A Well-Balanced Multi-Dimensional Reconstruction Scheme for Hydrostatic Equilibria

Roger Käppeli

Seminar for Applied Mathematics SAM

ETH

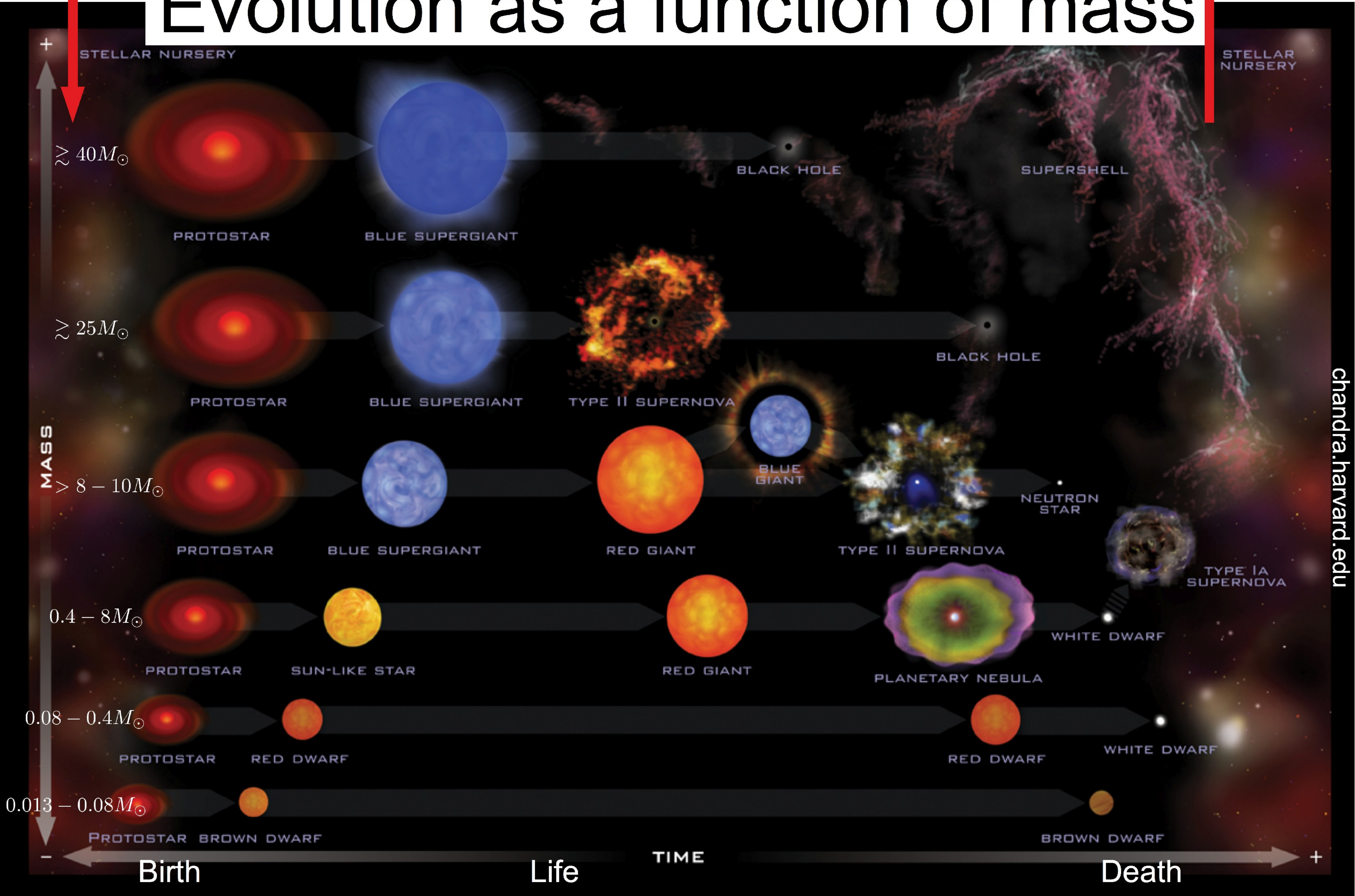
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Outline

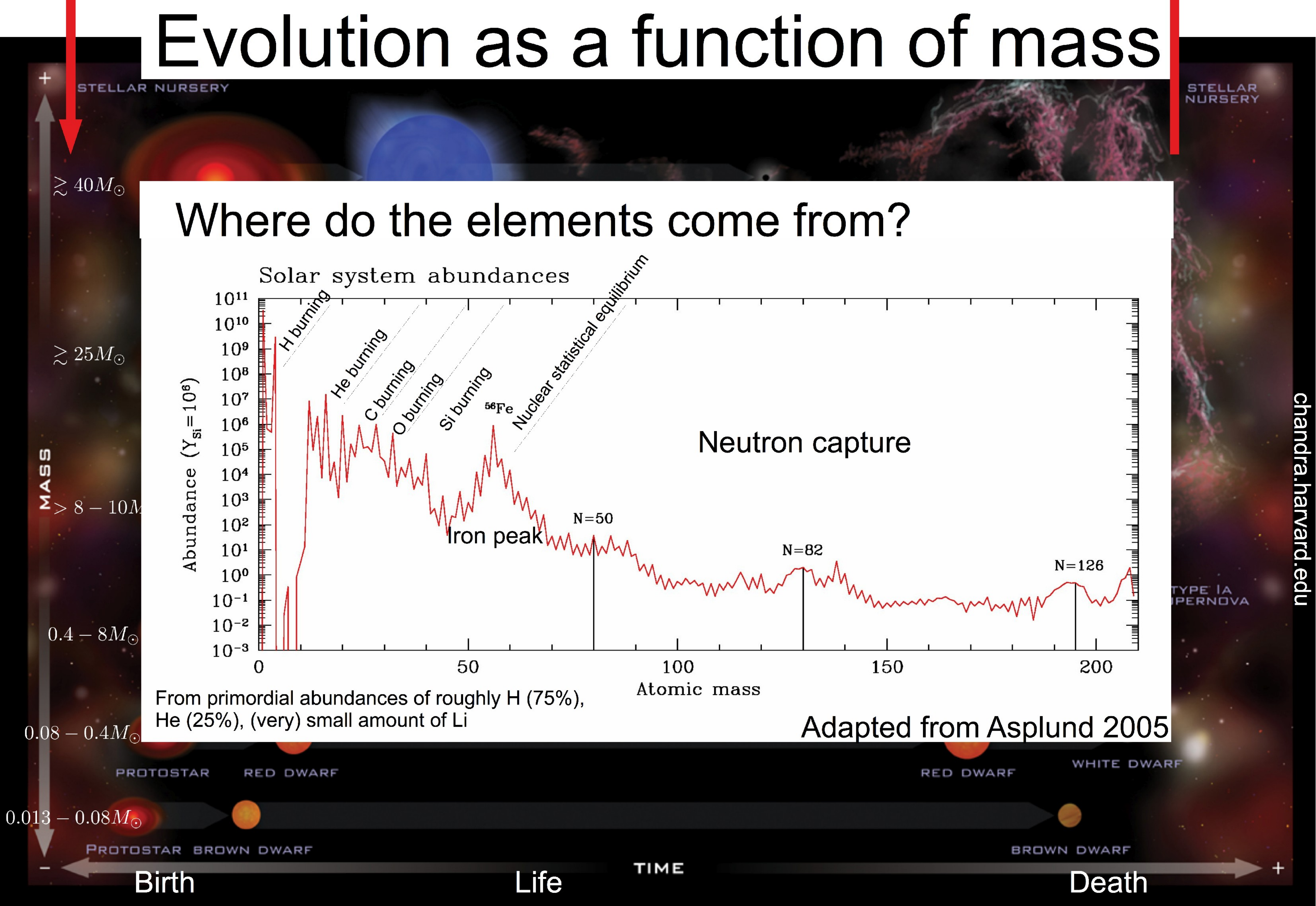
- **Introduction**
  - (Astro)Physical motivation
- **Well-balanced scheme for HydroStatic Equilibrium (HSE)**
  - First order
  - Second order
- **Multi-dimensional extension**
  - Limitations
- **Conclusion**

Stellar life cycle

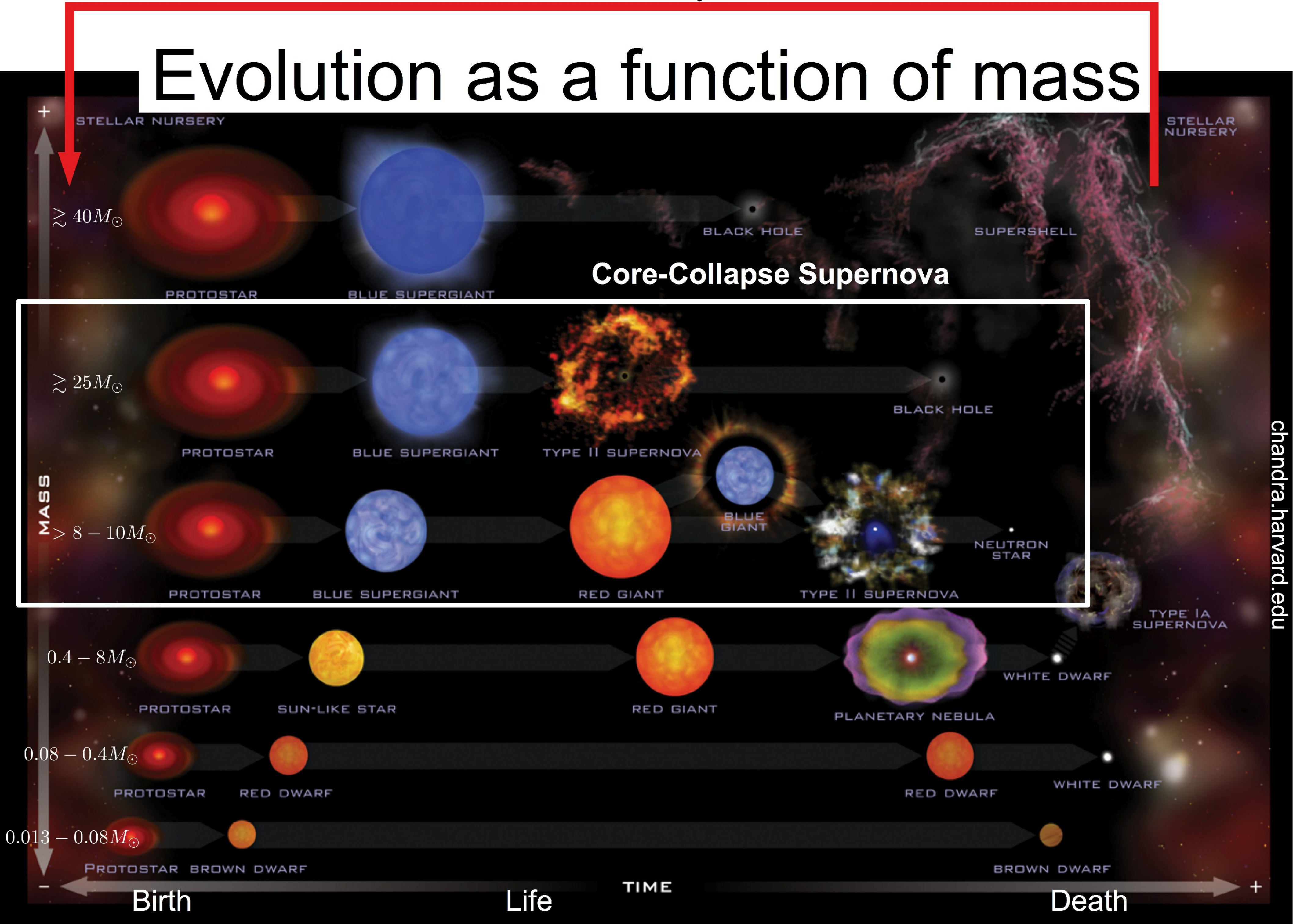
Evolution as a function of mass



Stellar life cycle



Stellar life cycle



Core-collapse supernova

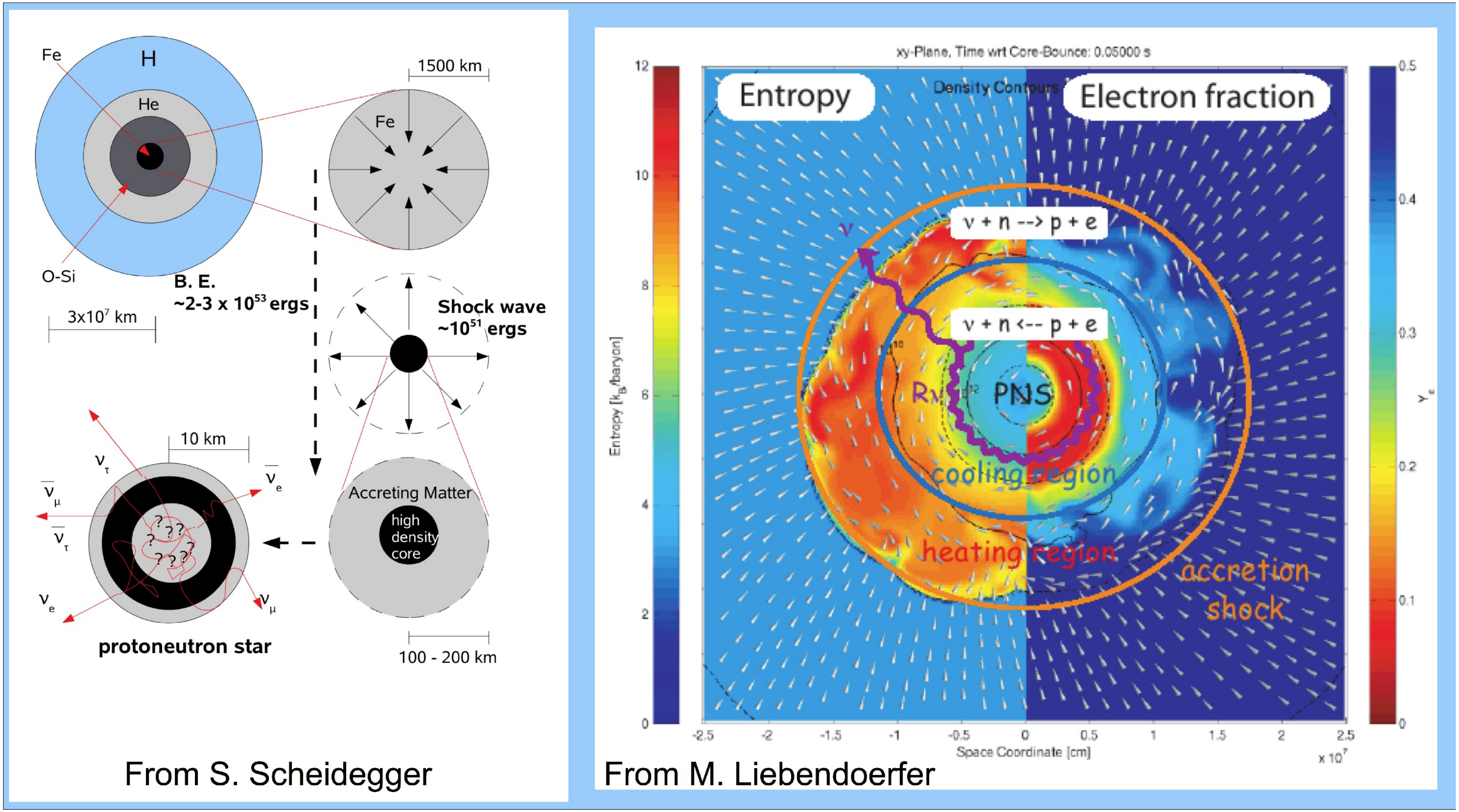
- General idea:
  - Implosion of iron core of massive $M \gtrsim 8M_\odot$ at the end of thermonuclear evolution
  - Explosion powered by gravitational binding energy of forming compact remnant:

$$E_b \approx 3 \times 10^{53} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{10\mathrm{km}}\right)^{-1} \mathrm{erg}$$

M Mass of remnant
R Radius of remnant

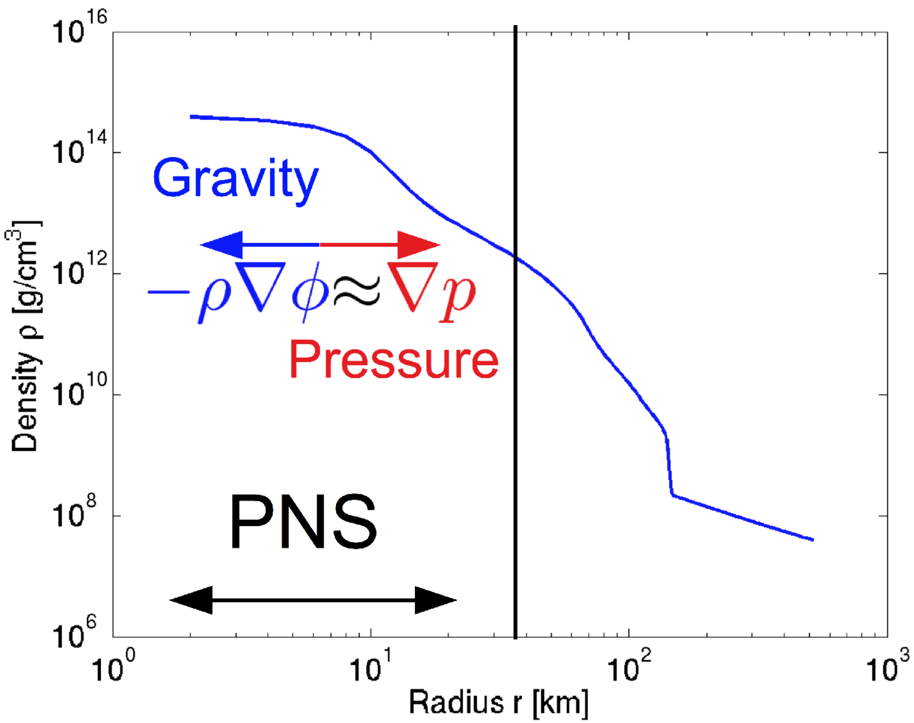
GRAVITY BOMB!

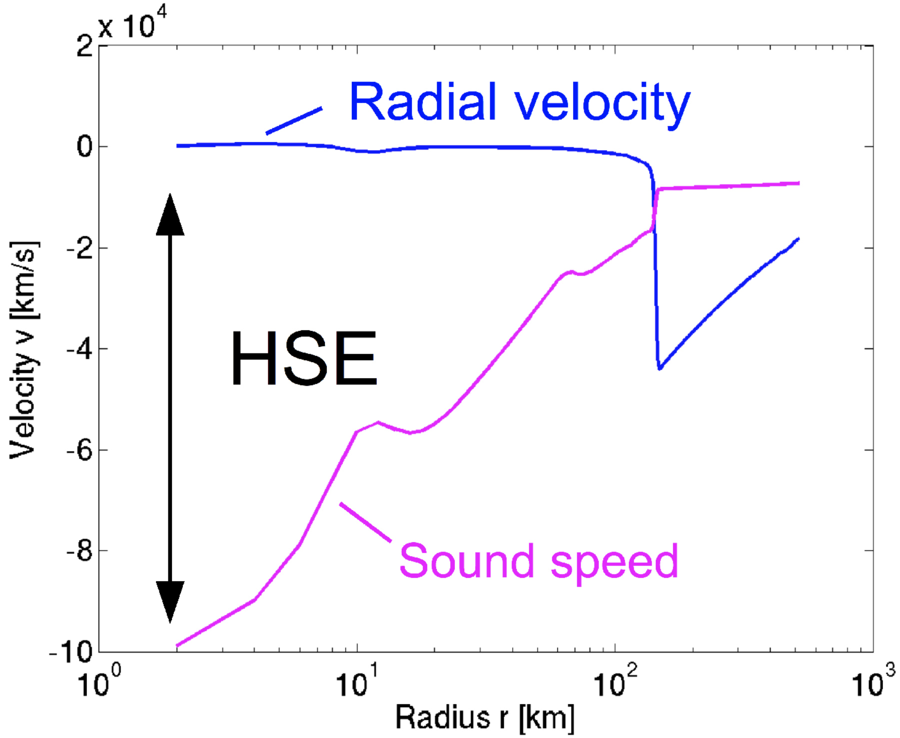
Core-collapse supernova



Radial profile

- The problem: (in our simulations)





Ability to maintain near hydrostatic equilibrium for a long time!

Outline

- **Introduction**
  - (Astro)Physical motivation
- **Well-balanced scheme for HydroStatic Equilibrium (HSE)**
  - First order
  - Second order
- **Multi-dimensional extension**
  - Limitations
- **Conclusion**

Well-balanced scheme for HSE

- Consider 1D hydrodynamics eqs with gravity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S}$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \qquad \boldsymbol{S} = -\begin{bmatrix} 0 \\ \rho \\ \rho v \end{bmatrix} \frac{\partial \phi}{\partial x}$$

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)

Well-balanced scheme for HSE (2)

- Classical solution algorithm:

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right) + \Delta t \boldsymbol{S}_i^n$$

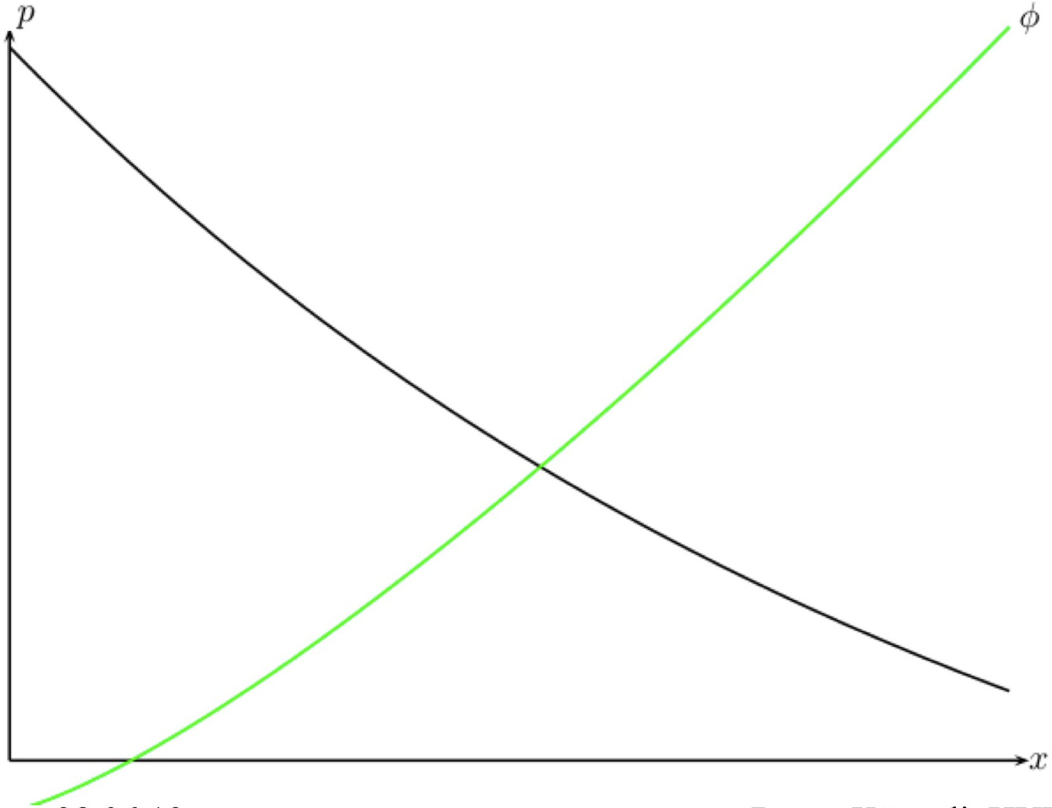
- Numerical flux $\boldsymbol{F}_{i\pm1/2}^n = \mathcal{F}(\boldsymbol{u}_{i\pm1/2}^{n,L}, \boldsymbol{u}_{i\pm1/2}^{n,R})$ from (approximate) Riemann solver, e.g.
  - (Local) Lax-Friedrichs Lax (1954), Rusanov (1961)
  - HLL (C) Harten, Lax and van Leer (1983), Toro et al. (1994)
  - Roe Roe (1981)

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



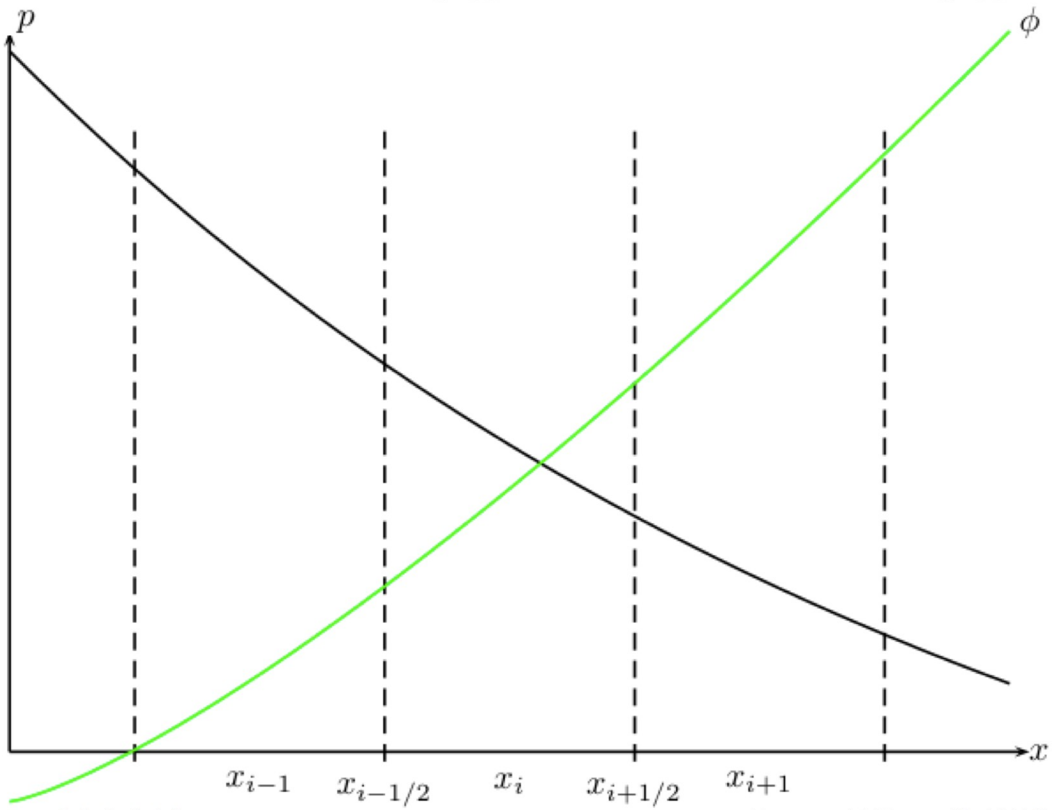
Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$

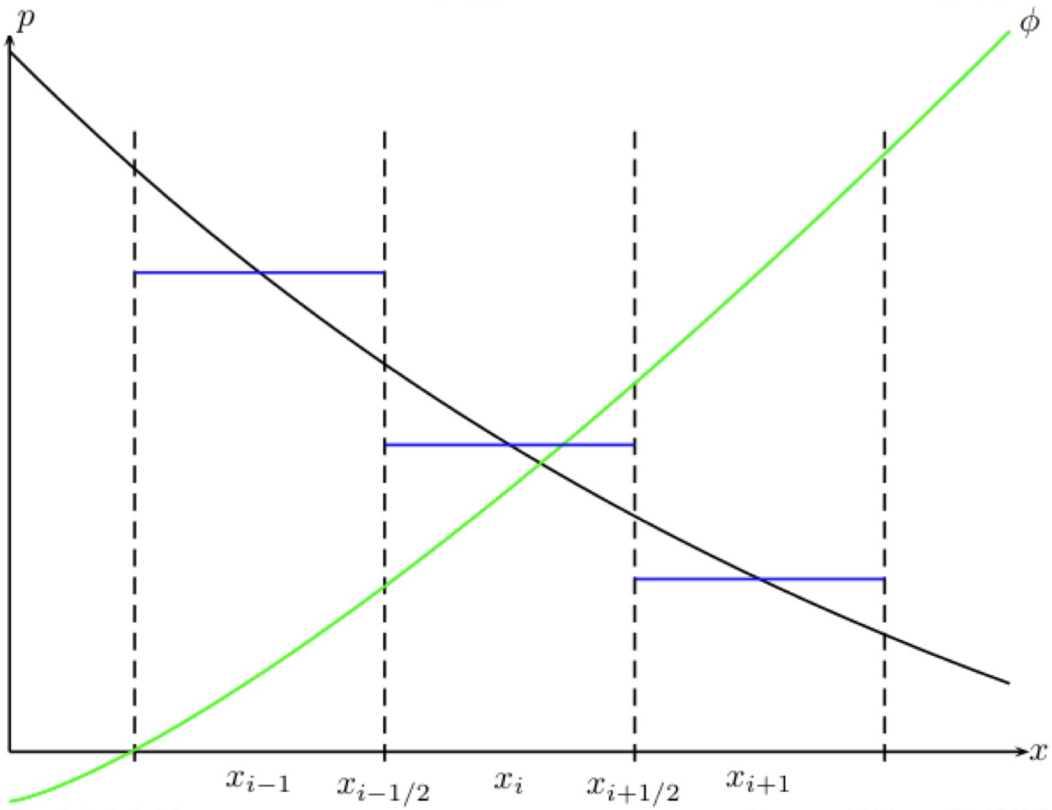
Discretise in cells $\left[x_{i-1/2}, x_{i+1/2}\right]$



Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium: $\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \quad \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$

EoS: $p = p(\rho, e)$



Discretise in cells $[x_{i-1/2}, x_{i+1/2}]$

Define cell averages

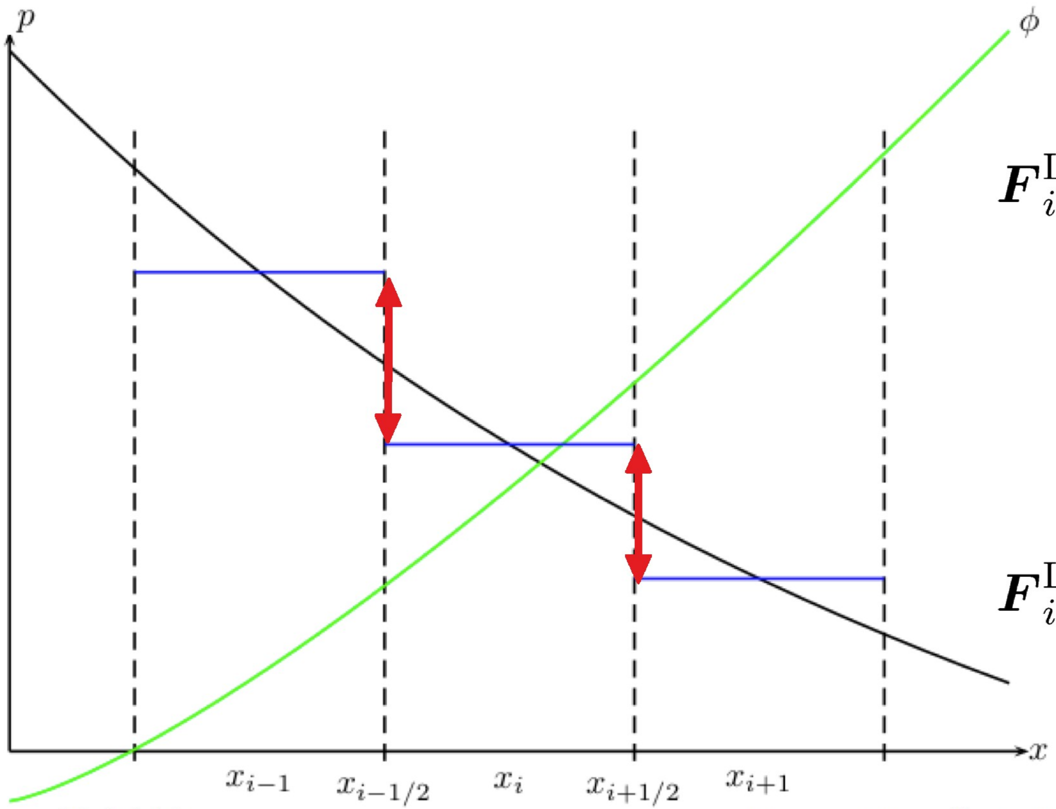
$$\boldsymbol{u}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}(x, t^n) \mathrm{d}x$$

$$\boldsymbol{S}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{S}\left(\boldsymbol{u}(x, t)\right) \mathrm{d}x$$

Well-balanced scheme for HSE (3)

Interested in hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) \stackrel{?}{=} \boldsymbol{S}^n_i$$



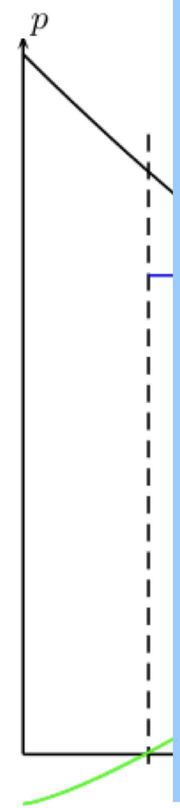
$$\boldsymbol{F}^{\mathrm{LxF}}_{i+1/2} = \frac{1}{2}\left(\boldsymbol{F}_i + \boldsymbol{F}_{i+1}\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i\right)$$

Contains also gravity induced gradient!

$$\boldsymbol{F}^{\mathrm{LxF}}_{i-1/2} = \frac{1}{2}\left(\boldsymbol{F}_{i-1} + \boldsymbol{F}_i\right) - \frac{S_{\max}}{2}\left(\boldsymbol{u}_i - \boldsymbol{u}_{i-1}\right)$$

Well-balanced scheme for HSE (3)

Intere
equil



Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \qquad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \qquad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0, 2]$$

Error in pressure:

(after 2 sound crossing times)

N	1st	2ndTVD
32	9.3E-02	1.3E-03
64	4.6E-02	3.2E-04
128	2.3E-02	8.0E-05
256	1.2E-02	2.0E-05
512	5.7E-03	5.1E-06

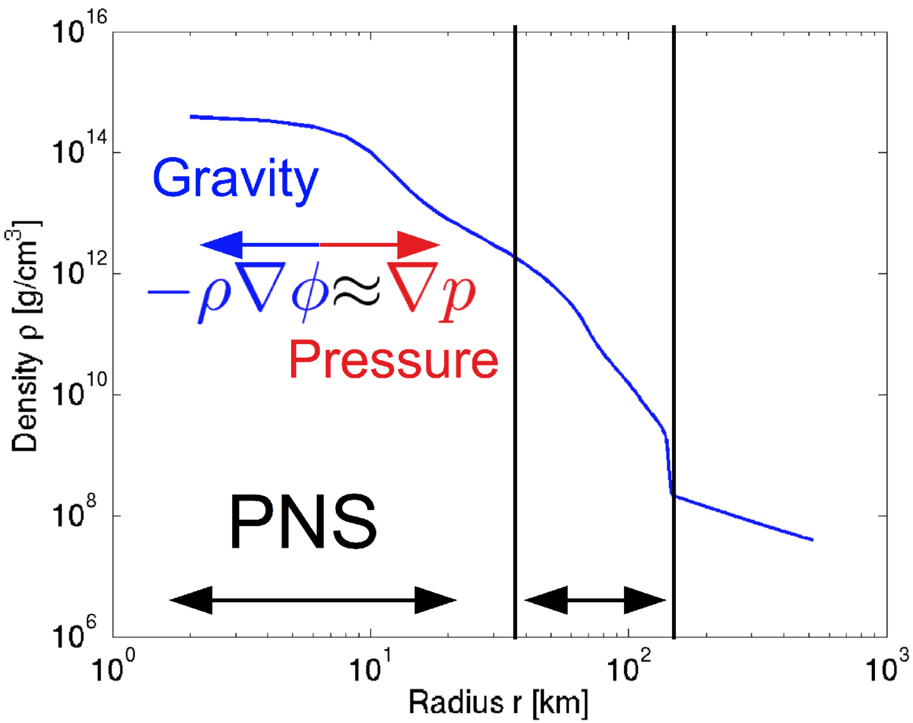
$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

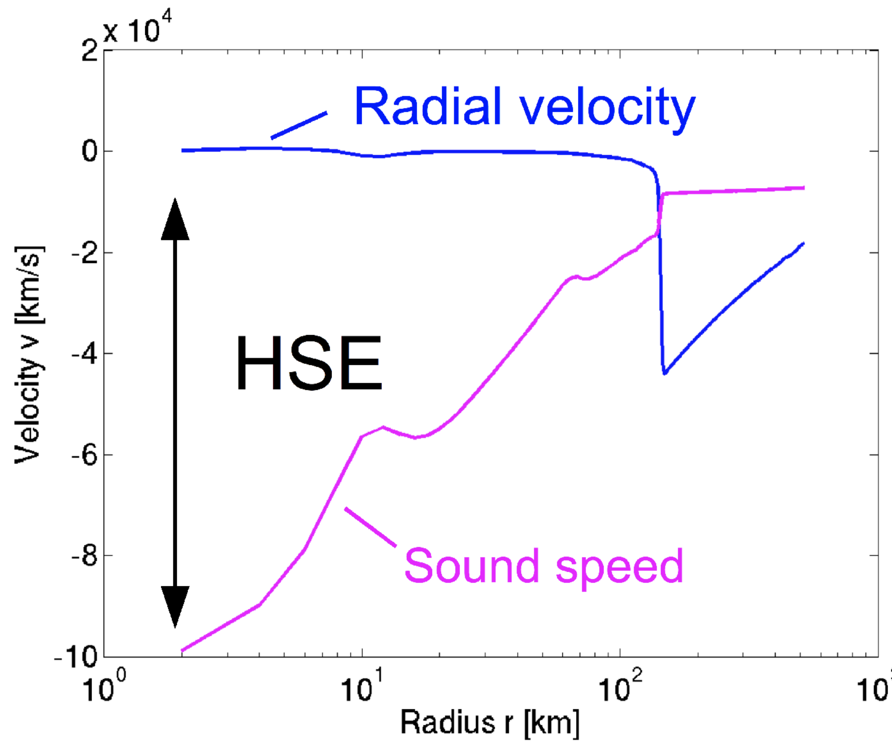
HLLC numerical flux

avity
!

Well-balanced scheme for HSE (4)

- The problem: (in our simulations)





Ability to maintain near hydrostatic equilibrium for a long time!

Well-balanced scheme for HSE (5)

- Solutions:
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(r)$
  - Steady state preserving reconstructions, well-balanced schemes e.g. LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010)

Note: there are many, many more... especially for shallow-water eqs!!!

Well-balanced scheme for HSE (5)

- Solutions:
  - Define a **global** stationary state $u_0(r)$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(r)$
  - Steady state preserving reconstructions, well-balanced schemes e.g. LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010)

Requirements

- Equilibrium not known in advance (self-gravity)
- Extensible for general EoS
- (At least) second order accuracy

Well-balanced scheme for HSE (6)

Interested in **numerical** hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) = \boldsymbol{S}^n_i$$

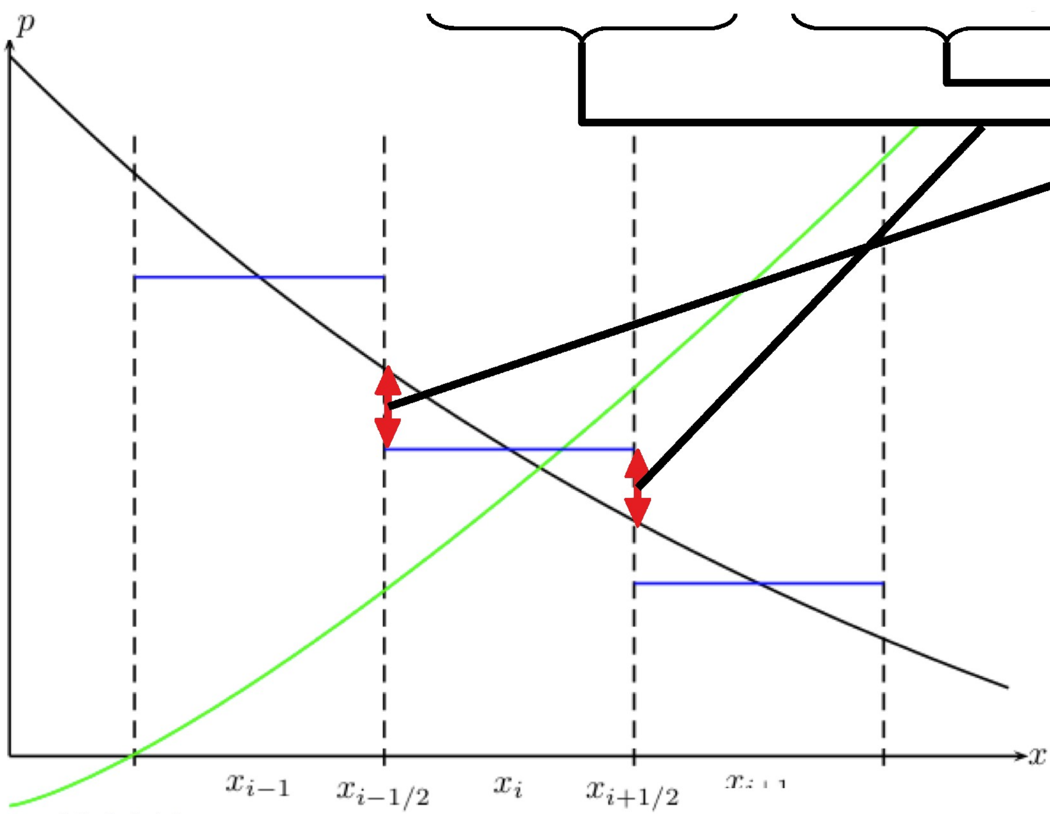
$$\frac{\partial p}{\partial x} + O(\Delta x^2) = \frac{p_{i+1/2} - p_{i-1/2}}{\Delta x} = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)$$

$$\frac{(p_{i+1/2} - p_i) - (p_{i-1/2} - p_i)}{\Delta x} = -\frac{\rho_i}{2} \frac{(\phi_{i+1} - \phi_i) - (\phi_{i-1} - \phi_i)}{\Delta x}$$

Well-balanced scheme for HSE (6)

Interested in **numerical** hydrostatic equilibrium:

$$\frac{p_{i+1/2}-p_i}{\Delta x}-\frac{p_{i-1/2}-p_i}{\Delta x}=-\frac{\rho_i}{2}\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}-\frac{\phi_{i-1}-\phi_i}{\Delta x}\right)$$



Equilibrium reconstruction:

$$p_{i+1/2}=p_i+\frac{\Delta x}{2}\Delta p_i^+$$

$$p_{i-1/2}=p_i-\frac{\Delta x}{2}\Delta p_i^-$$

Equilibrium differences:

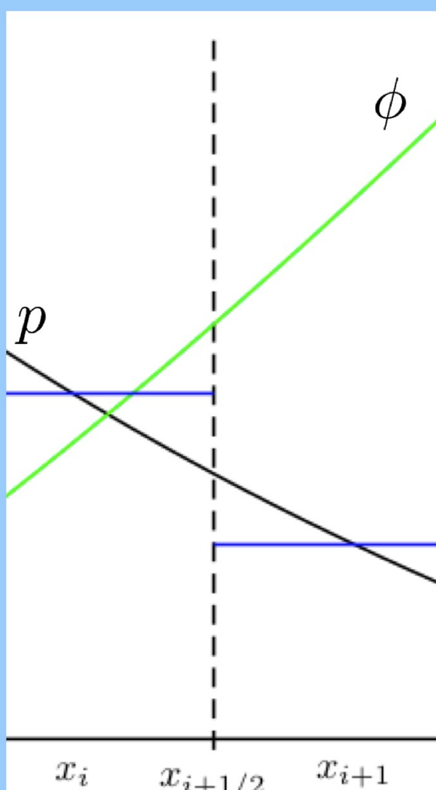
$$\Delta p_i^+=-\rho_i\frac{\phi_{i+1}-\phi_i}{\Delta x}$$

$$\Delta p_i^-=-\rho_i\frac{\phi_i-\phi_{i-1}}{\Delta x}$$

Well-balanced scheme for HSE (6)

Interested in **numerical** hydrostatic equilibrium:

Equilibrium?



$$p^L_{i+1/2} \overset{!}{=} p^R_{i+1/2}$$

$$p_i + \frac{\Delta x}{2} \Delta p_i^+ = p_{i+1} - \frac{\Delta x}{2} \Delta p_{i+1}^-$$

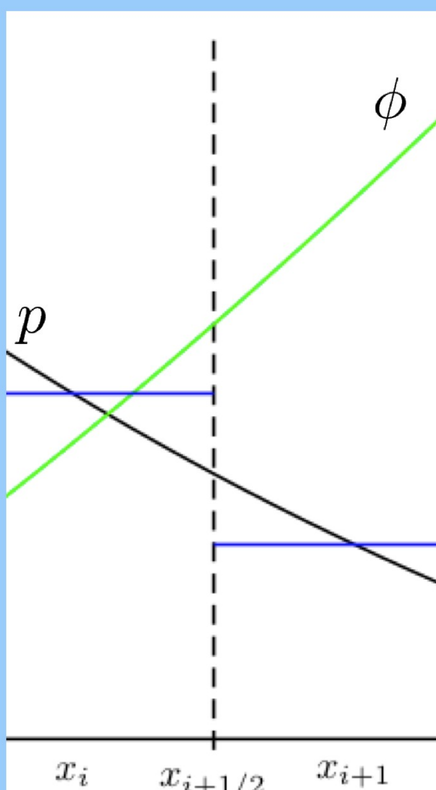
$$\frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

Discrete HydroStatic Equilibrium

Well-balanced scheme for HSE (6)

Interested in **numerical** hydrostatic equilibrium:

Equilibrium?



$$p^L_{i+1/2} \overset{!}{=} p^R_{i+1/2}$$

Requirement on Riemann solver:

$$\boldsymbol{F}^n_{i\pm1/2} = \boldsymbol{\mathcal{F}}\left(\begin{bmatrix} \rho^L_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix}, \begin{bmatrix} \rho^R_{i+1/2} \\ 0 \\ p_{i+1/2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ p_{i+1/2} \\ 0 \end{bmatrix}$$

e.g. HLLC, Roe

Discrete HydroStatic Equilibrium

28.

Well-balanced scheme for HSE (7)

- Second order extension:

$$\tilde{\boldsymbol{\sigma}}_i = \varphi\left(\frac{\boldsymbol{u_i} - \boldsymbol{u}_{i-1}}{\Delta x} - \frac{\boldsymbol{\Delta u}_{i-1}^{+} + \boldsymbol{\Delta u}_{i}^{-}}{2}, \frac{\boldsymbol{u_{i+1}} - \boldsymbol{u}_{i}}{\Delta x} - \frac{\boldsymbol{\Delta u}_{i}^{+} + \boldsymbol{\Delta u}_{i+1}^{-}}{2}\right)$$

$$\boldsymbol{u}_{i+1/2}^{L} = \boldsymbol{u}_i + \left(\boldsymbol{\Delta u}_i^{+} + \tilde{\boldsymbol{\sigma}}_i\right)\frac{\Delta x}{2}$$

$$\boldsymbol{u}_{i+1/2}^{R} = \boldsymbol{u}_{i+1} - \left(\boldsymbol{\Delta u}_{i+1}^{-} - \tilde{\boldsymbol{\sigma}}_{i+1}\right)\frac{\Delta x}{2}$$

$$\boldsymbol{\Delta u}_i^{\pm} = \left[\begin{array}{c} 0 \\ 0 \\ \frac{\Delta p_i^{\pm}}{\gamma - 1} \end{array}\right]$$

Reconstruction in deviation from equilibrium

Similar to Botta et al. 2004,Fuchs et al. 2010

- Time stepping:

$$\boldsymbol{u}^{*} = \boldsymbol{u}^{n} + \Delta t^{n}\boldsymbol{L}(\boldsymbol{u}^{n})$$

$$\boldsymbol{u}^{**} = \boldsymbol{u}^{*} + \Delta t^{n}\boldsymbol{L}(\boldsymbol{u}^{*})$$

$$\boldsymbol{u}^{n+1} = \frac{1}{2}\left(\boldsymbol{u}^{n} + \boldsymbol{u}^{**}\right)$$

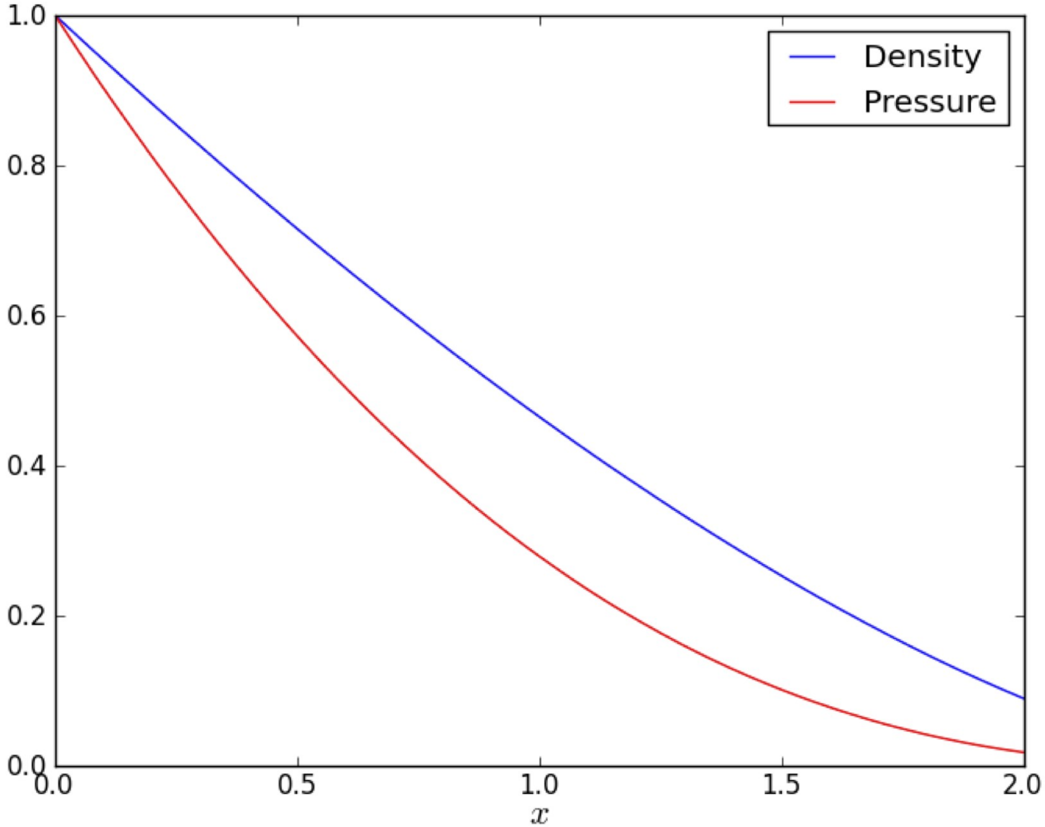
Strong Stability Preserving
Runge-Kutta,
Gottlieb et al. 2001

Example 1

Hydrostatic atmosphere in a constant gravitational field

$$\phi_i = g x_i \qquad \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} \qquad p_i = K_i \rho_i^\gamma$$

$$x \in [0, 2] \qquad K = const.$$ ~ entropy



Error in pressure:

N	1st	2ndTVD
32	9.3E-02 / **4.0E-16**	1.3E-03 / **2.0E-16**
64	4.6E-02 / **2.2E-15**	3.2E-04 / **8.7E-17**
128	2.3E-02 / **2.9E-15**	8.0E-05 / **8.1E-16**
256	1.2E-02 / **2.9E-15**	2.0E-05 / **3.8E-16**
512	5.7E-03 / **8.1E-14**	5.1E-06 / **1.8E-15**

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

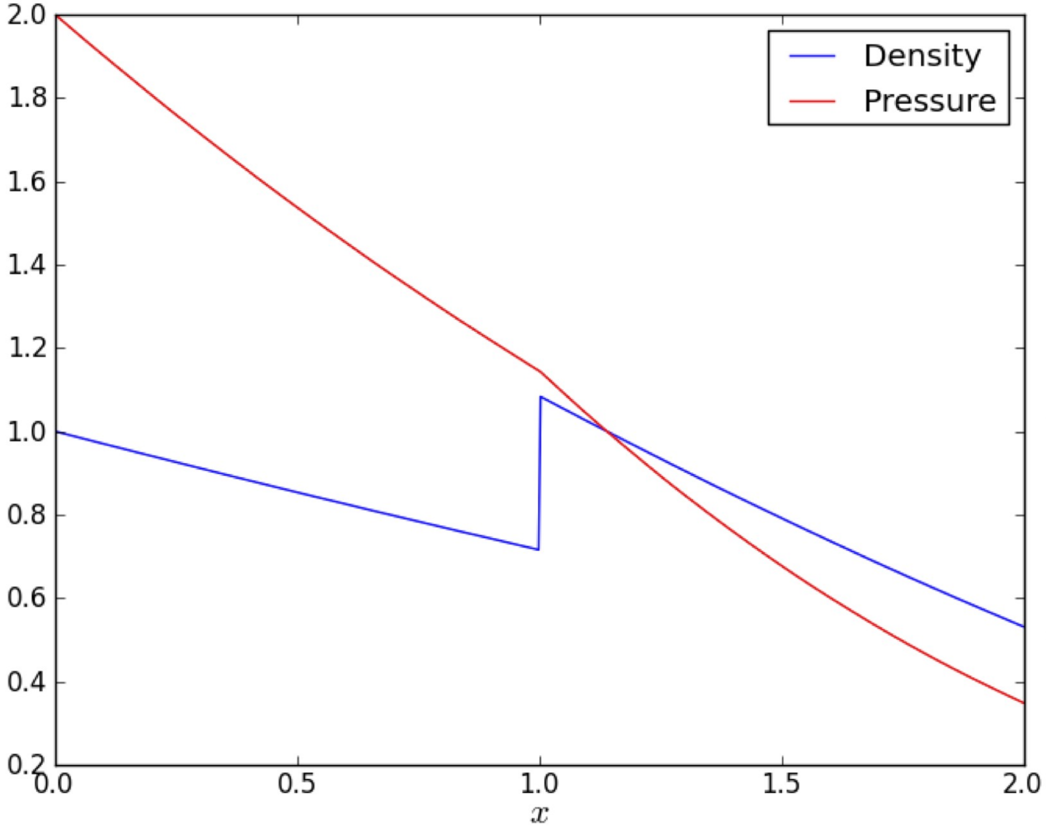
Example 2

Hydrostatic atmosphere in a constant gravitational field

$$\phi_i = g x_i \qquad \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{\phi_{i+1} - \phi_i}{\Delta x} \qquad p_i = K_i \rho_i^\gamma$$

$$x \in [0, 2]$$

$$K = \begin{cases} 2 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$ ~ entropy



Error in pressure:

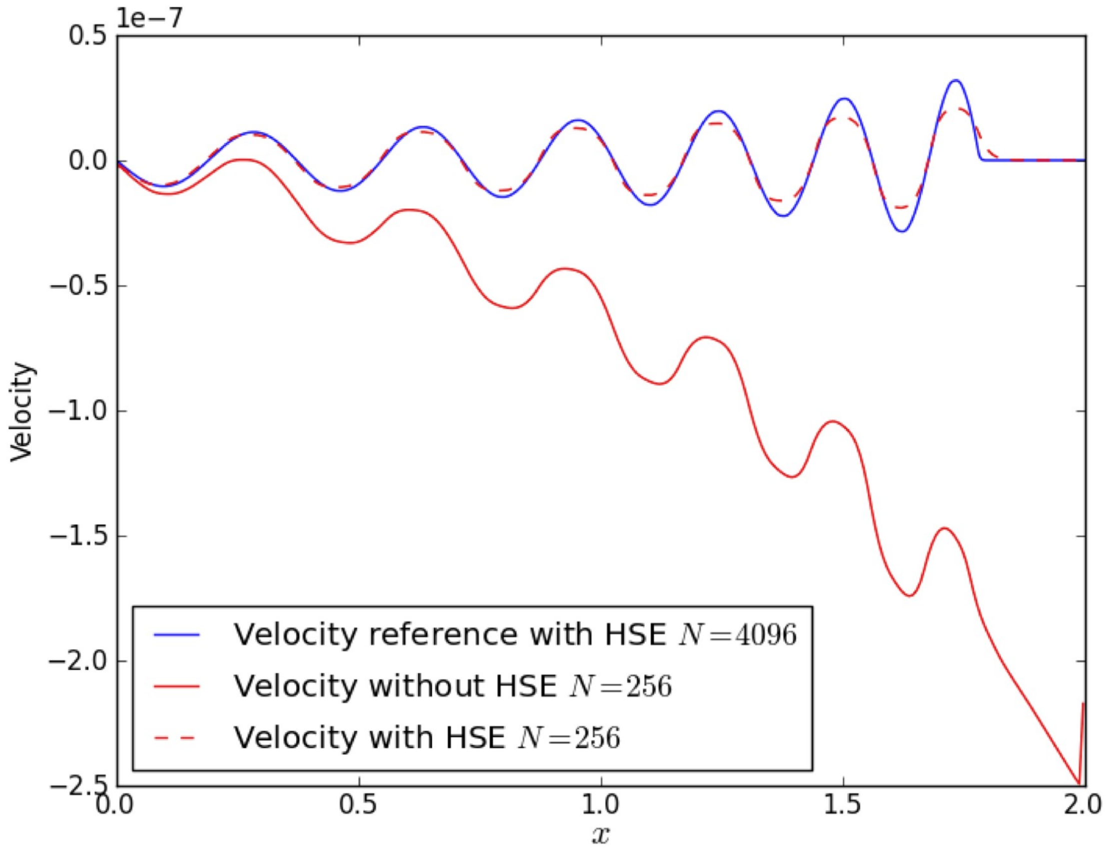
N	1st	2ndTVD
32	6.3E-02 / **3.3E-16**	6.2E-04 / **1.3E-16**
64	3.2E-02 / **3.8E-15**	1.6E-04 / **4.6E-16**
128	1.6E-02 / **6.1E-15**	4.2E-05 / **8.8E-16**
256	8.0E-03 / **7.0E-15**	1.1E-05 / **6.7E-16**
512	4.0E-03 / **1.1E-13**	2.7E-06 / **3.4E-15**

$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

Example 3

Hydrostatic atmosphere in a constant gravitational field
+ **small** amplitude waves

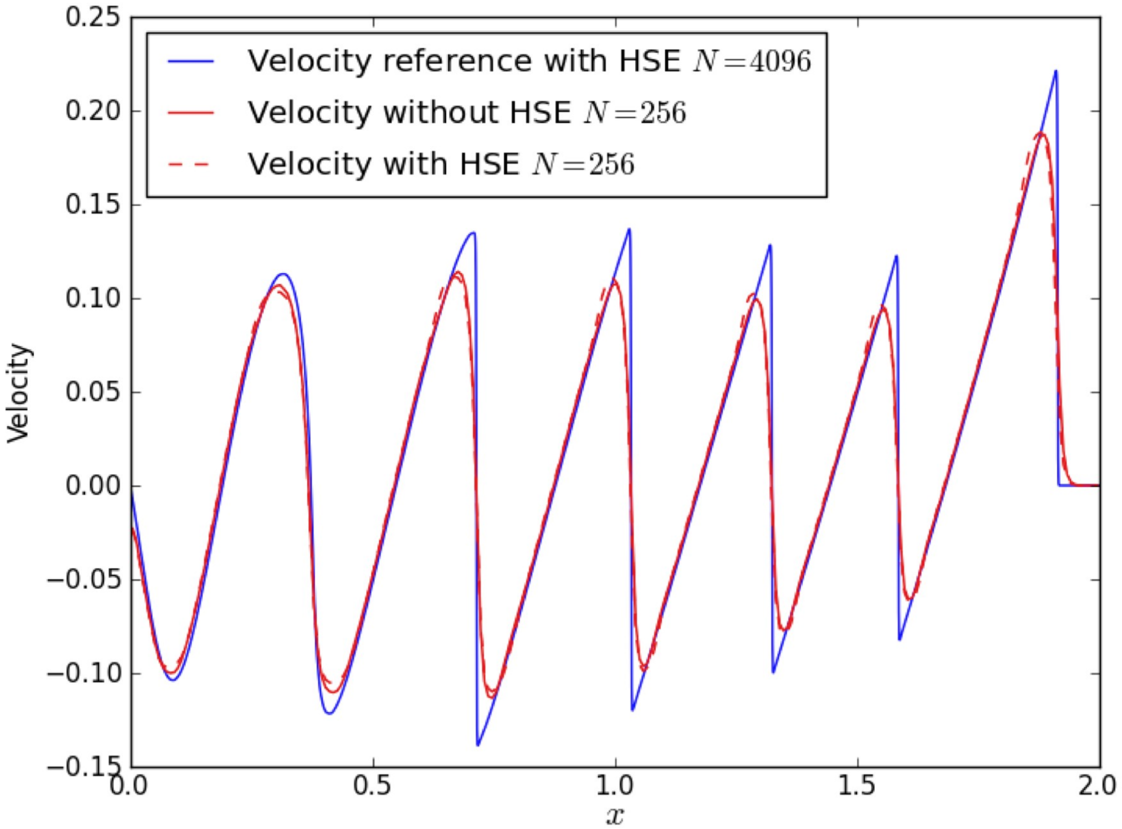
Velocity perturbation: $v(t, x=0) = 10^{-8} \sin\left(\frac{6}{1.8} 2\pi t\right)$



Example 3 (2)

Hydrostatic atmosphere in a constant gravitational field
+ **large** amplitude waves

Velocity perturbation: $v(t, x = 0) = 10^{-1} \sin\left(\frac{6}{1.8} 2\pi t\right)$



Example 6

Polytrope: model star (e.g. main sequence stars, white dwarfs, neutron stars)

Euler equations in spherical symmetry:

$$\frac{\partial(r^2 \boldsymbol{u})}{\partial t} + \frac{\partial(r^2 \boldsymbol{F})}{\partial r} = r^2 \boldsymbol{S}$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} \qquad \boldsymbol{F} = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \qquad \boldsymbol{S} = \frac{2p}{r} - \begin{bmatrix} 0 \\ \rho \\ \rho v \end{bmatrix} \frac{\partial \phi}{\partial x}$$

Poisson equation in spherical symmetry:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G \rho$$

Example 6 (2)

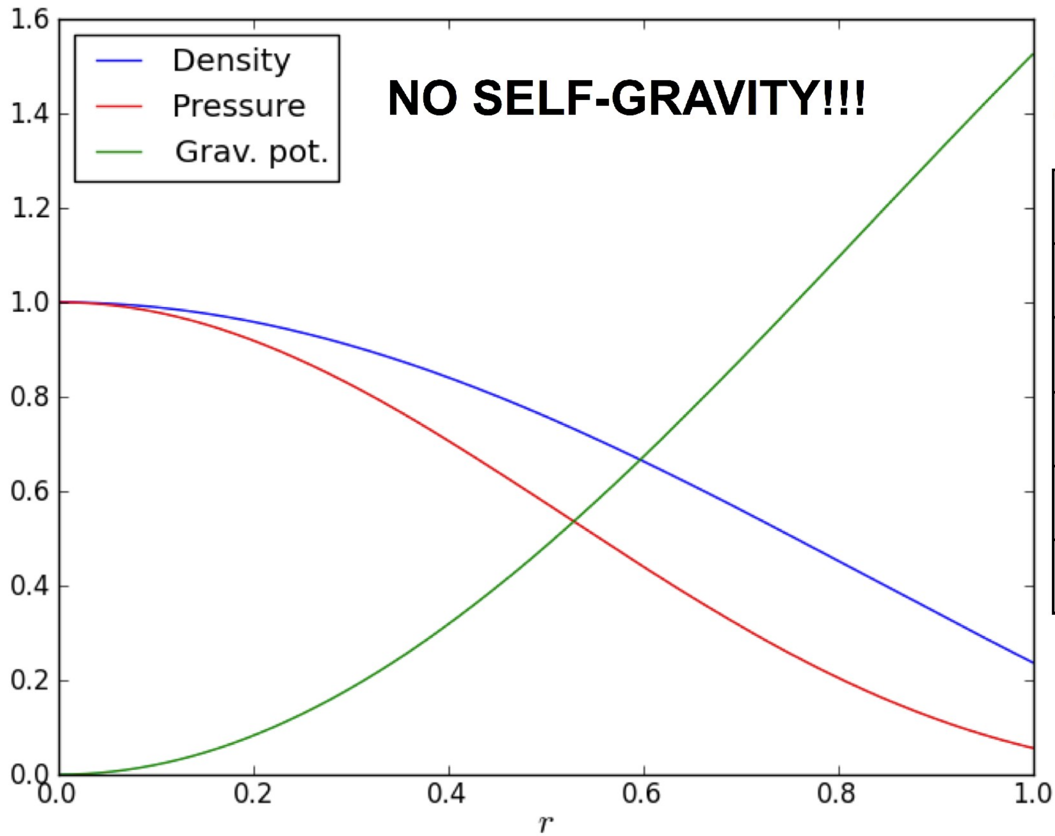
Polytrope: model star $\gamma = 2$ ~ neutron stars

HSE: $$\frac{p_{i+1} - p_i}{\Delta r} = -\frac{\rho_i + \rho_{i+1}}{2} \frac{GM_{1+1/2}}{r^2_{i+1/2}}$$

Poisson: $$\frac{\phi_{i+1} - \phi_i}{\Delta r} = \frac{GM_{1+1/2}}{r^2_{i+1/2}}$$

$$p_i = K_i \rho_i^\gamma \quad K = const.$$

$$M_{i+1/2} = \int_0^{r_{i+1/2}} 4\pi r^2 \mathrm{d}r$$



Error in pressure:

N	1st	2ndTVD
32	2.3E-02 / **2.7E-16**	1.8E-03 / **1.4E-16**
64	1.1E-02 / **5.7E-16**	4.4E-04 / **5.0E-16**
128	5.7E-03 / **4.3E-16**	1.1E-04 / **3.3E-16**
256	2.8E-03 / **1.0E-15**	2.8E-05 / **7.7E-16**
512	1.4E-03 / **6.6E-13**	6.9E-06 / **3.5E-16**

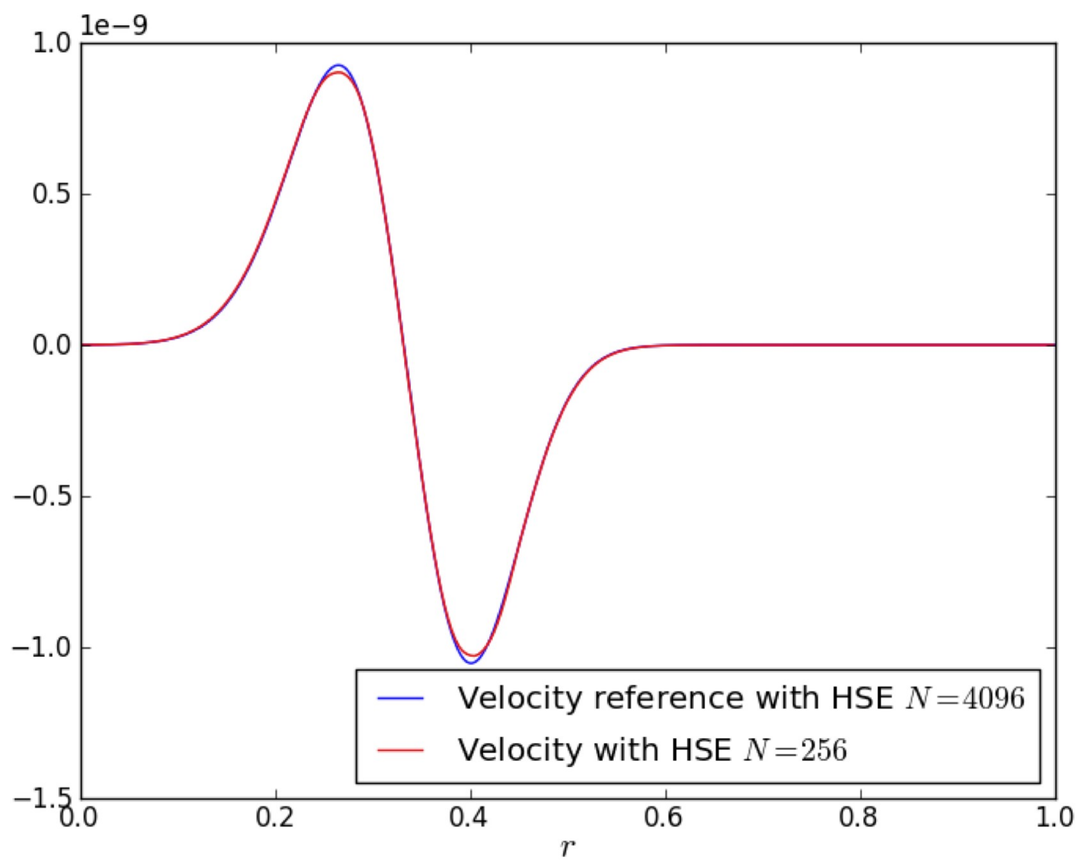
$$Err = \frac{1}{N} \sum_i |p_i - p_i^0|$$

Example 6 (3)

Polytrope: model star $\gamma = 2$ ~ neutron stars

+ density perturbation

$$\rho(r) = \rho(r)\left(1 + 10^{-6}\exp(-100r^2)\right)$$



Without HSE reconstruction the velocity is way off!

Outline

- **Introduction**
  - (Astro)Physical motivation
- **Well-balanced scheme for HydroStatic Equilibrium (HSE)**
  - First order
  - Second order
- **Multi-dimensional extension**
  - Limitations
- **Conclusion**

Multi-dimensional extension

- Straight forward directional application of HydroStatic Reconstruction

$$\frac{\mathrm{d}\boldsymbol{u}_{i,j}}{\mathrm{d}t} = \boldsymbol{L}(\boldsymbol{u}) = -\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2,j} - \boldsymbol{F}_{i-1/2,j}\right) - \frac{1}{\Delta y}\left(\boldsymbol{G}_{i,j+1/2} - \boldsymbol{G}_{i,j-1/2}\right) + \boldsymbol{S}_{i,j}$$

- Numerical equilibrium:

$$\frac{p_{i+1,j} - p_{i,j}}{\Delta x} = -\frac{\rho_{i,j} + \rho_{i+1,j}}{2}\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x}$$

$$\frac{p_{i,j+1} - p_{i,j}}{\Delta y} = -\frac{\rho_{i,j} + \rho_{i,j+1}}{2}\frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y}$$

3D analogous...

Example 7

Polytrope: model star (e.g. main sequence stars, white dwarfs, neutron stars)

HSE: $\nabla p = -\rho \nabla \phi$ Poisson equation: $\nabla^2 \phi = -4\pi G \rho$

Equation of state $p = K\rho^\gamma$ $K = 1$

Take $\gamma = 2$ ~ neutron stars

Central density

Then there's an exact solution:

$$\rho(\boldsymbol{x}) = \rho_c \frac{\sin(\alpha r)}{r}$$

$$\phi(\boldsymbol{x}) = -\gamma K \rho(\boldsymbol{x})$$

$$\alpha = \sqrt{\frac{2K}{4\pi G}} \quad r = \sqrt{x^2 + y^2 + z^2}$$

Example 7

Po

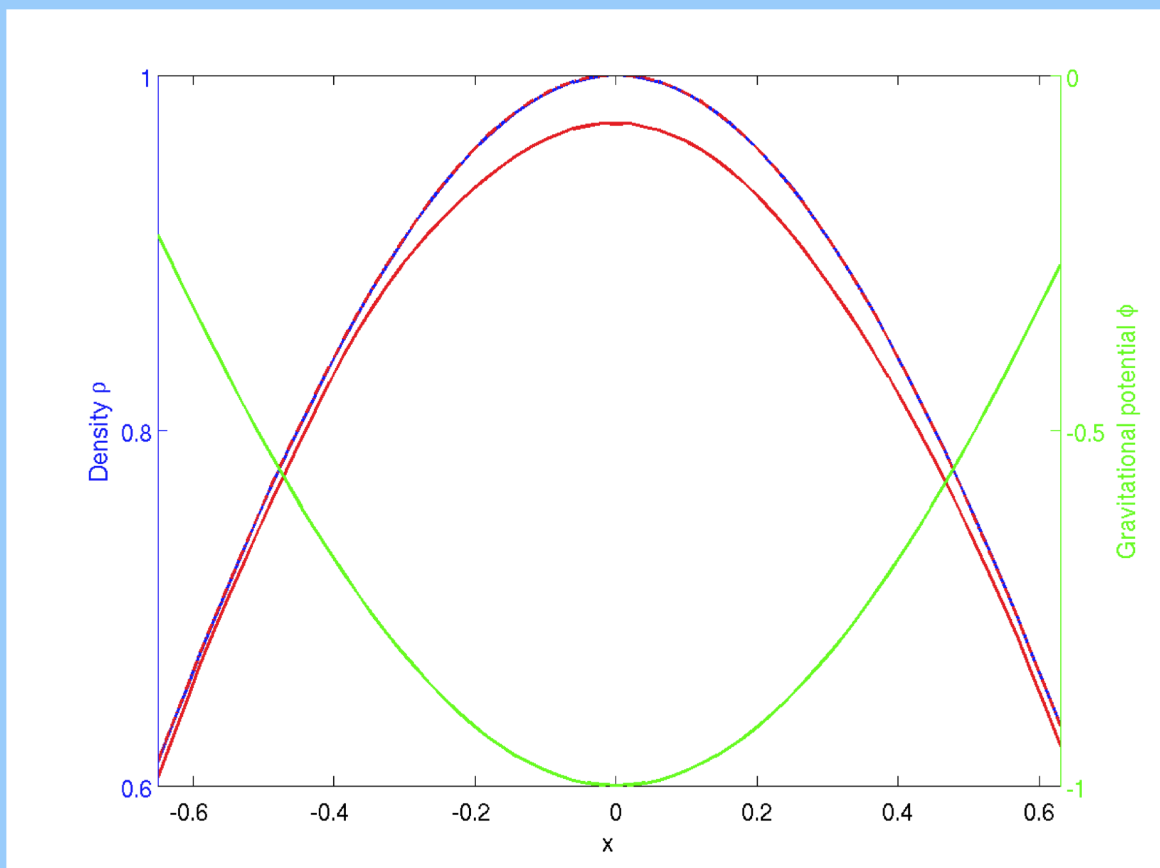
H ρ

E

Ta

Th

Evolution for 20 “sound crossing” times



NO HSE

WITH HSE

Reference

Error in density:

N	2ndTVD
64	1.9E-02 / **1.2E-15**
128	4.9E-03 / **3.4E-15**

$$Err = \frac{1}{N}\sum_i |\rho_i - \rho_i^0|$$

Example 7

Po

H

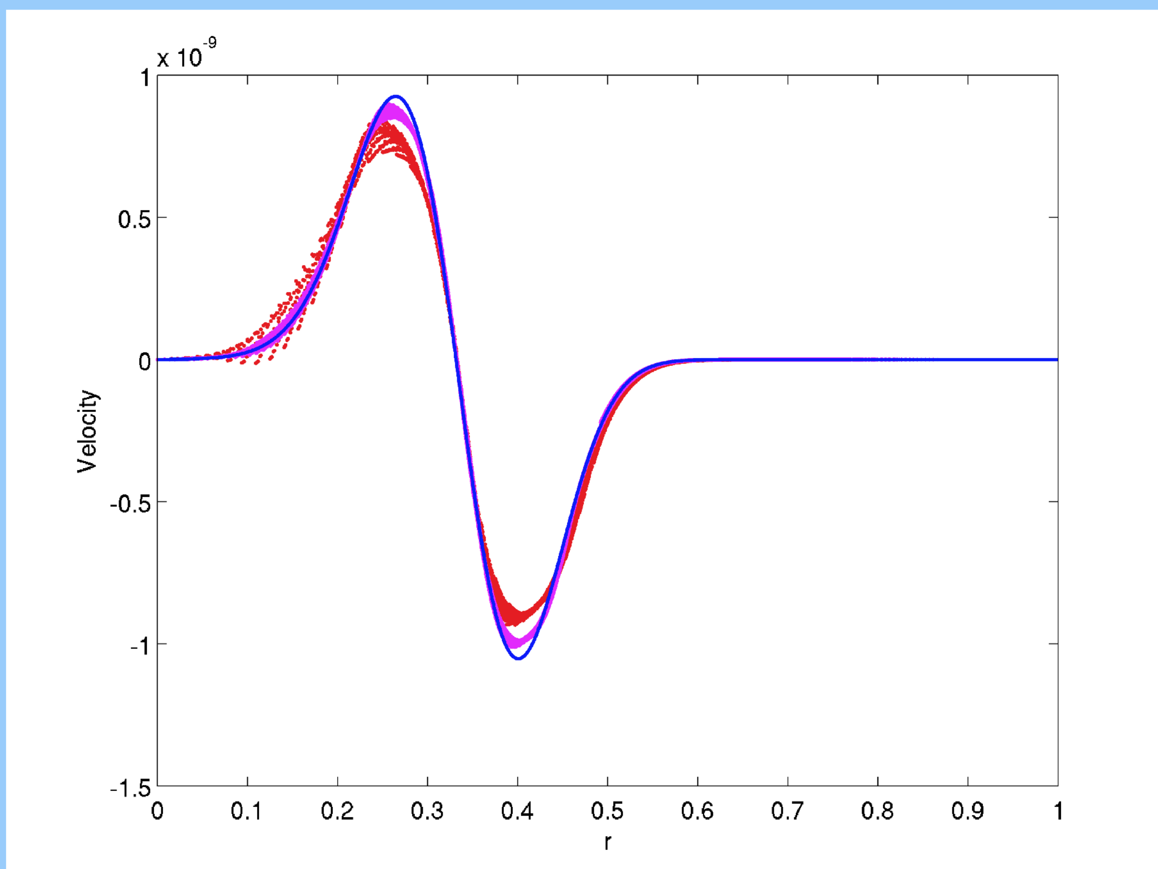
E

Ta

Th

ρ

(Central) density perturbation



WITH HSE N=64^3

WITH HSE N=128^3

Reference

Without HSE reconstruction the velocity is way off!

Conclusions

- 1D well-balanced scheme for hydrostatic equilibrium (for any Equation of State EoS)
- Extension to higher-order? Non-zero velocity steady state?
- Multi-D well-balanced scheme for hydrostatic equilibrium
  - Unfortunately with limitations (so far...)
    Although not exactly well-balanced for general EoS, the ability to maintain HSE is greatly increased

Thank you for you attention!!!