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Extensions for systems of conservation laws.

(Conservation laws with prescribed eigenframe.)

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Hyperbolic system of conservation laws

$$u_t + f(u)_x = 0\,. \tag{1}$$

- n equations on n unknown functions $u(x,t) \in \Omega \subset \mathbb{R}^n$.
- one space-variable $x \in \mathbb{R}$; one time-variable: $t \in \mathbb{R}$.
- $f(u) : \Omega \longrightarrow \mathbb{R}^n$ smooth flux.
- $\forall u \in \Omega$ the Jacobian $D_u f$ is diagonalizable over reals

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- $\lambda^1(u), \ldots, \lambda^n(u)$ are eigenfunctions of $D_u f$.
- $(R_1(u), \ldots, R_n(u))$ are eigen vectors of $D_u f$, independent $\forall u \in \Omega$

$\mathcal{R} = \Big(R_1(u), \ldots, R_n(u)\Big)$ is called an eigenframe

Conservation laws with prescribed eigenframe

What properties of $u_t + f(u)_x = 0$ are determined by an eigenframe?

Two specific problems: Given a frame $\mathcal{R}$:

1. find all fluxes f for which $\mathcal{R}$ is an eigenframe.
2. find companion conservation laws (extensions) that all systems $u_t + f(u)_x = 0$ with this eigenframe possess.

For both problems:

How does the degree of freedom depend on the properties of $\mathcal{R}$?

Problem 1: Fluxes with prescribed eigenframe

Given:

(i) an affine coordinate system $u = (u_1, \ldots, u_n)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

(ii) a frame $\mathcal{R} = (R_1, \ldots, R_n)$ on Ω.

Find all: vector valued maps $f = (f^1, \ldots, f^n) \colon \Omega \to \mathbb{R}^n$, s. t.$\forall \bar{u} \in \Omega$ $R_1|_{\bar{u}}, \ldots, R_n|_{\bar{u}}$ are right eigenvectors of the Jacobian matrix $D_u f(\bar{u})$.

Equivalently. Let $R = [R_1(u) \,|\, \cdots \,|\, R_n(u)]$ and $L(u) := R(u)^{-1}$.

<u>Find all</u> $\Lambda(u) := \mathrm{diag}[\lambda^1(u), \ldots, \lambda^n(u)]$,

such that $\boxed{J(u) := R(u)\Lambda(u)L(u)}$ is the Jacobian matrix of some map $f : \mathcal{U} \to \mathbb{R}^n$.

$\boxed{(\lambda^1(u), \ldots, \lambda^n(u)) \text{ are eigenfunctions of } J = D_u f.}$

Problem 1: Fluxes with prescribed eigenframe

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(i) an affine coordinate system $u = (u_1, \ldots, u_n)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

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is the Jacobian matrix of some map $f : \mathcal{U} \to \mathbb{R}^n$.

Jacobian problem

$\boxed{(\lambda^1(u), \ldots, \lambda^n(u)) \text{ are eigenfunctions of } J = D_u f.}$

Examples

- $\forall\mathcal{R}$ constant eigenfunctions $\lambda^1(u) = \cdots = \lambda^n(u) = \bar\lambda \in \mathbb{R}$ work.

$$R(u)\,\bar\Lambda\,L(u) = \bar\Lambda = Df \quad \text{for} \quad f = \bar\lambda u + \bar v, \quad \bar v \in \mathbb{R}^n.$$

- $\exists\mathcal{R}$ with only constant eigenfunctions:

$$R_1 = [u_1,\, u_2,\, 0]^T,\; R_2 = [-u_2,\, u_1,\, 0]^T,\; R_3 = [-u_2,\, u_1,\, 1]^T.$$

- $\exists\mathcal{R}$ with large family of possible eigenvalues:

$$R_1 = [u_1,\, u_2,\, 0]^T, \quad R_2 = [-u_2,\, u_1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

$\big(\lambda^1(u),\, \lambda^2(u)\,, \lambda^3(u)\big)$ depend on 3 arbitrary functions of 1 variable.

- $\exists\mathcal{R}$ with small family of possible eigenvalues:

$$R_1 = [-1,\, 0,\, u_2{+}1]^T,\; R_2 = \left[\frac{u_3}{(u_2)^2 - 1},\, -1,\, u_1\right]^T,\; R_3 = [1,\, 0,\, 1{-}u_2]^T$$

$\big(\lambda^1(u),\, \lambda^2(u)\,, \lambda^3(u)\big)$ depend on 2 constants:

$$\lambda^1 = -C_1 + C_2, \qquad \lambda^2 = C_1\, u_2 + C_2 \qquad \lambda^3 = C_1 + C_2\,.$$

Given a frame, can we predict the degree of freedom for prescribing eigenfunctions? Strategy

- Write down a system of differential equations for $\lambda^1(u), \ldots, \lambda^n(u)$.
- Use integrability theorems (Frobenius, Darboux and Cartan-Kähler) to determine the degree of freedom of the general solution.

Differential system for $\left(\lambda^1(u), \ldots, \lambda^n(u)\right)$

- A matrix $J(u) = (J^i_j(u))$ is a Jacobian iff

$$\frac{\partial J^i_j(u)}{\partial u_k} = \frac{\partial J^i_k(u)}{\partial u_j} \text{ for all } i, j, k = 1, \ldots, n \text{ with } j < k\,,$$

- $J(u) = R(u)\Lambda(u)L(u)$ is a Jacobian iff

$$\sum_{m=1}^{n} \left[B^i_{mj}\partial_k\lambda^m - B^i_{mk}\partial_j\lambda^m + \lambda^m\left(\partial_k B^i_{mj} - \partial_j B^i_{mk}\right)\right] = 0\,,$$
$$i, j, k = 1, \ldots, n \text{ with } j < k,$$

where $B^i_{mj}(u) := R^i_m(u)L^m_j(u), \qquad \partial_i = \frac{\partial}{\partial u_i}$

- A system of $\frac{n^2(n-1)}{2}$ linear PDE's on λ's.
- $n = 1$ the system is empty; $n = 2$ determined compatible system (Dafermos (1995)) also covered by our analysis of rich frames.
- For $n > 2$ this is an overdetermined system. How many arbitrary function and constant are in general solution?

An equivalent differential-algebraic system (the $\lambda(\mathcal{R})$-system) *

A vector field can be identified with a linear first order differential operator

$$R_i = \begin{pmatrix} R_i^1 \\ \vdots \\ R_i^n \end{pmatrix} \quad \leftrightarrow \quad r_i := R_i(u)^1 \frac{\partial}{\partial u_1} + \cdots + R_i^n(u) \frac{\partial}{\partial u_n}$$

$n(n-1)$ linear, 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ algebraic equations:

$$\begin{cases} r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j) & i \neq j, \qquad (\lambda(\mathcal{R})\text{-diff}) \\ \Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k) & i < j,\ i \neq k,\ j \neq k \quad (\lambda(\mathcal{R})\text{-alg}), \end{cases}$$

where $\boxed{\Gamma^k_{ij} := L^k(D_u R_j) R_i\,.}$

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*In different contexts the λ-system appeared in Sévannec (1994), Tsarëv (1985).

An advantage the $\lambda(\mathcal{R})$-system

$$\begin{cases} r_i(\lambda^j) = \Gamma^j_{ji}(\lambda^i - \lambda^j) & i \neq j, \qquad (\lambda(\mathcal{R})\text{-diff}) \\ \Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k) & i < j,\, i \neq k,\, j \neq k \quad (\lambda(\mathcal{R})\text{-alg}). \end{cases}$$

Γ^k_{ij} are the Christoffel symbols of the trivial connection $\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = 0$ computed relative to the frame $\mathcal{R}$, i.e. $\nabla_{r_i} r_j = \sum_{k=1}^{n} \Gamma^k_{ij} r_k$.

- Underlying geometry helps to check compatibility conditions:
  1. symmetry $\Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij}$, where c^k_{ij} are structure coefficients:*

     $$[r_i, r_j] := r_i \circ r_j - r_j \circ r_i = \sum_{k=1}^{n} c^k_{ij}\, r_k.$$

  2. flatness $r_m\left(\Gamma^j_{ki}\right) - r_k\left(\Gamma^j_{mi}\right) = \sum_{s=1}^{n} \left(\Gamma^j_{ks}\Gamma^s_{mi} - \Gamma^j_{ms}\Gamma^s_{ki} - c^s_{km}\Gamma^j_{si}\right).$
- Analysis of the $(\lambda(\mathcal{R})\text{-alg})$ provides a starting point for the classification.

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* c^k_{ij} are also called interaction coefficients.

The rank of the algebraic part:

$$\Gamma^k_{ji}(\lambda^i - \lambda^k) = \Gamma^k_{ij}(\lambda^j - \lambda^k), \quad i < j,\ i \neq k,\ j \neq k \qquad (\lambda(\mathcal{R})\text{-alg})$$

$$0 \leq \text{rank}(\lambda(\mathcal{R})\text{-alg}) \leq (n-1).$$

Extreme cases:

- $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = (n-1) \quad \Rightarrow \lambda^1(u) = \cdots = \lambda^n(u) = \bar{\lambda} \in \mathbb{R}$
  (trivial case)

- $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0 \quad \Longleftrightarrow \quad \Gamma^k_{ji} = 0,\ \forall i, j, k$ distinct $\Rightarrow \ c^k_{ji} = 0$
  $\forall i, j, k$ distinct (rich case)*

- only trivial solutions $\not\Rightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = n - 1\,.$

- $\mathcal{R}$ is rich $\not\Rightarrow$ $\text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0\,.$

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*equivalently, $[r_i, r_j] \in \text{span}\{r_i, r_j\}$.

Rich frame with empty $\lambda(\mathcal{R})$-alg

General solution of the $\lambda(\mathcal{R})$ system depends on n functions of one variable

There are strictly hyperbolic conservative systems with eigenframe $\mathcal{R}$.

- all $n = 2$ frames belong to this case.
- rich orthogonal frames belong to this case.

Example: (frame defines cylindrical coordinates)

$$R_1 = [u_1,\, u_2,\, 0]^T, \quad R_2 = [-u_2,\, u_1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

$$\lambda^1 = F_1(v), \quad \lambda^2 = \frac{1}{\sqrt{v}} \int_*^{\sqrt{v}} F_1(\tau^2)\, d\tau + \frac{1}{u_1} F_2\left(\frac{u_2}{u_1}\right), \quad \lambda^3 = F_3(u_3),$$

where $v = (u_1)^2 + (u_2)^2$ and F_1, F_2 and F_3 are arbitrary functions of 1 variable.

Rich system with non-trivial $\lambda(\mathcal{R})$-alg

- $\lambda(\mathcal{R}) \quad \Rightarrow$ multiplicity conditions on eigenvalues $\Rightarrow$

  $\nexists$ strictly hyperbolic systems with eigenframe $\mathcal{R}$

- General solution of $\lambda(\mathcal{R})$ depends on $s < n$ functions of one variable,

  where s is the number of distinct eigenvalues of multiplicity 1,

Example

$R_1 := (u_1, u_2, u_3)^T, \qquad R_2 = (u_1, u_2, 0)^T, \qquad R_3 = (u_1, 0, u_3)^T.$

$$\lambda^1 = F\left(\frac{u_3\, u_2}{u_1}\right) + \frac{u_3\, u_2}{u_1} F'\left(\frac{u_3\, u_2}{u_1}\right), \quad \lambda^2 = \lambda^3 = F\left(\frac{u_3\, u_2}{u_1}\right),$$

where F is an arbitrary function of 1 variable.

$\lambda(\mathcal{R})$-system for arbitrary frames $n = 3$

I. rank($\lambda(\mathcal{R})$-alg) $= 0 \quad \Rightarrow \quad \mathcal{R}$ is rich.

The general solution of $\lambda(\mathfrak{R})$ depends on 3 functions of one variable.
$\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$.

II. rank($\lambda(\mathcal{R})$-alg) $= 1$ (a single algebraic constraint):

IIa. All three λ^i appear in the algebraic constraint $\quad \Rightarrow \quad \mathcal{R}$ is not rich.

The solution of $\lambda(\mathcal{R})$ is either trivial or depends on 2 arbitrary constants
In the latter case,
$\exists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$

IIb. Exactly two λ^i appear in the algebraic constraint $\Rightarrow$ two λ^i coincide;
$\Rightarrow \quad \nexists$ strictly hyperbolic conservative system with eigenframe $\mathcal{R}$.
The general solution is either trivial or depends on 1 arbitrary function of 1 variables. $\mathcal{R}$ may or may not be rich

III. rank($\lambda(\mathcal{R})$-alg) $= 2 \Rightarrow$ only trivial solutions $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \bar{\lambda} \in \mathbb{R}$.

Extensions and entropies:

Definition: a smooth function $\eta\colon \Omega \to \mathbb{R}$ is called an extension of

$$u_t + f(u)_x = 0$$

if $\exists$ smooth function $q\colon \Omega \to \mathbb{R}$ such that $\boxed{\operatorname{grad} q = \operatorname{grad} \eta\,(D_u f)}$.

Proposition:

- $\eta(u)_t + q(u)_x = 0$, whenever u is a smooth solution of $u_t + f(u)_x = 0$.

- η is an extension iff:

  for each pair $1 \le i \ne j \le n$: $\boxed{\lambda^j = \lambda^i \text{ or } R_i^T \left(D_u^2 \eta\right) R_j = 0}$.

An extension η is called an entropy if $D_u^2 \eta$ is positive semidefinite and is called strict entropy if $D_u^2 \eta$ is positive definite.

Problem 2: Extensions with prescribed eigenframe

Given:

(i) an affine coordinate system $u = (u_1, \ldots, u_n)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

(ii) a frame $\mathcal{R} = (R_1, \ldots, R_n)$ on Ω.

Find all: functions $\eta \colon \Omega \to \mathbb{R}$, such that $\boxed{R_i^T \left(D_u^2 \eta\right) R_j = 0}$

(frame $\mathcal{R}$ is orthogonal with respect to the inner product defined by the Hessian matrix $D_u^2 \eta$).

Equivalently. Let $R = [R_1(u) \mid \cdots \mid R_n(u)]$ and $L(u) := R(u)^{-1}$.

Find all $\mathcal{B}(u) := \mathrm{diag}[\beta^1(u), \ldots, \beta^n(u)]$

such that $\boxed{H(u) := L^T(u)\mathcal{B}(u)L(u)}$ is the Hessian matrix of some function $\eta \colon \Omega \to \mathbb{R}$.

β^i is the "length" of R_i relative to the inner product H.

Problem 2: Extensions with prescribed eigenframe

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(i) an affine coordinate system $u = (u_1, \ldots, u_n)$ on an open, smoothly contractible to a point $\Omega \subset \mathbb{R}^n$;

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Equivalently. Let $R = [R_1(u) \mid \cdots \mid R_n(u)]$ and $L(u) := R(u)^{-1}$.

Find all $\mathcal{B}(u) := \mathrm{diag}[\beta^1(u), \ldots, \beta^n(u)]$

Hessian problem

such that $\boxed{H(u) := L^T(u)\mathcal{B}(u)L(u)}$ is the Hessian matrix of some function $\eta \colon \Omega \to \mathbb{R}$.

β^i is the "length" of R_i relative to the inner product H.

Comparison of Jacobian and Hessian problems:

Given: an invertible at each point of Ω $n \times n$-matrix $L(u)$.

Jacobian problem: Find all possible $\Lambda = \mathrm{diag}[\lambda^1, \dots, \lambda^n]$, s.t.

$\boxed{J(u) = L^{-1}(u)\,\Lambda(u)\,L(u)}$ is a Jacobian matrix.

Hessian problem: Find all possible $\mathcal{B} = \mathrm{diag}[\beta^1, \dots, \beta^n]$, s.t.

$\boxed{H(u) = L^{T}(u)\,\mathcal{B}(u)\,L(u)}$ is a Jacobian matrix.*

For orthonormal frames: Hessian problem = Jacobian problem. For orthogonal frames: Hessian problem $\sim$ Jacobian problem.

*on a smoothly contractible to a point domain, a symmetric matrix is a Hessian if and only if it is a Jacobian.

The $\beta(\mathcal{R})$-system *

$n(n-1)$ linear 1st order PDEs and $\frac{n(n-1)(n-2)}{2}$ linear algebraic equations.

$$\begin{cases} r_i(\beta^j) = \beta^j\,(\Gamma^j_{ij} + c^j_{ij}) - \beta^i\,\Gamma^i_{jj} & i \neq j, \quad \beta(\mathcal{R})\text{-diff} \\ \beta^k\,c^k_{ij} + \beta^j \Gamma^j_{ik} - \beta^i\,\Gamma^i_{jk} = 0 & i < j,\ i \neq k,\ j \neq k \quad \beta(\mathcal{R})\text{-alg} \end{cases}$$

The rank of $\beta(\mathcal{R})$-alg:

- $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 0 \iff \text{rank}(\lambda(\mathcal{R})\text{-alg}) = 0 \iff$ (rich frame, admitting strictly hyperbolic conservative systems).

  Well known!

  The general solution of the β-system depends on n arbitrary functions of one variable.

- $\text{rank}(\beta(\mathcal{R})\text{-alg}) = \text{rank}(\lambda(\mathcal{R})\text{-alg})$ for $n \leq 3$.

- in general $\text{rank}(\beta(\mathcal{R})\text{-alg}) \neq \text{rank}(\lambda(\mathcal{R})\text{-alg})$ for $n > 3$.

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*In a different context the β-system appeared in Conlon and Liu (1981)

$\beta(\mathcal{R})$-system for $n = 3$

$$0 \leq \text{rank}(\beta(\mathcal{R})\text{-alg}) = \text{rank}(\lambda(\mathcal{R})\text{-alg}) \leq 2.$$

I. $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 0 \Rightarrow \mathcal{R}$ is rich; a general solution of $\beta(\mathfrak{R})$ depends on 3 functions of 1 variable; $\exists$ hyperbolic conservative system with eigenframe $\mathcal{R}$, each of them posses strict entropies.

II. $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 1$ (a single algebraic constraint): see Jenssen and Kogan *Communications in PDE's* (2012)

III. $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 2 \Rightarrow \text{rank}(\lambda(\mathcal{R})\text{-alg}) = 2 \Rightarrow$ only trivial solutions of $\lambda(\mathcal{R})$: $\lambda^1(u) = \lambda^2(u) = \lambda^3(u) = \bar{\lambda} \in \mathbb{R}$ with trivial fluxes $f = \bar{\lambda} u + \bar{v}$, where $\bar{v} \in \mathbb{R}^n$.

The size of the solution of $\beta(\mathcal{R})$-system may vary, but any function η is an extension because the first condition is satisfied:

$$\forall i \neq j: \quad \boxed{\lambda^j = \lambda^i \ \text{ or } \ R_i^T \left(D_u^2 \eta\right) R_j = 0}.$$

Some consequences of the classification (n=3)

A frame $\mathcal{R}$ admits strictly hyperbolic conservation laws in two cases

1. $\mathcal{R}$ is rich and $\Gamma^k_{ij} = 0$ for all i, j, k distinct. The families of fluxes and extensions depend on 3 arbitrary functions of 1 variables.

2. $\mathcal{R}$ is not rich and $\lambda(\mathcal{R})$-alg is equivalent to $\alpha_1(\lambda^1 - \lambda^3) + \alpha_2(\lambda^2 - \lambda^3) = 0$, where $0 \neq \alpha_1 = -\alpha_2 \neq 0$.*

   The family of fluxes for a given $\mathcal{R}$ depends on 2 constants and a vector:

   $$\{c_1 f + c_2 u + \bar{v} \,|\, c_1, c_2 \in \mathbb{R}, \bar{v} \in \mathbb{R}^n\}.$$

   The family of extensions depend on at most 2 arbitrary functions of 1 variable.

   If strict entropies exist then the degree of freedom is at most 1 arbitrary functions of 1 variable (Euler system for 1 dimensional compressible flow).

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*This condition can be expressed in terms of Γ^k_{ij}.

Summary of the results

Jacobian problem – we provide a classification of “the number” of conservative systems with the given eigenframe

- for rich frames (n is arbitrary)
- arbitrary frames for $n = 3$.

Hessian problem – classification of “the number” of extensions that are admitted by all conservative systems with the given eigenframe due to orthogonality condition $R_i^T \left(D_u^2 \eta \right) R_j = 0$:

- the answer for rich frames admitting strictly hyperbolic conservative system was well known.
- we provide a classification for arbitrary frames for $n = 3$.

For orthogonal frames Hessian problem $\sim$ Jacobian problem.

Maple code with examples.

http://www.math.ncsu.edu/~iakogan/symbolic/geometry_of_conservation_laws.html

The talk is based on:

1. Jenssen, H. K., Kogan, I. A., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010) pp. 211– 254.

2. Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140

More questions for the future ...

- What properties of the solutions of $u_t + f(u)_x = 0$ are encoded in other "geometric" objects associated with the system (e.g. Hugoniot locus)? Geometric classification of hyperbolic conservation laws?

- Could we use geometric properties for carrying out more or less structured search of systems that admit amplitude or total variation blowup of the solutions in finite time? *

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*see Young (1999), Jenssen(2000), Baiti and Jenssen (2001), Jenssen and Young (2004)

Thank you! †

†Additional slides follow.

$\beta(\mathcal{R})$-system when $n = 3$ and $\text{rank}(\beta(\mathcal{R})\text{-alg}) = 1$. (complete classification)

(1) Only the trivial solution: $\beta^1 = \beta^2 = \beta^3 \equiv 0$ ($\exists$ rich frames in this class)

(2) Exactly two β^i are zero and the third depends on 1 arbitrary function of one variable. ($\exists$ rich frames in this class)

(3) Exactly one β^i is zero and the other two β^i depend on

- **(3a)** 2 arbitrary functions of one variable. ($\exists$ rich frames in this class)
- **(3b)** 1 common arbitrary constant.

(4) There are non-degenerate solutions (all β^i are non-zero) depending on

- **(4a)** 1 arbitrary function of one variable and 1 arbitrary constant.
- **(4b)** 2 arbitrary constants.
- **(4c)** 1 arbitrary constant.

The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

$v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $\boxed{p(v,S) > 0}$ is pressure as a given function of v and S, s.t $\boxed{p_v < 0}$.

- $U_t + f(U)_x = 0$, where $U = (v,u,S)$ and $f(U) = (-u, p(v,S), 0)^T$.

- eigenvalues of Df are $\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}$.

- eigenvectors of D_f are $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $R_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

Inverse problem: Coordinates $U = (v, u, S)$

- For a given pressure function $p = p(v, S) > 0$, with $p_v < 0$ define a frame $\mathcal{R}$:

  $R_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T,\ R_2 = [\,-p_S,\ 0,\ p_v\,]^T,\ R_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

- determine the class of conservative systems with eigenfields $\mathcal{R}$ by solving the λ-system for λ^1, λ^2, λ^3.

- Observation: frame is rich $\quad\Leftrightarrow\quad \left(\frac{p_S}{p_v}\right)_v \equiv 0 \Leftrightarrow p(v, S) = \Pi(v + F(S))$.

Solution of the $\lambda(\mathcal{R})$-system:

in the non-rich case:

- $\lambda(\mathcal{R})$-alg consists of:

$$\frac{p_v}{4}\left(\frac{p_S}{p_v}\right)_v(\lambda^1+\lambda^3-2\lambda^2)=0 \Leftrightarrow \lambda^2=\frac{1}{2}(\lambda^1+\lambda^3)$$

  that involves all three λ's (case IIa) $\Rightarrow$ the general solution depends on two constants.

- from the differential part of λ-system we obtain:

$$\lambda^1=-C_1\sqrt{-p_v}+C_2\,,\quad \lambda^2=C_2\,,\quad \lambda^3=C_1\sqrt{-p_v}+C_2\,.$$

$$f(U)=C_1\begin{pmatrix}-u\\ p(v,S)\\ 0\end{pmatrix}+C_2\begin{pmatrix}v\\ u\\ S\end{pmatrix}+\bar{v}.$$

Solution of the $\beta(\mathcal{R})$-system:

in the non-rich case:

- $\beta(\mathcal{R})$-alg consists of:

$$\left(\frac{p_S}{p_v}\right)_v \left(\beta^1 - \beta^3\right) = 0 \Leftrightarrow \beta^1 = \beta^3$$

- The general solution depends on 1 function of 1 variable and 1 constant (K_2 can be absorbed into arbitrary function):

$$\begin{aligned} \beta^1 &= \beta^3 = K_1\, p_v\,, \\ \beta^2 &= \frac{K_1 p_v^2}{2} \left(\int_{K_2}^{v} p_{SS}(\tau, S)\, d\tau - \frac{p_S^2}{p_v}(v, S) + F(S) \right)\,. \end{aligned}$$

- $\exists$ strict entropies.

Blowup example

$$R_1 = [-1,\, 0,\, u_2 + 1]^T\,, \qquad R_2 = \left[\frac{u_3}{(u_2)^2 - 1},\, -1,\, u_1\right]^T,$$
$$R_3 = [1,\, 0,\, 1 - u_2]^T\,.$$

- non-rich frame
- $\mathsf{rank}(\lambda(\mathcal{R})\text{-alg}) = \mathsf{rank}(\beta(\mathcal{R})\text{-alg}) = 1$

Solution of the λ-system

$\lambda(\mathcal{R})$-alg: $2\lambda^2 = (1 - u_2)\lambda^1 + (1 + u_2)\lambda^3$ involves all three λ's (case IIa)
$\Rightarrow$ the general solution depends on two constants:

$$\lambda^1 = -C_1 + C_2, \qquad \lambda^2 = C_1\, u_2 + C_2 \qquad \lambda^3 = C_1 + C_2\,.$$

fluxes:

$$f(u) = C_1 \begin{pmatrix} u_1\, u_2 + u_3 \\ \frac{1}{2}\,(u_2)^2 \\ u_1 - u_1\,(u_2)^2 - u_2\, u_3 \end{pmatrix} + C_2 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \bar{v}$$

Solution of the β-system

$\beta(\mathcal{R})$-alg: $(u_2 - 1)\beta^1 = (u_2 + 1)\beta^3$ the general solution depends on one arbitrary function of one variable:

$$\beta^1 = 0, \qquad \beta^2 = F(u_2), \qquad \beta^3 = 0,$$

extensions (modulo affine parts):

$$\eta(u_1, u_2, u_3) = G(u_2)\,, \qquad \text{where } G'' = F.$$

Rich orthogonal frame

$$R_1 = [u_1,\, u_2,\, 0]^T, \quad R_2 = [-u_2,\, u_1,\, 0]^T, \quad R_3 = [0,\, 0,\, 1]^T.$$

$\lambda(\mathcal{R})$-alg and $\beta(\mathcal{R})$-alg are empty $\Rightarrow$ general solutions of $\lambda(\mathcal{R})$ and $\beta(\mathcal{R})$ depend on 3 arbitrary functions of 1 variable:

$$\lambda^1 = F_1(v), \qquad \lambda^2 = \frac{1}{\sqrt{v}} \int_*^{\sqrt{v}} F_1(\tau_2)\, d\tau + \frac{1}{u_1} F_2\left(\frac{u_2}{u_1}\right),$$

$$\lambda^3 = F_3(u_3);$$

$$\beta^1 = v\, G_1(v), \qquad \beta^2 = \sqrt{v} \int_*^{\sqrt{v}} G_1(\tau_2)\, d\tau + u_1\, G_2\left(\frac{u_2}{u_1}\right),$$

$$\beta^3 = G_3(u_3), \qquad \text{where } v = v = (u_1)^2 + (u_2)^2.$$

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any solution of λ-system can be combined with any solution of β-system.