Finite Difference Schemes for Keyfitz-Kranzer System.[1]

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[1] Joint work with Nils Henrik Risebro.

Outline of the talk

- Keyfitz-Kranzer System:
  - Symmetric System.
  - Non-symmetric System.
- Semi-discrete Finite difference scheme.
- Convergence to weak solutions.
- Fully-discrete Finite difference scheme.
- Entropy solution.

K-K Equation

- Symmetric Keyfitz-Kranzer System:

$$\begin{cases} u_t + (u\phi(|u|))_x = 0, & (x,t) \in \Omega = \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

- $u = \left(u^{(1)}, \dots, u^{(n)}\right) : \mathbb{R} \times [0,T) \to \mathbb{R}^n$ is the unknown vector map.
- $|u| = \sqrt{{u^{(1)}}^2 + \cdots + {u^{(n)}}^2}$.
- $\phi : \mathbb{R} \to \mathbb{R}$ is given (sufficiently smooth) scalar function.

Applications

- elasticity theory.
- magnetohydrodynamics.
- enhanced oil recovery.

E-values

- For $F(u) = u\phi(|u|)$, $B(u) = dF(u)$ is the matrix

$$B_{i,j}(u) = \phi(|u|)\delta_{i,j} + \phi'(|u|)\frac{u_i u_j}{|u|}, \quad i,j = 1,2,\cdots,n,$$

- First eigenvalue of $B(u)$: $\lambda_1 = \phi(|u|) + \phi'(|u|)\,|u|$
- Other $n-1$ eigenvalues are $\lambda_i = \phi(|u|)$, $i = 2,3,\cdots,n$.
- Not strictly hyperbolic.
- Formally $r = |u|$ satisfies

$$r_t + (r\phi(r))_x = 0,$$

Weak solution

- We say $u(x,t)$ a weak solution to KK system if
  - $u(x,t) \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$.
  - For all test functions $\psi \in C_0^\infty(\mathbb{R} \times [0,\infty))$

$$\iint_{\mathbb{R}\times\mathbb{R}^+} u\psi_t + u\phi(|u|)\psi_x \, dxdt + \int_{\mathbb{R}} u_0\psi(x,0)\, dx = 0,$$

- Existence (Panov)

$$u_t^\varepsilon + \left(u^\varepsilon\phi(|u^\varepsilon|)\right)_x = \varepsilon u_{xx}^\varepsilon,$$

- First show the existence of a *measure-valued* solution $\nu_{(t,x)}$ of the Cauchy problem.
- Indeed $\nu_{(t,x)}$ is regular: $\nu_{(t,x)}(u) = \delta(u - u(t,x)), u(t,x) \in L^\infty(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^n)$.

Idea of the proof

- For 2×2 system:

$$u_t^\varepsilon + (u^\varepsilon \phi(r^\varepsilon))_x = \varepsilon u_{xx}^\varepsilon,$$
$$v_t^\varepsilon + (v^\varepsilon \phi(r^\varepsilon))_x = \varepsilon v_{xx}^\varepsilon.$$

- Look at $r^\varepsilon = \sqrt{(u^\varepsilon)^2 + (v^\varepsilon)^2}$. Indeed it satisfies

$$r_t^\varepsilon + f(r^\varepsilon)_x = \varepsilon r_{xx}^\varepsilon - 2\varepsilon \left((u_x^\varepsilon)^2 + (v_x^\varepsilon)^2 \right)$$

- Also look at $\psi^\varepsilon := \tan^{-1}\left(\frac{v^\varepsilon}{u^\varepsilon}\right)$.
- Suppose $r^\varepsilon \to r$ and $\psi^\varepsilon \to \psi$.
- Then $u^\varepsilon = r^\varepsilon \cos(\psi^\varepsilon)$ and $v^\varepsilon = r^\varepsilon \sin(\psi^\varepsilon)$.

Semi discrete scheme

- Given $\Delta x > 0$, we set $x_j = j\Delta x$ for $j \in \mathbb{Z}$ and $u_j = u(x_j)$.
- Scheme:

$$u_j'(t) + D_- \left(\phi(r_j(t))u_j(t)\right) = 0, \text{ for } j \in \mathbb{Z}, \ t > 0,$$

  with initial values

$$u_j(0) = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\, dx,$$

  where $r_j(t) = |u_j(t)|$.
- Existence and uniqueness of differentiable solutions.

Compensated Compactness lemma

- Theorem

Assume that $\{u^\varepsilon\}$ is a sequence of uniformly bounded functions such that $|u^\varepsilon| \le M$ for all ε. Let $u^\varepsilon \overset{\star}{\rightharpoonup} u$ and $f(u^\varepsilon) \overset{\star}{\rightharpoonup} v$, and set

$$(\eta_1(s), q_1(s)) = (s-k, f(s)-f(k)),$$
$$(\eta_2(s), q_2(s)) = \left(f(s)-f(k), \int_k^s (f'(\theta))^2 \, d\theta\right),$$

where k is an arbitrary constant. If

$$\eta_i(u^\varepsilon)_t + q_i(u^\varepsilon)_x \quad \text{is in a compact set of } H^{-1}_{\mathrm{loc}}(\Omega) \text{ for } i = 1,\, 2,$$

then

1. $v = f(u)$, *a.e.* (x,t),
2. $u^\varepsilon \to u$, *a.e.* (x,t) *if* $\mathrm{meas}\,\{u \,|\, f''(u) = 0\} = 0$.

Lemma

- Let $\{u_j(t)\}$ be defined by the scheme, and let $r_j(t) = |u_j(t)|$. Then

$$\begin{aligned}
\|r_{\Delta x}(t)\|_{L^1(\mathbb{R})} &\leq \|r_{\Delta x}(0)\|_{L^1(\mathbb{R})}\,, \\
\|r_{\Delta x}(t)\|_{L^2(\mathbb{R})} &\leq \|r_{\Delta x}(0)\|_{L^2(\mathbb{R})}\,, \\
\|r_{\Delta x}(t)\|_{L^\infty(\mathbb{R})} &\leq \|r_{\Delta x}(0)\|_{L^\infty(\mathbb{R})}\,.
\end{aligned}$$

Furthermore, there is a constant C, independent of Δx and T, such that

$$\begin{aligned}
\int_0^T \sum_j \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right) \phi'(s)\, ds \\
+ \Delta x \sum_j \phi_{j-1} \Delta x \left|D^- u_j\right|^2\, dt \leq C.
\end{aligned}$$

Proof

- Let η be a differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$, take the inner product of the scheme with $\nabla_u \eta(u_j)$.

$$\frac{d}{dt}\eta(u_j) + D^- \left(\phi_j \eta(u_j)\right) + \left[(\nabla_u \eta(u_j), u_j) - \eta(u_j)\right] D^- \phi_j + \phi_{j-1} \frac{\Delta x}{2} d_u^2 \eta_{j-1/2} \left(D^- u_j, D^- u_j\right) = 0.$$

- $\eta(u) = |u|$ gives

$$\frac{d}{dt} r_j + D^-(r_j \phi(r_j) \leq 0.$$

- Multiplying by Δx and summing over j:

$$\left\| r_\Delta x(t) \right\|_{L^1(\mathbb{R})} \leq \left\| |u_0| \right\|_{L^1(\mathbb{R})}.$$

Contd..

- Max principle: $0 \leq r_j(t) \leq \sup_j |u_j(0)|$.
- $\eta(u) = |u|^2$ gives

$$\frac{d}{dt} r_j^2(t) + D^- \left(r_j^2 \phi_j \right) + r_j^2 D^- \phi_j + \phi_{j-1} \Delta x \left| D^- u_j \right|^2 = 0.$$

- Observe

$$D^- \left(r_j^2 \phi_j \right) + r_j^2 D^- \phi_j = D^- g(r_j) + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2 \right) \phi'(s)\, ds,$$

- where

$$g(r) = 2 \int_0^r s\phi(s) + s^2 \phi'(s)\, ds.$$

Contd..

- Hence

$$\|r_\Delta x(t)\|_{L^2(\mathbb{R})} \leq \||u_0|\|_{L^2(\mathbb{R})},$$

- Furthermore

$$\int_0^T \sum_j \int_{r_{j-1}}^{r_j} (r_j^2 - s^2)\, \phi'(s)\, ds + \Delta x \sum_j \phi_{j-1} \Delta x \left| D^- u_j \right|^2 \, dt \leq C,$$

- Let δ be a positive constant, and let e be some unit vector in $\mathbb{R}^n$. Choose

$$\eta(u) = \max\{\delta\,|u| - (e,u), 0\}.$$

CC Lemma

- We have
$$\sum_j \eta(u_j(t)) \leq \sum_j \eta(u_j(0)).$$
- We have that $\eta(u) = 0$ if u is in the cone $\Gamma_\delta = \{u \,|\, \delta\,|u| \leq (e, u)\}$.
- If $u_0 \in \{u \,|\; |u| \leq R\} \cap \Gamma_\delta$, then $u_{\Delta x}(x, t)$ is also in this set.

- Lemma

Let $u_{\Delta x}$ be generated by the scheme and $r_{\Delta x} = |u_{\Delta x}|$. Then
$$\{\eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})_x\}_{\Delta x > 0} \text{ is compact in } H^{-1}_{\mathrm{loc}},$$
where η_i and q_i are entropy and entropy flux functions.

Proof

- Let $i = 1$ or $i = 2$, and ψ is a test function.

$$
\begin{aligned}
\langle \mathcal{L}_i, \psi \rangle &= \langle \eta_i(r_{\Delta} x)_t + q_i(r_{\Delta x})_x, \psi \rangle \\
&= \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\frac{d}{dt}\eta_i(r_j) + D^- q_i(r_j) \right) \psi(x,t)\, dx\, dt \\
&+ \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \big(\psi(x_{j-1/2}, t) - \psi(x,t)\big)\, D^- q_i(r_j)\, dx\, dt \\
&=: \langle \mathcal{L}_{i,1}, \psi \rangle + \langle \mathcal{L}_{2,i}, \psi \rangle.
\end{aligned}
$$

- Regarding $\mathcal{L}_{i,2}$ we have

$$
\begin{aligned}
|\langle \mathcal{L}_{2,i}, \psi \rangle| &= \Big| \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{x_{j-1/2}}^{x} \psi_x(y,t)\, dy\, D^- q_i(r_j)\, dx\, dt \Big| \\
&\le \int_0^T \sum_j \Delta x^{3/2} \Big(\int_{x_{j-1/2}}^{x_{j+1/2}} (\psi_x(x,t))^2 \, dx \Big)^{1/2} \left\| q_i' \right\|_{L^\infty} \left| D^- r_j \right| \, dt \\
&\le C \int_0^T \Big(\sum_j \Delta x \int_{x_{j-1/2}}^{x_{j+1/2}} (\psi_x(x,t))^2 \, dx \Big)^{1/2} \Big(\Delta x^2 \sum_j \left(D^- r_j \right)^2 \Big)^{1/2} dt \\
&\le C \sqrt{\Delta x} \Big(\iint_\Omega (\psi_x(x,t))^2 \, dx dt \Big)^{1/2} \Big(\int_0^T \Delta x \sum_j \Delta x \left(D^- r_j \right)^2 \, dt \Big)^{1/2} \\
&\le C \sqrt{\Delta x} \left\| \psi \right\|_{H^1(\Omega)} .
\end{aligned}
$$

- Observe

$$\frac{d}{dt}\eta_1(u_j) + D^-(q_1(r_j)) + e_{1,j} = 0,$$

where

$$f(r) = r\phi(r), \quad q_1(r) = f(r) - f(k) \text{ and}$$

$$e_{1,j} = \phi_{j-1}\Delta x \left(D^- u_j\right)^T \frac{1}{r_{j-1/2}} \left(I - \frac{u_{j-1/2} \otimes u_{j-1/2}}{r_{j-1/2}^2} \right) \left(D^- u_j\right).$$

- Also

$$\frac{d}{dt}r_j + f'(r_j)D^- r_j - \frac{\Delta x}{2} f''\left(r_{j-1/2}\right)\left(D^- r_j\right)^2 + e_{1,j} = 0.$$

Contd..

- Rewrite

$$\frac{d}{dt}\eta_2(r_j) + D^- q_2(r_j) \\ + \frac{\Delta x}{2} q_2''(r_{j-1/2}) \left(D^- r_j\right)^2 - f'(r_j)\,(e_{2,j} - e_{1,j}) = 0.$$

- where

$$e_{2,j} = \frac{\Delta x}{2} f''\left(r_{j-1/2}\right) \left(D^- r_j\right)^2.$$

- Using $\int_0^T \Delta x \sum_j \Delta x \left|D^- u_j\right|^2 \leq C$, one can show that e_1 and e_2 are in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

Convergence

- There exists a subsequence of $\{\Delta x\}$ (not relabeled) and a function r such that $r_{\Delta x} \to r$ a.e. $(x,t) \in \Omega$. We also have that $r \in C([0,T]; L^1(\mathbb{R}))$. Furthermore, r satisfies

$$\begin{cases} r_t + f(r)_x \leq 0, & x \in \mathbb{R},\ t > 0, \\ r = |u_0|, & x \in \mathbb{R},\ t = 0, \end{cases}$$

in the distributional sense.

BV estimate

- Define

$$\tau_j^{(i)} = \frac{u_j^{(i)} - (u_j, e)}{(u_j, e)},$$

if $u_j \neq 0$. If $u_j = 0$ set $\tau_j^{(i)} = \tau_{j+1}^{(i)}$.

- Observe

$$D^- \tau_j^{(i)} = \frac{\left(D^- u_j^{(i)}\right)(u_j, e) - u_j^{(i)} (D^- u_j, e)}{(u_j, e)(u_{j-1}, e)}.$$

- We have

$$\frac{d}{dt} \tau_j^{(i)} = -\phi_{j-1} \frac{(u_{j-1}, e)}{(u_j, e)} D^- \tau_j^{(i)}.$$

Lemma

- If

$$\left|\tau_{\Delta x}^{(i)}(\cdot,0)\right|_{B.V.(\mathbb{R})} \leq C$$

for some constant C which is independent of Δx, then there is a subsequence of $\{\Delta x\}$ (not relabeled) and functions $\tau^{(i)}$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\tau_{\Delta x}^{(i)}(\cdot,t) \to \tau^{(i)}(\cdot,t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof

- Set $\theta_j = D^- \tau_j^{(i)}$.
- Then θ_j satisfies

$$\frac{d}{dt}\theta_j + \lambda_{j-1} D^- \theta_j + \theta_j D^- \lambda_j = 0.$$

- Let $\eta_\alpha(\theta)$ be a smooth approximation to $|\theta|$ such that

$$\eta_\alpha''(\theta) \geq 0 \quad \text{and} \quad \lim_{\alpha\to 0} \eta_\alpha(\theta) = \lim_{\alpha\to 0} \left(\theta \eta_\alpha'(\theta)\right) = |\theta|\,.$$

- Hence

$$\frac{d}{dt}\eta_\alpha(\theta_j) + D^- \left(\lambda_j \eta_\alpha(\theta_j)\right) \leq \left(\eta_\alpha(\theta_j) - \theta_j \eta_\alpha'(\theta_j)\right) D^- \lambda_j.$$

Contd..

- Now let $\alpha \to 0$ to obtain

$$\frac{d}{dt}\left|\theta_j\right| + D^{-}\left(\lambda_j \left|\theta_j\right|\right) \leq 0.$$

- Hence

$$\left|\tau^{(i)}_{\Delta x}(\cdot, t)\right|_{B.V} \leq \left|\tau^{(i)}_{\Delta x}(\cdot, 0)\right|_{B.V.} \leq C$$

- $\tau^{(i)}_{\Delta x}(\cdot, t)$ is also L^1_{loc} Lipschitz continuous in t.
- Observe

$$u^{(i)}_{\Delta x} = r_{\Delta x} \sin\left(\varphi^{(i)}_{\Delta x}\right),$$

where

$$\varphi^{(i)}_{\Delta x} = \tan^{-1}\left(\tau^{(i)}_{\Delta x}\right)$$

and $\tau \mapsto \tan^{-1}(\tau)$ is a continuous function.

Main Theorem

- Theorem

  Under some assumptions, let $u_{\Delta x}$ *be defined by the scheme. Let* $|u_0| \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and*

$$\left|\tau_{\Delta x}^{(i)}\right|_{B.V.(\mathbb{R})} \leq C \quad \textit{for } i = 1, \ldots, n$$

  and C *is independent of* Δx*, then there exists a function* u *in* $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ *such that* $u_{\Delta x} \to u$ *as* $\Delta x \to 0$*. The function* u *is a weak solution to the original system.*

Fully discrete scheme

- Consider:

$$D_t^+ u_j^n + D^- \left(u_j^n \phi \left(\left| u_j^n \right| \right) \right) = 0,$$

with initial values

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) \, dx.$$

- Suppose $u_0 \in L^2(\mathbb{R})^n \cap L^\infty(\mathbb{R})^n$, and

$$\frac{\Delta t}{\Delta x} \leq \min\left\{\frac{1}{\|f'\|_{L^\infty}}, \frac{1}{C_\phi\left(1+\|u_0\|_{L^\infty(\mathbb{R})}\right)^2}\right\},$$

hold. Then

$$\|u_{\Delta x}(\cdot, t_n)\|^2_{L^2(\mathbb{R})^n} \leq \|u_0\|^2_{L^2(\mathbb{R})^n},$$
$$\|u_{\Delta x}(\cdot, t_n)\|^2_{L^\infty(\mathbb{R})^n} \leq \|u_0\|^2_{L^\infty(\mathbb{R})^n},$$

for all $n > 0$, furthermore

$$\Delta t \Delta x \sum_{n=0, j\in\mathbb{Z}}^{N-1} \Delta x \left|D^- u_j^n\right|^2 \leq 2\|u_0\|^2_{L^2(\mathbb{R})^n},$$

where $N = \mathrm{ceil}(T/\Delta t)$.

Strong convergence of $R_{\Delta x} := r_{\Delta x}^2$

- Suppose

$$\lim_{r \downarrow 0} \frac{\phi'(r)}{r} < \infty$$

holds and define $G(R) = g(\sqrt{R})$. Then there is a subsequence of $\{R_\Delta x\}$ (not relabeled) and a function R such that $R_{\Delta x} \to R$ a.e. $(x,t) \in \Omega$. We have that $R \in L^\infty([0,T]; L^1(\mathbb{R}))$. Furthermore, R satisfies

$$\begin{cases} R_t + G(R)_x \leq 0, & x \in \mathbb{R},\ t > 0, \\ R = |u_0|^2, & x \in \mathbb{R},\ t = 0, \end{cases}$$

in the distributional sense.

Conclusion

- Limit u is a distributional solution of

$$u_t + (u\phi(r))_x = 0.$$

- To conclude that u is a weak solution one has to show that $|u| = r$.
- One can show $|u| \leq r$.
- Lack of BV estimate!

Entropy solution

- A bounded measurable function $u(x,t)$ is called an entropy solution if
  - For all test functions $\psi \in C_0^\infty(\mathbb{R} \times [0,\infty))$

$$\iint_{\mathbb{R}\times\mathbb{R}^+} u\psi_t + u\phi(|u|)\psi_x \, dxdt + \int_{\mathbb{R}} u_0\psi(x,0)\, dx = 0,$$

  - $r = |u|$ is an entropy solution (in the sense of Kružkov) of the scalar conservation law

$$\begin{cases} r_t + (r\phi(r))_x = 0, & t > 0, \\ r(x,0) = |u_0(x)|. \end{cases}$$

Numerical Method

- Define $v \in S^{n-1}$ by $vr = u$.
- Extended system

$$
\begin{aligned}
r_t + (r\phi(r))_x &= 0 \\
(rv)_t + (r\phi(r)v)_x &= 0 \\
v_t + \phi(r)v_x &= 0.
\end{aligned}
$$

- To show that $u = rv$ is an entropy solution, one must show (for the approximations) that the limit of $|v| = 1$ if $|v_0| = 1$.

A scheme which enforces the entropy condition

- Consider:

$$\begin{cases} r_j^{n+1} = r_j^n - \Delta t D^- f_j^n, & n \geq 0, \\ r_j^0 = \left| u_j^0 \right|, \end{cases}$$

and

$$\begin{cases} w_j^{n+1} = w_j^n - \Delta t \phi_j^n D^- w_j^n, & n \geq 0, \\ r_j^0 w_j^0 = u_j^0. \end{cases}$$

Properties of $r_{\Delta x}$

- Assume that the CFL condition holds and $r_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then for each $\Delta x > 0$ we have that
  (a) $-M \leq r_{\Delta x}(x,t) \leq M$, for all x and $t > 0$.
  (b) For $n \geq 0$ the functions

$$n \mapsto \Delta x \sum_{j\in\mathbb{Z}} \left| r_j^n \right|, \quad n \mapsto \sum_{j\in\mathbb{Z}} \left| r_j^n - r_{j-1}^n \right|, \quad n \mapsto \sum_{j\in\mathbb{Z}} \left| r_j^{n+1} - r_j^n \right|$$

  are non-increasing. In particular this means that the family $\{r_{\Delta x}\}_{\Delta x > 0}$ is (uniformly in Δx) bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R})) \cap \mathrm{BV}(\mathbb{R} \times \mathbb{R}^+)$.
  (c) Moreover $r_\Delta x(\cdot, t) \to r(\cdot, t)$ strongly in $L^1(\mathbb{R})$ for all $t \geq 0$, where $r \in \mathrm{Lip}([0,T]; L^1(\mathbb{R}))$ and is the unique entropy (in the sense of Kružkov) solution of the conservation law

$$\begin{cases} r_t + f(r)_x = 0, \\ r(x,0) = \left| u(x,0) \right|. \end{cases}$$

- For w:

$$w_j^{n+1} = \left(1 - \lambda \phi_j^n\right) w_j^n + \lambda \phi_j^n w_{j-1}^n.$$

- Hence

$$\inf_j w_j^0 \leq w_j^n \leq \sup_j w_j^0, \quad n > 0.$$

and

$$|w_\Delta x(\cdot, t)|_{B.V.(\mathbb{R})^n} \leq |w_{\Delta x}(\cdot, 0)|_{B.V.(\mathbb{R})^n}.$$

- Furthermore

$$\begin{aligned} \Delta x \sum_j \left| w_j^{n+1} - w_j^n \right| &\leq \Delta t \, \|\phi\|_{L^\infty} \sum_j \left| w_j^n - w_{j-1}^n \right| \\ &\leq C \Delta t \, |w_{\Delta x}(\cdot, t_n)|_{B.V.(\mathbb{R})^n}. \end{aligned}$$

- Hence the map $t \mapsto w_{\Delta x}(\cdot, t)$ is L^1- Lipschitz continuous.

- Multiply the r equation with that w equation to get

$$\begin{aligned} r_j^{n+1} w_j^{n+1} = r_j^n w_j^n - \Delta t D^- \left(f_j^n w_j^n \right) \\ + \Delta t \left(f_j^n - f_{j-1}^n \right) D^- w_j^n \left(\lambda \phi_j^n - 1 \right). \end{aligned}$$

- Then we have

$$\begin{aligned} D_t^+ \left(r_j^n w_j^n \right) + D^- \left(f_j^n w_j^n \right) &= \Delta t \left(\lambda \phi_j^n - 1 \right) \left(f_j^n - f_{j-1}^n \right) D^- w_j^n \\ &=: e_j^n. \end{aligned}$$

- Multiply the above equation by a test function ψ and integrate over Ω.

- One can show (r, rw) is a weak solution to

$$\begin{aligned} r_t + f(r)_x &= 0, \\ (r\,w)_t + (\phi(r) r\,w)_x &= 0. \end{aligned}$$

- Regarding the right hand side:

$$\begin{aligned} &\Big| \sum_{n,j} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} e_j^n \psi \, dxdt \Big| \\ &\leq \Delta t \, \|\psi\|_{L^\infty(\Omega)} \left(\lambda \, \|\phi\|_{L^\infty} + 1 \right) \Delta t \sum_{n,j} \left| f_j^n - f_{j-1}^n \right| \left| w_j^n - w_{j-1}^n \right| \\ &\leq \Delta t C \, \|\psi\|_{L^\infty(\mathbb{R})} \, \|w_\Delta x(x,0)\|_{L^\infty(\mathbb{R})^n} \, T \, \|r_{\Delta x}(x,0)\|_{B.V.(\mathbb{R})} \, . \end{aligned}$$

Wagner transformation

- There is a one-to-one correspondence between bounded Lebesgue measurable solutions $(r, r\theta)$ to the system

$$
\begin{aligned}
r_t + (r\phi(r))_x &= 0, \\
(r\,\theta)_t + (\phi(r) r\,\theta)_x &= 0.
\end{aligned}
$$

  and weak solutions $(\tau, \tilde{\theta})$ to the system

$$
\begin{aligned}
\tau_t - (\phi(1/\tilde{\tau}))_y &= 0, \\
\tilde{\theta}_t &= 0
\end{aligned}
$$

- where

$$
\tilde{\theta} = \theta \circ T^{-1}.
$$

Contd..

- Transformation: $T : (x, t) \to (y(x, t), t)$ defined by

$$\frac{\partial y}{\partial x}(x, t) = r(x, t),$$
$$\frac{\partial y}{\partial t}(x, t) = -\phi(r(x, t))r(x, t),$$
$$y(0, 0) = 0.$$

- Same argument concludes $|\theta| = 1$.
- Assume that $u_0 \in B.V.(\mathbb{R})$. If $\lambda = \Delta t/\Delta x$ satisfies the CFL-condition $\lambda < \sup_x f'(|u_0(x)|)$, and $u_\Delta x$ is defined by scheme, then $u = \lim_{\Delta x \to 0} u_\Delta x$ is the unique entropy solution.

Thanks !