DECAY RATE OF DEGENERATE CONVECTION DIFFUSION EQUATIONS

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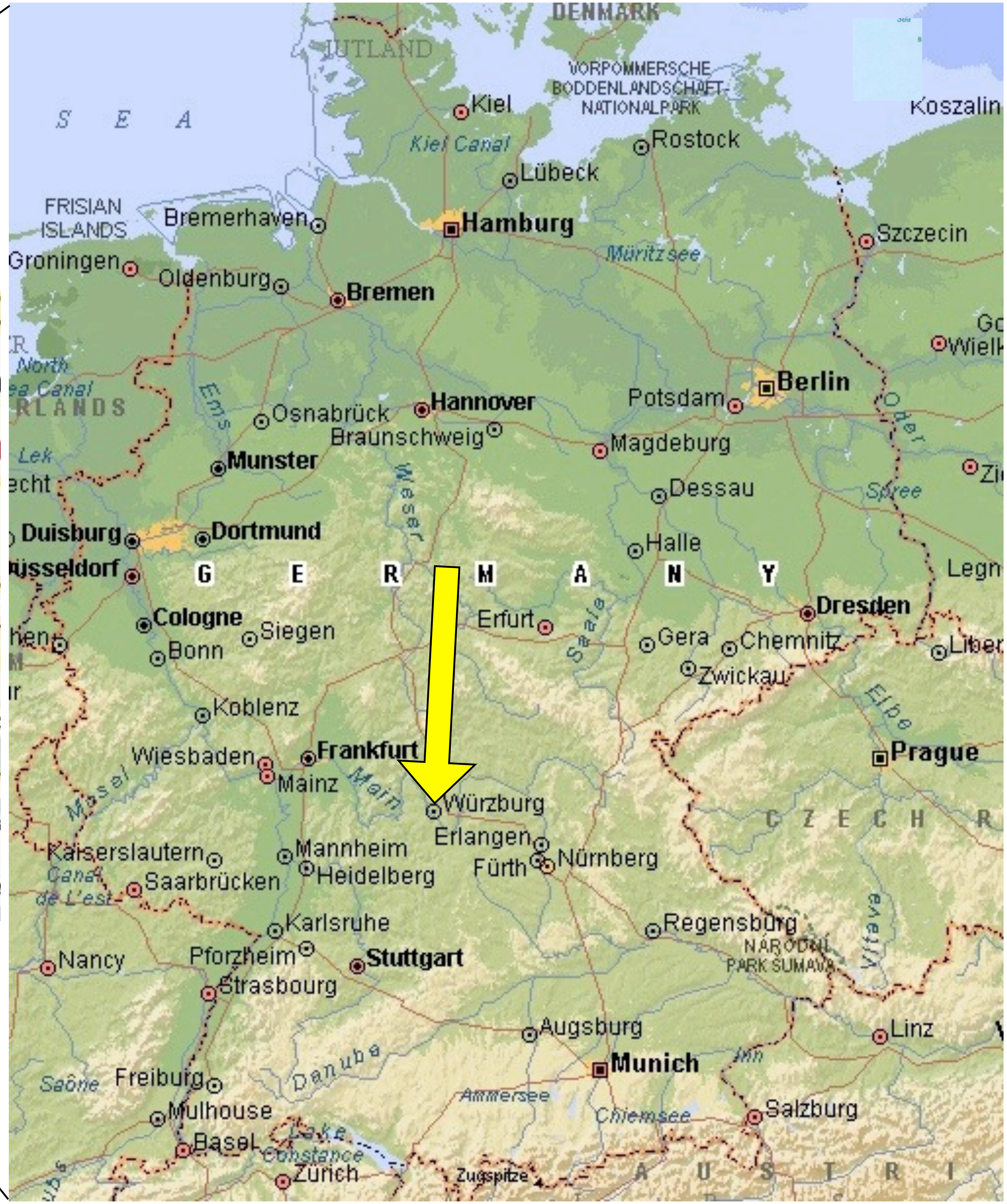
where is Würzburg?



As you all know, for the heat equation in one space dimension

$$u_t = u_{xx}$$

for an initial value probem with compact support

$$u(x,t) \quad \to \quad 0 \quad \text{as} \quad t \to \infty \qquad \text{with decay rate} \quad \frac{1}{\sqrt{t}}$$

pointwise

you can read it off the explicit solution formula:

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t > 0) \end{aligned}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}$$

Consider the large time behaviour of the initial value problem of the inviscid Burgers equation

$$u_t + (u^2)_x = 0 \qquad\qquad u(x,0) = u_0(x)$$

$u_0(x)$ periodic $\qquad u(x,t) \quad \longrightarrow \quad \bar{u}$ as $t \to \infty$ (mean value)

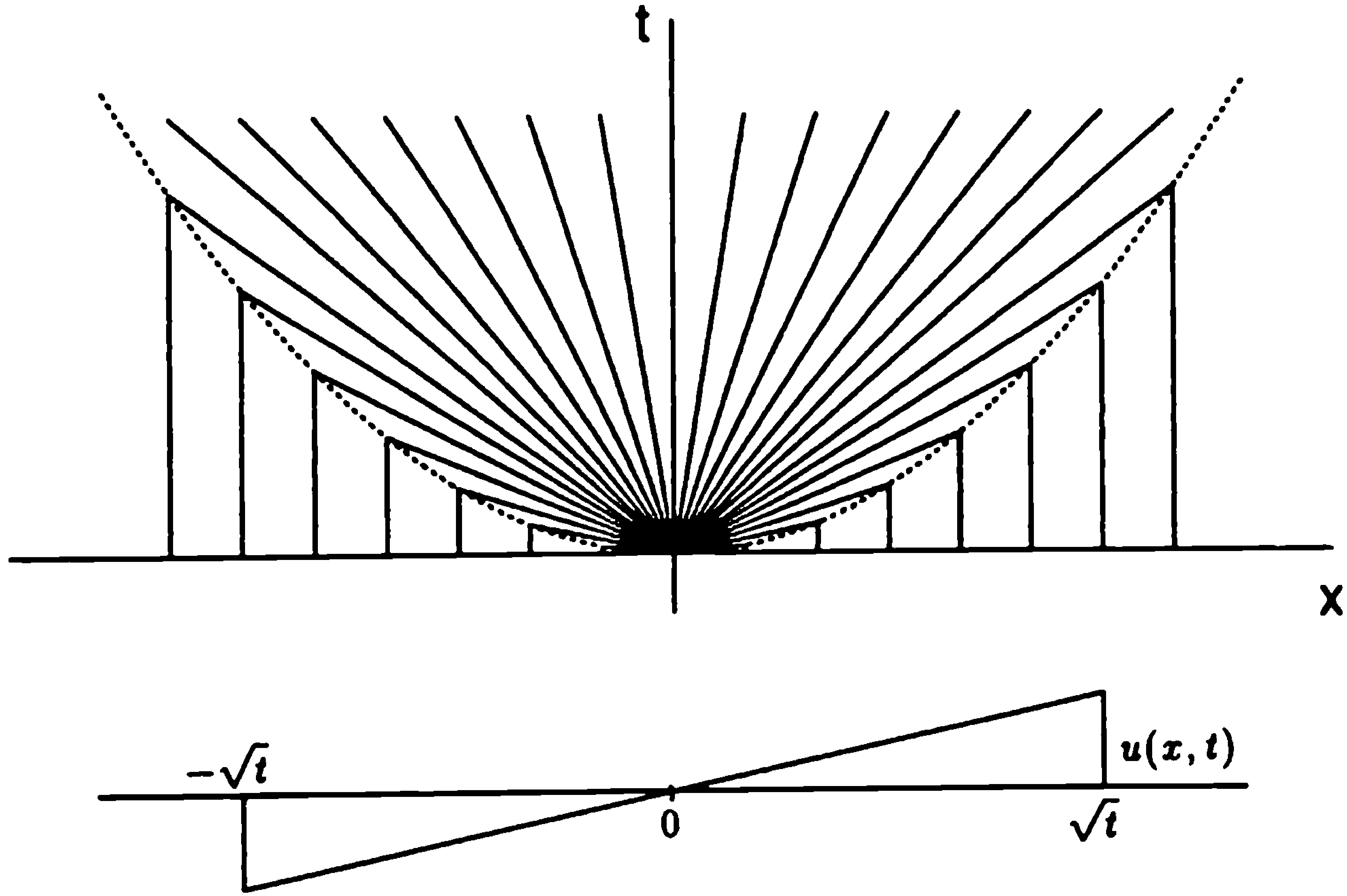
with decay rate $\dfrac{1}{t}$

$u_0(x)$ compactly supported
or $\quad u_0(x) \in L^1(\mathcal{R})$

$$u(x,t) \overset{L^1(\mathcal{R})}{\longrightarrow} 0 \quad \text{as} \quad t \to \infty$$

with decay rate $\dfrac{1}{\sqrt{t}}$

illustrating the effect of "energy dissipation" via the shock waves



N-wave solution to Burgers equation

moving on to the viscous Burgers equation

$$u_t + (u^2)_x = \nu u_{xx}$$

with compactly supported initial data

$$u(x,t) \quad \rightarrow \quad 0 \quad \text{as} \quad t \rightarrow \infty$$

pointwise

with decay rate $\frac{1}{\sqrt{t}}$

this can be seen this with help of the Cole-Hopf transformation resulting in

$$u(x,t) = -2\nu \frac{\partial}{\partial x} \ln \left\{ (4\pi\nu t)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-x')^2}{4\nu t} - \frac{1}{2\nu} \int_0^{x'} u(x'',0)dx'' \right] dx' \right\}.$$

here two time decay mechanisms are at play:

- *dissipation from the parabolic term*
- *"dissipation" from "within" the shocks coming from the nonlinear flux*

joining forces to the same decay rate

What happens if we use a more general flux but keep strictly positive diffusion?

consider a viscous conservation law with a non-Burgers flux

$$u_t + (u^q)_x = u_{xx} \qquad \text{for} \quad u_0(x) \in L^1(\mathcal{R})$$

the large time decay rate depending on exponent q:

for q > 2 $\qquad \|u(t)\|_{L^\infty} \leq K_\infty t^{-1/2}$

for 1 < q < 2 $\qquad \|u(t)\|_{L^\infty} \leq K_\infty t^{-1/q}$

faster than $\frac{1}{\sqrt{t}}$

case q = 2 is the viscous Burgers equation the last result also holds true without viscosity

$$u_t + u_x^q = 0 \qquad \text{for} \quad u_0(x) \in L^1(\mathcal{R})$$

for 1 < q < 2

we obtain $\dfrac{1}{t^q}$ decay in $L^1(\mathcal{R})$ *Ph. Laurencot 1998*

so this large time decay rate is faster than $\dfrac{1}{\sqrt{t}}$

How does this work if we use degenerate diffusion? $u(x,0) = u_0(x) \geq 0$

lets first look at the modified heat equation

$$u_t = (u^m)_{xx} \qquad \text{for} \quad u_0(x) \in L^1(\mathcal{R})$$

$$m \in \mathcal{N}$$

gives a large time decay rate:

$$\|u(t)\|_{L^\infty} \leq C_\infty t^{-1/(m+1)}$$

here C_∞ depends on m and initial data

say m=2 $\quad u_t = (u^2)_{xx} \quad u(x,t)$ decays like $\frac{1}{t^{\frac{1}{3}}}$ *slower than* $\frac{1}{\sqrt{t}}$

for m = 1 this is the heat equation result

summary:

$$u_t + (u^q)_x = u_{xx}$$

for q ≥ 2 $\frac{1}{\sqrt{t}}$

for 1 < q < 2 $\frac{1}{t^q}$

faster

$$u_t = (u^m)_{xx} \qquad m \in \mathcal{N}$$

$$\frac{1}{t^{m+1}}$$

slower

let us combine both degenerate diffusion and non-Burgers fluxes

We shall consider a scalar equation of the type

$$u_t + F(u, x, t)_x + H(u, x, t) = G(u)_{xx}$$

where G(u) may lead to a degenerate elliptic operator

Thus the solutions are not smooth, they are weak solutions.

our technique requires manipulating with the solutions as if they were smooth

we deal with degenerary of the parabolic term by considering a regularisation of type

$$u_t + f(u)_x = ((g(u) + \epsilon)u_x)_x \qquad g(u) \geq 0$$

this gives uniform parabolicity

one can get regularity estimates of the solution independent of epsilon

For

$$u_t + F(u,x,t)_x + H(u,x,t) = G(u)_{xx}$$

we identify conditions on F, H and G for which we obtain large time behaviour of the form

$$||u(x,t)||_{L^\infty} \leq \frac{C}{\sqrt{t}}$$

G(u) may be pointwise degenerate

This includes all the above cases and includes new one's.

our strategy

- find bounds for u of the type $u_x < M(u, x, t)$
- then obtain time decay for u

$$u(x,t) \leq \frac{C}{\sqrt{t}}$$

our result in more detail:

consider solution to

$$u_t + F(u,x,t)_x + H(u,x,t) = (g(u)u_x)_x \quad (x,t) \in \mathbb{R} \times (0,T)$$

where the coefficients of the PDE satisfy

$$F(u,x,t) = F_1(u,x,t) + F_2(u,x,t)$$

$$(F_1)_{uu} \geq 0, \quad (F_1)_{xu} \geq 0, \quad (F_1)_{xx} \geq 0, \quad f_2 \geq 0, \quad (f_2)_t \geq 0,$$
$$H_u f_2 - (f_2)_u H \geq 0, \quad (f_2)_u (F_1)_x - (F_1)_u (f_2)_x \leq 0.$$

$$f_2(u,x,t) := \frac{F_2(u,x,t)}{g(u)}$$

we conclude that the solution satisfies

$$u_x \leq f_2(u,x,t)$$

now suppose the solution to

$$u_t + F(u, x, t)_x + H(u, x, t) = (g(u)u_x)_x$$

where the conditions on the previous in particular satisfy

$$u_x \leq \frac{M}{t}$$

we can conclude

$$u(x, t) \leq \frac{M}{\sqrt{t}}$$

pointwise

sketch of the proof

1.) *gradient decay:* $u_x \leq f_2(u, x, t)$

here the bound is a combination of the coefficients of the PDE.

set $v = u_x - f_2(u, x, t)$

assume that at initial time this is negative

and then proceed to show that

$$v_t < 0$$

this uses properties of the PDE plus the assumptions on the coefficients we made

2.) *obtain a particular gradient decay of the solution in time*:

for our PDE we introduce the new variable

$$w = ut$$

we get an evolution for w

$$w_t - \frac{w}{t} + F(w,x)_x + H(w,x) = w_{xx}$$

applying our result in part 1.) to this equation we conclude

$$w_x < \text{const.} \qquad \text{because} \quad f_2 \leq \text{const.}$$

Hence $w_x = u_x t \leq C$ or $u_x(x,t) \leq \frac{C}{t}$

3.) *form this time decay of the gradient conclude decay of solution*

we use the fact that we are in one space dimension

$$fix \quad \overset{pos.}{t} \quad and \quad x_0 \quad \text{put } A = u(x_0, t) \qquad (\textit{we know } u_x(x,t) \leq \frac{C}{t})$$

$$u(x,t) \geq \frac{1}{\sqrt{t}}(x - x_1) \quad \text{for} \quad 0 \leq x - x_1 \leq At$$

pos. in the inetrval (x_1, x_0)

$$x_1 = x_0 - At$$

integrating gives

$$M = \int_{\mathbb{R}} u(x,t)\,dx \geq \int_0^{At} \frac{x}{t}\,dx = \frac{t \;\; A^2}{2}$$

which implies

$$u(x,t) \leq \frac{M}{\sqrt{t}}$$

Now we consider time decay of the solution to porous media type equations with degenerate diffusion in higher space dimensions

$$u_t = \Delta u^m + \sum_{i=1}^{N} f_i(u)_{x_i} - S(u)$$

non-negative initial data

$$u(x,0) = u_0(x_1, x_2, \cdots, x_N) \geq 0$$

Because of parabolic degeneracy the solution may fail to be smooth.

So we perturb both the initial data to be strictly positive

$$u^{\varepsilon} \geq \varepsilon$$

and also the equation to be strictly parabolic.

One can get uniform bounds on the solution independent of ε .

This allows us to manipulate the equation.

$$u_t = \Delta u^m + \sum_{i=1}^{N} f_i(u)_{x_i} - S(u)$$

under certain assumptions on the coefficients of this PDE

we can prove bounds on the gradient of the solution

$$(u^2)_{x_i} \leq \frac{M}{(1+t)^{\frac{\alpha}{2}}}$$

any $\alpha \geq 1$

An example for which this holds true is

$$u_t = \Delta u^m \qquad u(x,0) \geq 0$$

the proof consists of long calculations

the idea is to do several changes of variables

first change of variables: $v = u^n$

which gives an evolution of v

$$v_t = nu^{n-1}\Delta(u^m) + \sum_i f_i'(u)v_{x_i} - nu^{n-1}S(u)$$

then set

$$w = \frac{1}{2}\sum_{i=1}^{N} v_{x_i}^2$$

next change of variables

which gives an evolution of w

$$\begin{aligned} w_t &= 2h'(v)(\Delta v)w + h(v)\Delta w - \sum_{i,j} h(v)v_{x_i x_j}^2 \\ &+ \frac{2m(m-n)}{n} v^{(m-n-1)/n} \sum_i v_{x_i} w_{x_i} + \frac{4m(m-n)(m-n-1)}{n^2} v^{(m-2n-1)/n} w^2 \\ &+ \sum_j f_j'(u) w_{x_j} + \sum_j \frac{f_j''(u)}{n u^{n-1}} v_{x_j} w - 2(n-1)\frac{S(u)}{u} w - 2S'(u)w. \end{aligned}$$

then set

$$\theta = (w - \frac{1}{t^\alpha})$$

next change of variables

which gives an evolution of θ

$$\theta_t + (\frac{1}{t^\alpha})_t \leq 2h'(v)(\Delta v)\theta + h(v)\Delta\theta - cmv^{(m-2n-1)/n}\frac{1}{t^{2\alpha}}$$

$$+ \frac{2m(m-n)}{n}v^{(m-n-1)/n}\sum_i v_{x_i}\theta_{x_i} + \frac{4m(m-n)(m-n-1)}{n^2}v^{(m-2n-1)/n}\theta^2$$

applying the maximum principle and our assumptions gives the desired gradient bound on u lets summarize:

for degenerate convection diffusion equations in one space dimension

$$u_t + F(u,x,t)_x + H(u,x,t) = (g(u)u_x)_x$$

we identified a certain class for which we can obtain time decay

$$u(x,t) \leq \frac{M}{\sqrt{t}}$$

for several space dimensions

$$u_t = \Delta u^m + \sum_{i=1}^{N} f_i(u)_{x_i} - S(u)$$

we identified equations for which we can show time decay of the derivative

$$(u_{x_i})^2 \leq \frac{M}{t}$$

Thank you for your attention!

here are examples of the type of equations for which we can show time decay of this type

$$u_t + (u^3)_x = (u^2)_{xx}$$

$$u_t + (u^2 + \frac{u^2 + 1}{t})_x = u_{xx}$$