# Dynamical stability of non-constant equilibria for the compressible Navier-Stokes equations in Eulerian coordinates

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# Motivation

- Logical step after proving local well-posedness
- Global strong solutions of the compressible Navier-Stokes equations
  - all derivatives are to be defined in $L_p$ w.r.t. time and space
  - bounded domains
  - exponential stability of equilibria $(0, \theta_\infty, \rho_\infty(x))$, $\theta_\infty > 0$
- Eulerian coordinates have not yet been fully understood.

  > *"The transformation from* $(1.3)$ *to* $(1.7)$ *is necessary, because our result is done in the* $L_p$*-framework and in this approach there is no possibility to treat eq.* $(1.3)$ *– so we consider* $(1.7)$*."*

  P.B. Mucha, W.M. Zajączkowski: Global existence of solutions of the Dirichlet problem for the compressible Navier-Stokes equations. Z. Angew. Math. Mech. 84, no. 6, 417-424 (2004).

- (CNSE) is a "model problem"; more complicated situations: two-phase fluids (NSAC, NSCH - up to now local well-posedness), free-boundary value problems, ...

$G \subset \mathbb{R}^n$ bdd. domain with $C^2$-boundary $\Gamma$. The fluid is characterized by its density $\rho$, and velocity field $u \in \mathbb{R}^n$.

$$\begin{aligned} \partial_t \rho + \nabla\cdot(\rho u) &= 0, & t > 0,\ x \in G, \\ \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) - \nabla\cdot\mathcal{T} &= \rho f_{ext}(\rho), & t > 0,\ x \in G, \end{aligned}$$

- Cauchy stress tensor: $\mathcal{T} = \mathcal{S}(u) - \pi\,\mathcal{I}$ with

$$\mathcal{S}(u) = 2\mu\,\mathcal{D}(u) + \lambda\nabla\cdot u\,\mathcal{I}, \quad \mathcal{D}(u) = \tfrac{1}{2}(\nabla u + (\nabla u)^T)$$

- $\mu$, $\lambda$ - viscosities (may depend on $\rho$ )
- $f_{ext}$ - external force
- pressure: $\pi(\rho) := \rho^2 \partial_\rho \psi(\rho)$ ($\psi$ – Helmholtz energy density)

# Boundary conditions

- velocity field $u$

$$\begin{aligned}
\text{non-slip:} \qquad & u(t,x) = 0, \quad t>0,\ x\in\Gamma,\\
\text{pure slip:} \qquad & (u(t,x)\,|\,\nu(x)) = 0, \quad t>0,\ x\in\Gamma,\\
& \mathcal{P}(\nu(x))\mathcal{S}(u(t,x))\cdot\nu(x) = 0, \quad t>0,\ x\in\Gamma,
\end{aligned}$$

  $(a\,|\,b) := \sum_i a_i b_i$, $\nu(x)$ – outer normal,
  $\mathcal{P}(\nu) := \mathcal{I} - \nu\otimes\nu$ projects a vector field on its tangential part
- density $\rho$ - no b.c. (characteristic case)
- initial conditions

$$u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \quad \forall x \in G$$

# Equilibria of interest

## The isothermal case

Seek steady states of the form: $w_\infty(x) := (u_\infty(x), \rho_\infty(x)) \equiv (0, \rho_\infty(x))$.

1. $f_{ext}(x, \rho_\infty(x)) = -\nabla\phi(x, \rho_\infty(x))$, $\phi$ smooth enough
2. $\int_G \rho_\infty(x)\, dx$ given (conservation of mass)
3. notation: $\mathcal{S}_\infty := 2\mu_\infty \mathcal{D}(u_\infty) + \lambda_\infty \nabla\cdot u_\infty \mathcal{I}$,
   $a_\infty := a(\rho_\infty)$, $a \in \{\mu, \lambda, \pi\}$
4. assume: $\mu_\infty > 0$, $2\mu_\infty + \lambda_\infty > 0$ for all $x \in \overline{G}$

$$
\begin{aligned}
\nabla\cdot(\rho_\infty u_\infty) &= 0, && x \in G,\\
\nabla\cdot(\rho_\infty u_\infty \otimes u_\infty) - \nabla\cdot\mathcal{S}_\infty + \nabla\pi_\infty + \rho_\infty \nabla\phi(\rho_\infty) &= 0, && x \in G,\\
u_\infty &= 0, && x \in \Gamma,\\
\int_G \rho_\infty\, dx &= m
\end{aligned}
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\end{aligned}
$$

One can show $u_\infty \equiv 0 \Rightarrow w_\infty \equiv (0, \rho_\infty)$.

# Equilibria of interest

## The isothermal case

The problem for $\rho_\infty$ is equivalent to

$$\begin{aligned} &\varphi(x, \rho_\infty(x)) = k_\infty, \quad x \in \overline{G}, \\ &\int_G \rho_\infty(x)\, dx = m > 0 \ \text{(is given)}, \end{aligned} \tag{1}$$

i.e. to determine the pair $(\rho_\infty, k_\infty)$, $k_\infty \in \mathbb{R}$, where

$$\varphi(x, \rho_\infty) := \psi(\rho_\infty) + \rho_\infty \psi'(\rho_\infty) + \phi(x, \rho_\infty).$$

Note: $(\psi(\rho_\infty) + \rho_\infty \psi'(\rho_\infty))' \equiv \frac{\pi(\rho_\infty)'}{\rho_\infty}$

Assumption: $\exists (\rho_\infty, k_\infty) \in \mathrm{H}^2_p(G) \times \mathbb{R}$ solving (1).

# Linearization

Introduce deviation

$$w(t,x) := (u(t,x), \varrho(t,x)) := (u(t,x), \rho(t,x) - \rho_\infty(x))$$

from equilibrium $(0, \rho_\infty)$, and then linearize around $(0, \rho_\infty)$:

$$\begin{aligned}
\partial_t \varrho + \nabla\cdot(\rho_\infty u) &= F_1(w), && (t,x) \in \mathbb{R}_+ \times G,\\
\rho_\infty \partial_t u - \nabla\cdot\mathcal{S}_\infty(u) + \rho_\infty \nabla(\varphi'_\infty \varrho) &= F_2(w), && (t,x) \in \mathbb{R}_+ \times G,\\
u &= 0, && (t,x) \in \mathbb{R}_+ \times \Gamma,\\
w &= w_0, && (t,x) \in \{0\} \times G,
\end{aligned}$$

with

$$F_1(w) := -\nabla\varrho \cdot u - \varrho \nabla\cdot u, \quad F_2(w) := \dots,$$

$$\mathcal{S}_\infty(u) := 2\mu_\infty \mathcal{D}(u) + \lambda_\infty \nabla\cdot u \mathcal{I}, \quad \varphi'_\infty = \frac{\pi'_\infty}{\rho_\infty} + \phi(x, \rho_\infty)',$$

$$w_0 = (\rho_0 - \rho_\infty, u_0).$$

1. momentum equation:

$$\rho_\infty \partial_t u - \mu_\infty \Delta u - (\mu_\infty + \lambda_\infty)\nabla\nabla\cdot u + \rho_\infty \varphi'_\infty \nabla \varrho = \dots$$

   Want to study in $\mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G;\mathbb{R}^n))$. The terms above indicate

$$\begin{aligned} u \in Z_2(\mathbb{R}_+) &:= \mathrm{H}^1_p(\mathbb{R}_+; \mathrm{L}_p(G;\mathbb{R}^n)) \cap \mathrm{L}_p(\mathbb{R}_+; \mathrm{H}^2_p(G;\mathbb{R}^n)), \\ \varrho &\in \mathrm{L}_p(\mathbb{R}_+; \mathrm{H}^1_p(G)) \end{aligned}$$

2. continuity equation:

$$\partial_t \varrho + \nabla\cdot(\rho_\infty u) = \dots.$$

   Want to study in $\mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{L}_p(G))$ and thus

$$\begin{aligned} \rho \in Z_1(\mathbb{R}_+) &:= \mathcal{Z}(\mathbb{R}_+) \cap \mathcal{Z}_\infty(\mathbb{R}_+), \\ \mathcal{Z}(\mathbb{R}_+) &:= \mathrm{H}^1_p(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{L}_p(\mathbb{R}_+; \mathrm{H}^1_p(G)), \\ \mathcal{Z}_\infty(\mathbb{R}_+) &:= \mathrm{C}^1(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{H}^1_p(G)). \end{aligned}$$

# The fixed point mapping

Define $\Phi : Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+) \to Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+)$ by $\Phi(\hat{w}) := w$ where

$$\begin{aligned} \partial_t \varrho + \nabla\cdot(\rho_\infty u) &= F_1(\hat{w}), && (t,x) \in \mathbb{R}_+ \times G, \\ \rho_\infty \partial_t u - \nabla\cdot\mathcal{S}_\infty(u) + \rho_\infty \nabla(\varphi'_\infty \varrho) &= F_2(\hat{w}), && (t,x) \in \mathbb{R}_+ \times G, \\ u &= 0, && (t,x) \in \mathbb{R}_+ \times \Gamma, \\ w &= w_0, && (t,x) \in \{0\} \times G. \end{aligned}$$

First task: Assume $\hat{w} = (\hat{\rho}, \hat{u}) \in Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+)$.
Prove that $\exists!\ w \in Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+)$. $\Rightarrow$ $\Phi$ well-defined.

- Can show:
  - $F_2(\hat{w}) \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G; \mathbb{R}^n))$
  - $F_1(\hat{w}) = -\nabla\hat{\varrho} \cdot \hat{u} - \hat{\varrho}\nabla\cdot\hat{u} \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{L}_p(G))$
- Expect a problem with $F_2(\hat{w})$, cf. remark of Mucha & Zajączkowski.

# Eulerian coord. vs. Lagrangian coord.

Relation between the Eulerian ($x$) and Lagrangian ($\xi$) coordinates

$$x = \xi + \int_{t_0}^{t} v(s,\xi)\,ds = x(t,\xi)$$

where

$$v(t,\xi) := u(t,x(t,\xi)), \quad \eta(t,\xi) := \varrho(t,x(t,\xi)), \quad \eta_\infty(t,\xi) = \rho_\infty(x(t,\xi)).$$

Euler: $\partial_t \varrho + \nabla_x \cdot (\rho_\infty u) = F_1^E = -\nabla_x \varrho \cdot u - \varrho \nabla_x \cdot u$

Lagrange: $\partial_t \eta + \nabla_\xi \cdot (\eta_\infty v) = F_1^L = \nabla_\xi \cdot (\eta_\infty v) - \nabla_u \cdot (\eta_\infty v) - \eta \nabla_u \cdot v$

with $\nabla_u \cdot := [\partial_x \xi_i]\partial_{\xi_i}$.

Different regularities: $F_1^E \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{L}_p(G))$

$F_1^L \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{H}_p^1(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{H}_p^1(G))$

# The idea

1. Apply $\nabla$ to $\partial_t \varrho + \nabla\cdot(\rho_\infty u) = -\nabla\cdot(\varrho u) =: F_1(w)$
2. Multiply by $\nabla\varrho|\nabla\varrho|^{p-2}$, $p \geq 2$
3. Integrate over $G$

$$\int_G \left[\partial_t \nabla \varrho(t) + \nabla\nabla\cdot(\rho_\infty u(t))\right]\cdot\nabla\varrho(t)|\nabla\varrho(t)|^{p-2}\,dx = F_3(w)(t),$$

with

$$\begin{aligned} F_3(w)(t) &:= \int_G \nabla F_1(w)(t)\cdot\nabla\varrho(t)|\nabla\varrho(t)|^{p-2}\,dx \\ &= -\int_G \nabla\nabla\cdot(\varrho(t)u(t))\cdot\nabla\varrho(t)|\nabla\varrho(t)|^{p-2}\,dx. \end{aligned}$$

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$$\frac{d}{dt}\frac{1}{p}\|\nabla\varrho(t)\|^p_{\mathrm{L}_p(G;\mathbb{R}^n)} + \int_G \nabla\nabla\cdot(\rho_\infty u(t))\cdot\nabla\varrho(t)|\nabla\varrho(t)|^{p-2}\,dx = F_3(w)(t),$$

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with

$$\begin{aligned} F_3(w)(t) \equiv -(1-1/p)\int_G \nabla\cdot u(t,x)|\nabla\varrho(t,x)|^p\,dx \\ -\int_G \left(\nabla\varrho(t,x)\,|\,\nabla u(t,x)\nabla\varrho(t,x)\right)|\nabla\varrho(t,x)|^{p-2}\,dx \\ -\int_G \varrho(t,x)\left(\nabla\nabla\cdot u(t,x)\,|\,\nabla\varrho(t,x)\right)|\nabla\varrho(t,x)|^{p-2}\,dx. \end{aligned}$$

If $(\varrho, u) \in Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+) \Rightarrow F_3(w) \in \mathrm{L}_1(\mathbb{R}_+)$!

# The idea

Conclusion: WE MUST NOT FORGET

$$\int_G \left[\partial_t \nabla \varrho + \nabla \nabla \cdot (\rho_\infty u)\right] \cdot \nabla \varrho |\nabla \varrho|^{p-2}\, dx \in \mathrm{L}_1(\mathbb{R}_+)$$

Studying model problem (linear problem, constant coeff., $\mathbb{R}^n$) shows:

## Character of this property

This equation is just another way to express regularity for the data in the continuity equation.

# The correct linearization

One has to study the linear problem

$$\begin{aligned}
\partial_t \varrho + \nabla\cdot(\rho_\infty u) &= f_1(t,x), && (t,x) \in \mathbb{R}_+ \times G,\\
\rho_\infty \partial_t u - \nabla\cdot \mathcal{S}_\infty(u) + \rho_\infty \nabla(\varphi'_\infty \varrho) &= f_2(t,x), && (t,x) \in \mathbb{R}_+ \times G,\\
u &= 0, && (t,x) \in \mathbb{R}_+ \times \Gamma,\\
(\varrho, u) &= (\varrho_0, u_0), && (t,x) \in \{0\} \times G,
\end{aligned}$$

1. $f_1 \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G)) \cap \mathrm{C}(\mathbb{R}_+; \mathrm{L}_p(G))$;
2. $f_2 \in \mathrm{L}_p(\mathbb{R}_+; \mathrm{L}_p(G; \mathbb{R}^n))$;
3. $(\varrho_0, u_0) \in \mathrm{H}^1_p(G) \times \mathrm{W}^{2-2/p}_p(G)$;
4. $\int_G \left[\partial_t \nabla \varrho + \nabla\nabla\cdot(\rho_\infty u)\right] \cdot \nabla \varrho |\nabla \varrho|^{p-2}\, dx \in \mathrm{L}_1(\mathbb{R}_+)$

# THE CORRECT LINEARIZATION

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3. $(\varrho_0, u_0) \in \mathrm{H}^1_p(G) \times \mathrm{W}^{2-2/p}_p(G)$;
4. $\int_G \left[\partial_t \nabla\varrho + \nabla\nabla\cdot(\rho_\infty u)\right] \cdot \nabla\varrho |\nabla\varrho|^{p-2}\, dx = g$ with $g \in \mathrm{L}_1(\mathbb{R}_+) / \mathrm{L}_1(\mathbb{R}_+)$

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## What does property 4. mean?

Eq. in 4. is equivalent to impose the regularity $(\partial_t + I)^{-1} f_1 \in \mathrm{C}(\mathbb{R}_+; \mathrm{H}^1_p(G))$

# The correct linearization

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## Theorem: existence of an isomorphism

Assumptions of domain and coefficients.

Suppose 1. through 4. $\Leftrightarrow$ $\exists!\ (\varrho, u) \in Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+)$.

# Main result

## The assumptions

1. Structure: $f_{ext}(x,\rho) = -\nabla\phi(x,\rho)$
2. Regularity: $\phi \in \mathrm{C}^2(\mathbb{R}_+; \mathrm{H}^1_p(G))$, $\mu$, $\lambda$, $\beta \in \mathrm{C}^2(\mathbb{R}_+)$, $\psi \in \mathrm{C}^3(\mathbb{R}_+)$
3. Positivity:

$$\mu(y) > 0, \quad 2\mu(y) + \lambda(y) > 0, \quad \forall y > 0$$
$$\partial_y \varphi(x,y) > 0, \quad \forall y > 0, x \in \overline{G}$$

   with $\varphi(x,y) := \psi(y) + y\psi'(y) + \phi(x,y)$.

4. $\forall m > 0$: $\exists\ (\rho_\infty, k_\infty) \in \mathrm{H}^2_p(G) \times \mathbb{R}$ of

$$\varphi(x, \rho_\infty(x)) = k_\infty, \quad x \in \overline{G}, \quad \int_G \rho_\infty(x)\, dx = m.$$

5. $w_0 := (\rho_0 - \rho_\infty, u_0) \in \mathcal{V} := \mathrm{H}^1_p(G) \times \mathrm{W}^{2-2/p}_p(G; \mathbb{R}^n)$
6. Compatibility conditions: $u_{0|\Gamma_0} = 0$

# Main result

**Theorem**

*Let $G$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with compact $C^2$-boundary $\Gamma$. Further, let $p \in (n+2, \infty)$ and the assumptions from 1 through 6 be satisfied. Then there exist constants $r > 0$ and $\delta_0 > 0$, such that, if*

$$\|(\rho_0 - \rho_\infty, u_0)\|_{\mathcal{V}} \leq r$$

*with $\delta \in [0, \delta_0)$, the problem (CNSE) possesses a unique global solution, i.e., $(\rho, u)$ belongs to $Z_1([0,T]) \times Z_2([0,T])$ for every $T > 0$ and $e^{\delta\cdot}(\rho - \rho_\infty, u)$ belongs to $Z_1(\mathbb{R}_+) \times Z_2(\mathbb{R}_+)$, for all $\delta \in [0, \delta_0)$. In particular, the equilibrium $(0, \rho_\infty)$ is exponentially stable in the phase space, i.e.,*

$$\lim_{t\to\infty} e^{\delta\cdot t}\|(\rho(t) - \rho_\infty, u(t))\|_{\mathcal{V}} = 0.$$

Thanks for your attention!