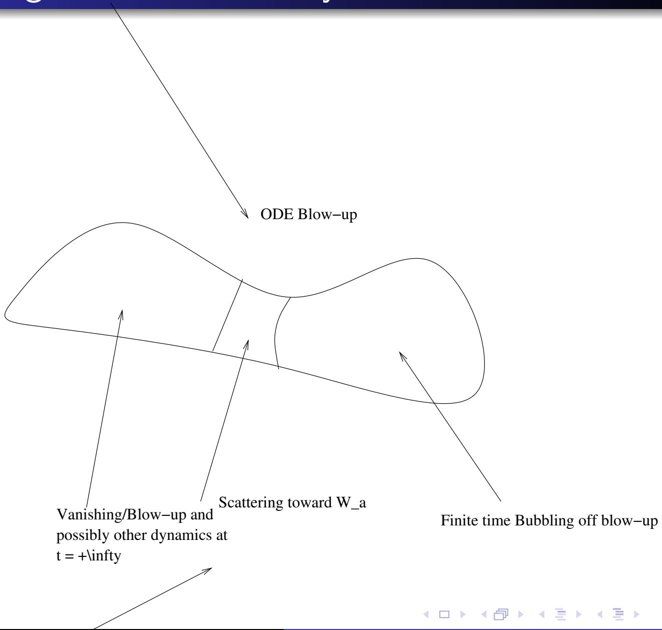
- This raises the question whether there is some co-dimension one set within a small neighborhood *in the energy topology* around $(W, 0)$ which comprises the data of all type II dynamics, and divides the data space into those resulting in finite time ODE blow up and those scattering to zero, as in the following picture :

Conjectural general threshold dynamics



Beyond the model case : critical Wave Maps I.

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- Basic examples : if $M = S^2$(standard sphere), then stereographic projection $Q : \mathbf{R}^2 \to S^2$ is the ground state. If $M = \mathbf{H}^2$(hyperbolic plane), then no such static map exists.

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- It is now known that in the defocussing case, no singularities may form (Sterbenz-Tataru '09). Indeed, by work of (K.-Schlag '09), for $M = \mathbf{H}^2$, critical Wave Maps exist globally, satisfy Strichartz estimates, scatter like free waves, and moreover admit profile decompositions in a suitable sense, analogously to the defocussing critical $\Box u = u^5$ in $3 + 1 - d$.

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- This leads to the question as to what happens in **focussing case**. To simplify the discussion, one reduces to symmetric targets M admitting a $SO(2)$-action. One can then talk about **equivariant Wave Maps**.
- For example, when $M = S^2$, so-called *co-rotational* Wave Maps lead to the scalar equation

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2}$$

Static soln. (ster. proj.) : $Q(r) = 2\arctan r$.

Beyond the model case : critical Wave Maps II.

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Theorem

(K.-Schlag-Tataru '06) For each $\nu > \frac{1}{2}$, there exists a blow up solution of the form

$$u(t,x) = Q_{\lambda(t)}(x) + \epsilon(t,x),\ \lambda(t) = t^{-1-\nu}$$

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- For solutions with energy strictly below that of Q, one has global existence and scattering (Struwe, Cote-Kenig-Merle, Sterbenz-Tataru).
- Not clear that there is an analogue of the stable manifold of (K.-Schlag '04) since no negative eigenvalue in spectrum of linearization.

Beyond the model case : critical Wave Maps II.

- However, as for critical NLW, it is expected that there is a quantized set of blow up rates corresponding to smooth data, in this case of the form $\nu \in \mathbf{N}$ up to logarithmic corrections.

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- We note that unlike for the critical NLW, there are no ODE-type blow up solutions for critical Wave Maps, and so there are probably stable type II solutions even under non-equivariant perturbations. This is poorly understood at this time.

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- It emerges that some of the phenomena revealed for specific examples have more universal character...

The method for producing type II solutions

- Back to $\Box u = -u^5$, our method is inspired by the blow-up constructions of K.-Schlag-Tataru. We recall here the basic setup. Here we explain how to construct blow up/vanishing at infinity (Donninger-K. '11)

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- Simple attempt $u(t, \cdot) = W_{\lambda(t)}(x) + \text{error}$, $\lambda(t) = t^{-(1-\nu)}$ leads to principal error term

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- This causes wave parametrix for $\partial_t^2 + \mathcal{L}$ to lead to t^2 growth : can't iterate !

The construction ; approximate solutions 1

- Instead, as for blow up solutions in (K.-Sch.-T.), we first attempt to construct an approximate solution by adding 'elliptic profile modifiers' :

$$u_{approx} = W_{\lambda(t)}(r) + v_1(r,t) + \ldots + v_{2k}(r,t)$$

The odd index v_l improve accuracy near $r = 0$, while the even index v_l improve it near characteristics (in blow up in-coming, here out-going).

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- To construct the correction terms v_i, introduce the auxiliary coordinates

$$R = \lambda(t)r,\ \tau = \int_{t_0}^{t} \lambda(s)\,ds + \frac{1}{\nu}t_0^{\nu} = \frac{1}{\nu}t^{\nu}$$

$$\lambda(t) = t^{-(1-\nu)},\ t \in [t_0, \infty)$$

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- To contrast this with the blow up solutions of (K.-S.-T.), there we had

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- For the sequel, we already note that while one can achieve arbitrary levels of accuracy for approximate blow up solutions by constructing sufficiently many of the v_i-corrections, **this is not the case in our situation** : we shall stop after the $k = 2$ stage, as later stages don't seem to help anymore.

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- To see this, we mimic here the procedure of (K.-S.-T.), encountering more singular expressions.

The construction ; approximate solutions 3

- Denote by e_i the error after the i-th correction, i. e.

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$$Lv_{2k-1} = \lambda^{-2} e_{2k-2},\ k \geq 1,\ L = -\partial_R^2 - \frac{2}{R}\partial_R - 5W^4$$

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- For even indices $i = 2k$, we replace the wave operator by

$$-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r$$

  This gives equation

$$t^2(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r)v_{2k} = t^2 e_{2k-1}$$

The construction ; approximate solutions 4

- One then introduces a new coordinate $a = \frac{r}{t}$; assuming $v_{2k} = \frac{\lambda^{\frac{1}{2}}}{(\lambda t)^{\beta}} W_{2k}(a)$, one finds a singular ODE of the form

  $$L_\rho W_{2k} = F,\ L_\rho = (1 - a^2)\partial_{aa} + 2(a^{-1} + a\rho - a)\partial_a - \rho^2 + \rho$$

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- Let $\mathcal{Q}_k$ be the ideal inside $\mathcal{Q}$ consisting of linear combinations of terms of the form with $2l_0 + \sum_j \lambda_j(2[j-1] + l_j) \geq 2(k-1)$

$$q_0(a)(1-a)^{\lambda_0(1+\beta_0)} q_1(a) \frac{1}{(t\lambda)^{2l_0}} \Pi_{j=1}^{N} [\frac{(1-a)^{(4j-3)(1+\beta_0)-2(j-1)}}{(\lambda t)^{2(j-1)+2l_j}} q_j(a)]$$

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- One also observes that increasing k by two, one pays $(1-a)^{-2\nu}$ but gains $(\lambda t)^{-2} = t^{-2\nu}$.
- This suggests that in contrast to the blow up construction in K. -S.-T. where iteration leads to arbitrarily accurate approximate solution, *here this process essentially stalls*.

The construction ; getting approximate sol. in light cone

- Fortunately, for us *it suffices to use only the first two corrections* v_1, v_2, where v_2, the correction near the cone, has essentially the form

$$v_2 = \lambda^{\frac{1}{2}} \frac{R^3}{(\lambda t)^4}(1-a)^{1+\beta_0}$$

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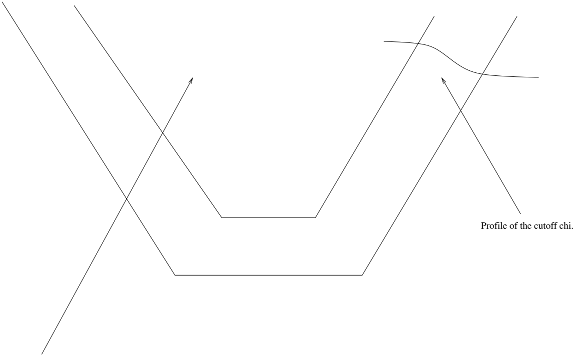
- To deal with this, *we truncate it near the light cone* : For $\chi(x) \in C^\infty(\mathbf{R})$ with $\chi(x) = 1$ for $|x| > 1$, $\chi(x) = 0$ on $|x| < \frac{1}{2}$, and $\chi_C(x) = \chi(\frac{x}{C})$, we replace v_2 by

$$\chi_C(t-r)v_2$$

  Note that this function does not have energy vanishing as $t \to \infty$! This is expected as our solution needs to have energy strictly above that of W by the results of Duyckaerts-Kenig-Merle.

The construction ; getting approximate sol. in light cone



- We first construct an exact solution here.
- Finally, we have the following theorem which gives an approximate solution in the inner forward light cone, albeit without arbitrary accuracy :

The construction ; getting approximate sol. in light cone

Theorem

There exist corrections v_1, v_2 *with the bounds*

$$\|\nabla_{t,r} v_1\|_{L^2_{r^2 dr}(K)} \lesssim \tau^{-\frac{1}{2}},\ \|\nabla_{t,r}[\chi_C(t-r)v_2]\|_{L^2_{r^2 dr}} \lesssim C^{-\frac{\nu}{2}}$$

such that the functions $u_{approx} = W_{\lambda(t)}(x) + v_1 + v_2$ *is an approximate solution with*

$$\|R\frac{e_2}{\lambda^2}\|_{L^1_{dR}} + \|\frac{e_2}{\lambda^2}\|_{H^{2\alpha}_{R^2 dr}(K)} \ll \tau^{-3},\ \alpha \in [0, \frac{1}{4}],\ \tau \gg 1$$

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- In particular, these approximate solutions have energy arbitrarily close to that of W if we pick C sufficiently large.

The construction ; getting exact sol. in light cone

- We now strive to construct an *exact solution*

$$u(t,x) = u_{approx}(t,x) + \epsilon(t,x)$$

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$$\epsilon_{tt} - \triangle\epsilon - 5\lambda^2(t)W^4(\lambda(t)r) = N(\epsilon) + e_2$$

Issue : operator $\Box_W = \Box - 5\lambda^2(t)W^4(\lambda(t)r)$ is time dependent.

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- We deal with this by using $\tau = \frac{1}{\nu}t^\nu$, $R = \lambda(t)r$, and

$$\epsilon(t,r) = v(\tau, R)$$

The construction ; distorted Fourier transform

- Then v solves the following problem with a time-independent Schrodinger operator :

$$(\partial_\tau+\frac{\dot{\lambda}}{\lambda}R\partial_R)^2 v+\frac{\dot{\lambda}}{\lambda}(\partial_\tau+\frac{\dot{\lambda}}{\lambda}R\partial_R)v-\triangle v-5W^4 v=\lambda^{-2}(\tau)[N(\epsilon)+e_2]$$

The construction ; distorted Fourier transform

- Then v solves the following problem with a time-independent Schrodinger operator :

$$(\partial_\tau + \frac{\dot{\lambda}}{\lambda} R\partial_R)^2 v + \frac{\dot{\lambda}}{\lambda}(\partial_\tau + \frac{\dot{\lambda}}{\lambda} R\partial_R)v - \triangle v - 5W^4 v = \lambda^{-2}(\tau)[N(\epsilon) + e_2]$$

- We solve this problem via the Fourier representation associated with the operator

$$\mathcal{L} := -\partial_{RR} - 5W^4(R)$$

This operator appears when replacing $v(\tau, R)$ by $\tilde{\epsilon} := Rv(\tau, R)$.

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Lemma

(K.-S.-T. '07) spec$(\mathcal{L})=\{\xi_d\}\cup[0,\infty)$. 0 *is a resonance, with resonant function* $\phi_0(R)=\frac{R(1-\frac{R^2}{3})}{(1+\frac{R^2}{3})^{\frac{3}{2}}}$. *Finally, the operator* $\mathcal{L}$ *is in limit-point case at infinity.*

-

The construction ; translating to Fourier side

- Standard theory for this type of operators then furnishes the existence of a Fourier basis $\{\phi(R,\xi)\}\cup\{\phi_d(R)\}$ such that we have the associated Fourier transform

$$\mathcal{F} : f \to \hat{f}$$

defined via

$$\hat{f}(\xi_d) = \int_0^\infty \phi_d(R)f(R)\,dR,\ \hat{f}(\xi) = \lim_{b\to\infty}\int_0^b \phi(R,\xi)f(R)\,dR,\ \xi \geq 0$$

Fundamental fact : *This is isometry from* $L^2(\mathbf{R}^+)$ *to* $L^2(\mathbf{R}^+,\rho)$*, and we have*

$$f(R) = \hat{f}(\xi_d)\phi_d(R) + \lim_{\mu\to\infty}\int_0^\mu \phi(R,\xi)\hat{f}(\xi)\rho(\xi)\,d\xi$$

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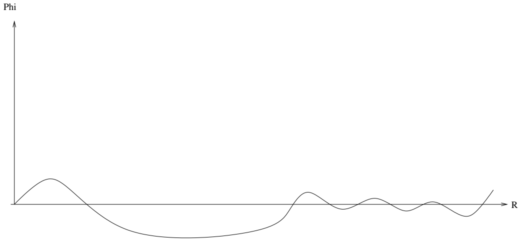
- Precise asymptotics $\rho(\xi)\sim\xi^{-\frac{1}{2}}$, $\xi \ll 1$, and $\rho(\xi)\sim\xi^{\frac{1}{2}}, \xi\to\infty$ extremely important for us.

The construction ; translating to Fourier side



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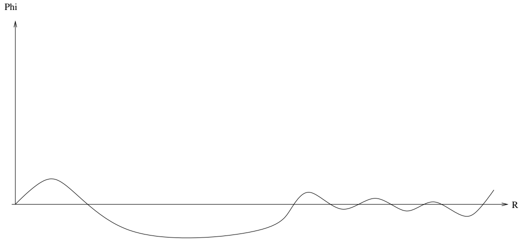
- 

- The Fourier basis $\phi(R,\xi)$ admits expansions of the form (K.-S.-T. '07)

$$\phi(R,\xi) = \phi_0(R) + R^{-1}\sum_{j=1}^{\infty}(R^2\xi)^j\phi_j(R^2),\ |\phi_j(u)| \leq \frac{C^j}{(j-1)!}|u|\langle u\rangle^{-\frac{1}{2}}$$

In particular, they 'behave non-oscillatory' in the region $R\xi^{\frac{1}{2}} \leq 1$ and 'become oscillatory' in the region $R\xi^{\frac{1}{2}} > 1$.

The construction ; translating to Fourier side

- Write

$$\tilde{\epsilon}(\tau, R) = x_d(\tau)\phi_d(R) + \int_0^\infty x(\tau,\xi)\phi(R,\xi)\rho(\xi)\, d\xi, \; \underline{x} = \begin{pmatrix} x_d \\ x \end{pmatrix},$$

and introducing the operator

$$\mathcal{D}_\tau = \partial_\tau - \frac{3}{2}\frac{\lambda_\tau}{\lambda} - \frac{\lambda_\tau}{\lambda}\begin{pmatrix} 0 & 0 \\ 0 & 2\xi\partial_\xi + \frac{\xi\rho'(\xi)}{\rho(\xi)} \end{pmatrix},$$

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- We get the transport equation

$$\Big(-\mathcal{D}_\tau^2 + \frac{\lambda_\tau}{\lambda}\mathcal{D}_\tau - \xi\Big)\underline{x} = 2\frac{\lambda_\tau}{\lambda}\mathcal{K}_{nd}\mathcal{D}_\tau\underline{x} + (\frac{\lambda_\tau}{\lambda})^2([\mathcal{K}_{nd}, \mathcal{K}_d] + \mathcal{K}_{nd}^2)\underline{x}$$
$$+ \Big(\partial_\tau(\frac{\lambda_\tau}{\lambda}) - (\frac{\lambda_\tau}{\lambda})^2\Big)\underline{x} - \underline{b}$$

Here $\underline{b} = \mathcal{F}\Big(\lambda^{-2}[N(\epsilon) + e_2]\Big)$.

The construction ; translating to Fourier side

- Here the factors $\frac{\lambda_\tau}{\lambda} \sim (1-\nu)\tau^{-2}$, so for $|\nu - 1| \ll 1$, one gains extra smallness.

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$$\mathcal{K}_{nd} = \begin{pmatrix} 0 & \mathcal{K}_{dc} \\ \mathcal{K}_{cd} & \mathcal{K}_0 \end{pmatrix}$$

where $\mathcal{K}_0$ is a 'Hilbert-transformation like' operator given by

$$\begin{aligned} \mathcal{K}_0 x(\eta) &= \langle \int_0^\infty x(\xi) R\partial_R \phi(R,\xi)\rho(\xi)\, d\xi,\ \phi(R,\eta)\rangle \\ &+ \langle \int_0^\infty 2\xi\partial_\xi x(\xi)\phi(R,\xi)\rho(\xi)\, d\xi,\ \phi(R,\eta)\rangle \end{aligned}$$

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- This means we expect to have good L^p-type bounds for $\mathcal{K}_0$.

The construction ; translating to Fourier side

- We solve this problem via iteration in a suitable Banach space. This space is defined via the norm

$$\|x(\tau,\cdot)\|_S := \|\frac{\xi^{\frac{1}{2+}}}{\langle\xi\rangle^{\frac{1}{2+}}}x(\tau,\xi)\|_{L^M_\xi} + \|\xi^{\frac{1}{2}}\langle\xi\rangle^{2\alpha}x(\tau,\xi)\|_{L^2_{d\rho}}$$

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Theorem

The preceding fixed point problem admits a solution $x(\tau,\xi)$ satisfying

$$\|x(\tau,\cdot)\|_S \lesssim \tau^{-2+\delta},\ \|x_d(\tau)\|_S \lesssim \tau^{-3+\delta}$$

for some small $\delta > 0$ (there is a playoff between the $2+$ and the $\delta > 0$).

-