Smoothing effect and Fredholm property for first-order hyperbolic PDEs

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Padova, 28 June 2012

The problem under consideration

$$\begin{aligned}
\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x,t)u &= f_j(x,t),\ j \le n,\ (x,t) \in (0,1) \times \mathbb{R} \\
u_j(x,0) &= \varphi_j(x), \qquad j \le n,\ x \in [0,1] \\
u_j(0,t) &= (Ru)_j(t), \quad 1 \le j \le m,\ t \in \mathbb{R}_+ \\
u_j(1,t) &= (Ru)_j(t), \quad m < j \le n,\ t \in \mathbb{R}_+
\end{aligned}$$

Motivation: traveling-wave models from

- laser dynamics
- population dynamics
- chemical kinetics

Smoothing, fredholmness and hyperbolic PDEs

- **Motivation**
  - bifurcation analysis of traveling-wave models
  - local investigations of semilinear and quasilinear hyperbolic problems via
    * Implicit Function Theorem
    * Lyapunov-Schmidt reduction
  - periodic synhronizations via
    * averaging procedure

- **Complications and related topics**
  - bad regularity properties of hyperbolic operator:
    * no smoothing effect in general
    * lost-of-smoothness effect
  - small denominators
  - non-strict hyperbolicity and constraints on lower-order terms

Examples

1. Smoothing effect. Heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u(x,0) = \varphi(x)$$

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(\frac{-|x-y|^2}{4t}\right) \varphi(y)\, dy$$

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2. No smoothing effect. Wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$u = \frac{\varphi(x-t) + \varphi(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(s)\, ds$$

Smoothing property (definition)

Definition.

A solution u is called smoothing if for any k there exists $T > 0$ such that u is C^k-smooth above the line $t = T$.

Remark.

Any t-periodic and smoothing solution immediately meets the C^∞-regularity in the entire domain.

Remark.

The domain of influence of the initial conditions is in general infinite and is determined by the boundary operator as well as the lower-order terms .

Smoothing effect I: Classical boundary conditions

$$
\begin{aligned}
\partial_t u_j + a_j(x,t)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x,t)u &= f_j(x,t),\ j \le n,\ (x,t) \in (0,1) \times \mathbb{R} \\
u_j(x,0) &= \varphi_j(x), \quad j \le n, \qquad x \in [0,1] \\
u_j(0,t) &= h_j(t), \quad 1 \le j \le m, \quad t \in \mathbb{R}_+ \\
u_j(1,t) &= h_j(t), \quad m < j \le n, \quad t \in \mathbb{R}_+
\end{aligned}
$$

Assumptions:

$a_j > 0$ for all $j \le m$ and $a_j < 0$ for all $j > m$,

$\inf_{x,t} |a_j| > 0$ for all $j \le n$,

for all $1 \le j \ne k \le n$ there exists $p_{jk} \in C^1([0,1] \times \mathbb{R})$
such that $b_{jk} = p_{jk}(a_k - a_j)$ and $p_{jk} = 0$
in the interior of the domain $\{(x,t) : a_j(x,t) = a_k(x,t)\}$.

Smoothing effect I: main result

Theorem. Assume that $a_j, b_{jk}, f_j, h_j \in C^\infty$ and $\varphi_j \in C$. Then any continuous solution is smoothing.

Operator representation of the problem:

$$u = CBu + Du + Ff,$$

where $(Cu)_j(x,t) = c_j(x_j,x,t)u_j\left(x_j, \omega_j(x_j;x,t)\right),$

$$c_j(\xi,x,t) = \exp\int_x^\xi \left(\frac{b_{jj}}{a_j}\right)(\eta,\omega_j(\eta;x,t))\,d\eta,$$

$$(Bu)_j(x,t) = \begin{cases} h_j(t) & \text{if } t > 0, \\ \varphi_j(x) & \text{if } t = 0, \end{cases}$$

$$(Du)_j(x,t) = \int_x^{x_j} \frac{c_j(\xi,x,t)}{a_j(\xi,\omega_j(\xi;x,t))} \sum_{k\neq j} \Big(b_{jk}u_k\Big)(\xi,\omega_j(\xi;x,t))d\xi$$

Proof idea

- **1-st step:** $u = CBu + Du + Ff$.

  Hence $u = (CBu + DCBu) + D^2u + (I + D)Ff$.

  If $T_1 > 0$ is sufficiently large, then

  $$u|_{t>T_1} = (C + DC)h + D^2u + (I + D)Ff.$$

  The operator D^2 maps $C(\Pi)^n$ continuously into $C_t^1(\Pi \cap t > T_1)^n$.

- **2-nd step:** Set $v = \partial_t u$. Then

  $$(\partial_t + a_j\partial_x)v_j + \sum_{k=1}^{n} b_{jk}v_k - \frac{\partial_t a_j}{a_j}v_j = G_j(f_j, \partial_t f_j, u)$$

  and we have the representation $v|_{\overline{\Pi}_{T_2}} = \tilde{C}h' + \tilde{D}v + \tilde{F}G(f, \partial_t f, u)$.

- Further we use **induction** on the order of regularity.

Smoothing effect II: Integral boundary conditions in population models

$$
\begin{aligned}
(\partial_t + \partial_x + \mu)u &= 0, \quad (x,t) \in (0,1) \times \mathbb{R}, \\
u(x,0) &= \varphi(x), \qquad x \in [0,1], \\
u(0,t) &= h\left(\int_0^1 \gamma(x) u(x,t)\, dx\right), \quad t \in \mathbb{R},
\end{aligned}
$$

Theorem. Assume that $h, \gamma \in C^\infty$ and $\varphi \in C$. Then any continuous solution is smoothing.

Proof idea

1-st step: Any continuous solution for large enough T_1 is given by the formula

$$
\begin{aligned}
u|_{t\geq T_1} &= CBu = CBCu \\
&= e^{-\mu x} h\left(\int_0^1 \gamma(\xi) e^{-\mu\xi} u(0, t-x-\xi)\, d\xi\right) \\
&= e^{-\mu x} h\left(\int_{t-x-1}^{t-x} \gamma(t-x-\tau) e^{\mu(x-t+\tau)} u(0,\tau)\, d\tau\right),
\end{aligned}
$$

2-nd step: Set $v = u_t$. Then for large enough T_2 we have

$$
\begin{aligned}
v|_{t\geq T_2} &= e^{-\mu x} h'\left(\int_0^1 \gamma(\xi) u(\xi, t-x)\, d\xi\right) \int_0^1 \gamma(\xi) e^{-\mu\xi} v(0, t-x-\xi)\, d\xi \\
&= e^{-\mu x} h'\left(\int_0^1 \gamma(\xi) u(\xi, t-x)\, d\xi\right) \int_{t-x-1}^{t-x} \gamma(t-x-\tau) e^{\mu(x-t+\tau)} v(0,\tau)\, d\tau.
\end{aligned}
$$

Smoothing effect III: Dissipative boundary conditions and periodic problems

$$\begin{aligned} \partial_t u + a(x,t)\partial_x u + b(x,t)u &= f(x,t), \quad (x,t) \in (0,1) \times \mathbb{R} \\ u(x,t+2\pi) &= u(x,t), \quad x \in [0,1] \\ u_j(0,t) &= h_j(z(t)), \quad 1 \le j \le m, \ t \in \mathbb{R}_+ \\ u_j(1,t) &= h_j(z(t)), \quad m < j \le n, \ t \in \mathbb{R}_+ \end{aligned}$$

with $z(t) = (u_1(1,t), \dots, u_m(1,t), u_{m+1}(0,t), \dots, u_n(0,t))$.

Theorem. Assume that $a_j, b_{jk}, f_j, h_j \in C^\infty$ and a_j, b_{jk}, f_j are 2π-periodic in t. If

$$\left|\partial_k h_j'(z)\right| \exp\left\{\int_x^{x_j} \left(\frac{b_{jj}}{a_j} - l\frac{\partial_t a_j}{a_j^2}\right)(\eta, \omega_j(\eta; x,t))\, d\eta\right\} < 1$$

for all $j,k \le n$, $x \in [0,1]$, $t \in \mathbb{R}$, $z \in \mathbb{R}^n$, and $l = 0,1,\dots,r$, then any continuous solution belongs to C^r.

Proof idea

- $u = Cu + Du + Ff, \quad u = CBu + Du + Ff \quad \Rightarrow$

$$u = CBu + (DC + D^2)u + (I + D)Ff$$

- The operator $I - CB \in \mathcal{L}\,(C_t^1)^n$ is bijective $\quad \Rightarrow$

$$u = (I - CB)^{-1}\Big[(DC + D^2)u + (I + D)Ff\Big]$$

- The operator $DC + D^2$ is smoothing (maps C into C_t^1)

Fredholmness: a problem from laser dynamics

$$\partial_t u_j + a_j(x)\partial_x u_j + \sum_{k=1}^{n} b_{jk}(x)u_k = f_j(x,t),\ j \le n,\ (x,t) \in (0,1) \times \mathbb{R}$$

$$\begin{aligned} u_j(x,t+2\pi) &= u_j(x,t), && j \le n,\ (x,t) \in [0,1] \times \mathbb{R} \\ u_j(0,t) &= \sum_{k=m+1}^{n} r^0_{jk} u_k(0,t), && j \le m,\ t \in \mathbb{R} \\ u_j(1,t) &= \sum_{k=1}^{m} r^1_{jk} u_k(1,t), && m+1 \le j \le n,\ t \in \mathbb{R}. \end{aligned}$$

Choice of function spaces

$$(u, f) \in (V^\gamma, W^\gamma)$$

Spaces for the right-hand sides:

$$W^\gamma = L^2\left(0,1; H^\gamma_{per}(\mathbb{R})\right)^n$$

Spaces for the solutions:

$$U^\gamma(a) = \left\{ u \in W^\gamma : \partial_x u \in W^{\gamma-1}, [\partial_t u_j + a_j \partial_x u_j]_{j=1}^n \in W^\gamma \right\}$$

$$\|u\|^2_{U^\gamma(a)} = \|u\|^2_{W^\gamma} + \left\| [\partial_t u_j + a_j \partial_x u_j]_{j=1}^n \right\|^2_{W^\gamma}.$$

$$V^\gamma(a) = \left\{ u \in U^\gamma(a) : u \text{ fulfills refl. bound. conditions } \right\}$$

Properties of the solution spaces

Lemma.

(i) The space $U^\gamma(\omega, a)$ is complete.

(ii) If $\gamma \geq 1$, then for any $x \in [0,1]$ there exists a continuous trace map $u \in U^\gamma(\omega, a) \mapsto u(x, \cdot) \in L^2\left((0, 2\pi); \mathbb{R}^n\right)$.

(iii) If $\gamma > 3/2$, then $U^\gamma(\omega, a)$ is continuously embedded into $C([0,1] \times [0, 2\pi]; \mathbb{R}^n)$.

Operator representation of the problem

$$\mathcal{A}(a, b^0)u + \mathcal{B}(b^1)u = f,$$

where $\mathcal{A}(a, b^0) \in \mathcal{L}(V^\gamma(a); W^\gamma)$ is defined by

$$\mathcal{A}(a, b^0)u = \left[\partial_t u + a_j \partial_x u_j + b_{jj} u_j\right]_{j=1}^n ,$$

and $\mathcal{B}(b^1) \in \mathcal{L}(W^\gamma)$ is defined by

$$\mathcal{B}(b^1)u = \left[\sum_{j \neq k} b_{jk} u_k\right]_{j=1}^n$$

Fredholm property

Theorem. (K., Recke, 2012) Assume that

$$a_j, b_{jj} \in L^\infty(0,1), \qquad \inf_{0\le x\le 1} |a_j(x)| > 0 \text{ for all } j \le n,$$

$$\left.\begin{array}{l}\text{for all } j \ne k \text{ there is } c_{jk} \in BV(0,1) \text{ such that} \\ b_{jk}(x) = c_{jk}(x)(a_j(x) - a_k(x)) \text{ for a.a. } x \in [0,1],\end{array}\right\}$$

$$R(a, b^0) \sum_{j,l=m+1}^{n} \sum_{k=1}^{m} |r^1_{jk} r^0_{kl}| < 1.$$

Then

(i) The operator $\mathcal{A}+\mathcal{B}$ is a Fredholm operator with index zero from $V^\gamma(a)$ into W^γ for all $\gamma \ge 1$.

(ii) (smoothing effect) The subspaces $\ker(\mathcal{A}+\mathcal{B})$ and $\ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$ do not depend on γ, $\ker(\mathcal{A}+\mathcal{B})^* = \ker(\tilde{\mathcal{A}}+\tilde{\mathcal{B}})$, and

$$im(\mathcal{A}+\mathcal{B}) = \left\{ f \in W^\gamma : \langle f, u\rangle_{L^2} = 0 \text{ for all } u \in \ker\left(\tilde{\mathcal{A}}+\tilde{\mathcal{B}}\right)\right\}.$$

Fredholmness and non-strict hyperbolicity

Fredholmness for non-strict hyperbolicity necessarily entails constraints on zero-order terms:

$$\begin{aligned} \partial_t u_1 + \partial_x u_1 = \partial_t u_2 + \partial_x u_2 + b u_1 &= 0, \\ u_1(x, t + 2\pi) - u_1(x, t) = u_2(x, t + 2\pi) - u_2(x, t) &= 0, \\ u_1(0, t) - r_0 u_2(0, t) = u_2(1, t) - r_1 u_1(1, t) &= 0. \end{aligned}$$

No small denominators: $r_0 r_1 < 1$. Assume that

$$b = \frac{r_0 r_1 - 1}{r_0},$$

covering arbitrarily large $|b|$. Then

$$u_1(x,t) = \sin l(t - x),\ u_2(x,t) = b\left(\frac{1}{1 - r_0 r_1} - x\right)\sin l(t - x),\ l \in \mathbb{N},$$

gives infinitely many linearly independent solutions. Hence, the Fredholmness is destroyed.

Fredholmness: proof idea

Theorem (Fredholmness criterion). Let W be a Banach space, I the identity in W, and $\mathcal{C} \in \mathcal{L}(W)$ with $\mathcal{C}^k$ being compact for some $k \in \mathbb{N}$. Then $I + \mathcal{C}$ is a Fredholm operator of index zero.

- $A \in \mathcal{L}(V^\gamma; W^\gamma)$ is an isomorphism, hence $A + B \in \mathcal{L}(V^\gamma; W^\gamma)$ is Fredholm iff $I + BA^{-1} \in \mathcal{L}(W^\gamma)$ is Fredholm.

- $(BA^{-1})^2 \in \mathcal{L}(W^\gamma)$ is compact.

- (right) parametrix or (right) regularizer: $A^{-1}(I - BA^{-1})$:

$$(A + B)\left[A^{-1}\left(I - BA^{-1}\right)\right] = I - \left(BA^{-1}\right)^2.$$

Thank you!