# Global Small Solutions of the 3D Kerr-Debye Model

Mohamed Kanso

Institut de Mathématiques de Bordeaux, UMR CNRS 5251

# Global Small Solutions of the 3D Kerr-Debye Model

- Models
- Properties and known results
- Main result
- Sketch of the proof

# Physical context

The propagation of electromagnetic waves in a homogeneous isotropic nonlinear material (cristal) is described by Maxwell's equations

$$\begin{aligned} \partial_t D - \mathrm{curl}\ H &= 0 \\ \partial_t B + \mathrm{curl}\ E &= 0 \end{aligned}$$

$$\mathrm{div} D = \mathrm{div} B = 0$$

$E$ : electric field
$H$ : magnetic field
$D$ : electric displacement
$B$ : magnetic induction

# Physical context

Maxwell's equations:

$$\begin{aligned} \partial_t D - \mathrm{curl}\ H &= 0 \\ \partial_t B + \mathrm{curl}\ E &= 0 \end{aligned}$$

$$\mathrm{div} D = \mathrm{div} B = 0$$

Kerr medium $\Rightarrow$ constitutive relations :

# Physical context

Maxwell's equations:

$$\begin{aligned} \partial_t D - \mathrm{curl}\ H &= 0 \\ \partial_t B + \mathrm{curl}\ E &= 0 \end{aligned}$$

$$\mathrm{div} D = \mathrm{div} B = 0$$

Kerr medium $\Rightarrow$ constitutive relations :

Kerr Model: instantaneous response

$$B = H \quad \text{and} \quad D = (1 + |E|^2)E$$

# Physical context

Maxwell's equations:

$$\begin{aligned} \partial_t D - \text{curl } H &= 0 \\ \partial_t B + \text{curl } E &= 0 \end{aligned}$$

$$\text{div} D = \text{div} B = 0$$

Kerr medium $\Rightarrow$ constitutive relations :

Kerr Model: instantaneous response

$$B = H \quad \text{and} \quad D = (1 + |E|^2)E$$

Kerr-Debye Model: finite response time

$$B = H \quad \text{and} \quad D = (1 + \chi)E$$

with

$$\partial_t \chi + \frac{1}{\tau}\chi = \frac{1}{\tau}|E|^2$$

$\tau$: relaxation parameter

**Y.- R. Shen,** The Principles of Nonlinear Optics, Wiley Interscience, 1994.
**R.W. Ziolkowski.** *The incorporation of microscopic material models into FDTD approach for ultrafast optical pulses simulations*, IEEE Transactions on Antennas and Propagation 45(3):375-391, 1997.

# Properties

Kerr-Debye is a relaxation model of Kerr in the sense of Chen-Levermore-Liu (CPAM 1994

Equilibrium manifold:

$$\mathcal{V} = \left\{ (D, H, \chi), \chi = |E|^2 = (1+\chi)^{-2}|D|^2 \right\}$$

Reduced system:
Kerr is the reduced system of Kerr-Debye on the equilibrium manifold

# Properties

Kerr-Debye is a relaxation model of Kerr in the sense of Chen-Levermore-Liu (CPAM 1994

Equilibrium manifold:

$$\mathcal{V} = \left\{(D, H, \chi), \chi = |E|^2 = (1+\chi)^{-2}|D|^2\right\}$$

Reduced system:
Kerr is the reduced system of Kerr-Debye on the equilibrium manifold

Entropy relations (electromagnetic energy): strictly convex

$$\mathbf{P}_K(D, H) = \frac{1}{2}(|E|^2 + |H|^2 + \frac{3}{2}|E|^4)$$

$$\mathbf{P}_{KD}(D, H, \chi) = \frac{1}{2}(1+\chi)^{-1}|D|^2 + \frac{1}{2}|H|^2 + \frac{1}{4}\chi^2$$

On the equilibrium manifold,

$$\mathbf{P}_{KD}(D, H, \chi(D)) = \mathbf{P}_K(D, H)$$

Kerr: hyperbolic symmetrizable system of conservation laws.

Kerr-Debye: hyperbolic symmetrizable, partially dissipative system

# Known results

## Local existence of smooth solutions

- Kato (1975) or Majda (1984): Cauchy problem
- Picard-Zajaczkowski (1995): Initial-Boundary Value Problem (IBVP)

# Known results

## Local existence of smooth solutions

- Kato (1975) or Majda (1984): Cauchy problem
- Picard-Zajaczkowski (1995): Initial-Boundary Value Problem (IBVP)

## Convergence of Kerr-Debye smooth solutions towards Kerr smooth solutions when $\tau \to 0$.

- Hanouzet-Huynh (2000): Cauchy problem using the results of Yong (1999).
- Carbou-Hanouzet (2009): Initial-Boundary Value Problem

# Known results

## Local existence of smooth solutions

- Kato (1975) or Majda (1984): Cauchy problem
- Picard-Zajaczkowski (1995): Initial-Boundary Value Problem (IBVP)

## Convergence of Kerr-Debye smooth solutions towards Kerr smooth solutions when $\tau \to 0$.

- Hanouzet-Huynh (2000): Cauchy problem using the results of Yong (1999).
- Carbou-Hanouzet (2009): Initial-Boundary Value Problem

## Global existence of smooth solutions ?

- global existence without smallness condition holds for the Cauchy problem as well as for the impedance IBVP for Kerr-Debye model in the 1D and 2D Tranverse Electric cases (Carbou-Hanouzet 2009).
- apparition of shocks for the Kerr model in both 1D and 2D-TE cases.

# Global existence in 3D

## For Kerr Model

Global small solution for the Cauchy problem: based on a decay estimate for the linear wave equation

**R. Racke**, Lectures on nonlinear evolution equations. Initial value problems. Aspects of Mathematics, E19. Friedr. Vieweg & Sohn, Braunschweig, 1992

# Global existence in 3D

## For Kerr Model

Global small solution for the Cauchy problem: based on a decay estimate for the linear wave equation

**R. Racke**, Lectures on nonlinear evolution equations. Initial value problems. Aspects of Mathematics, E19. Friedr. Vieweg & Sohn, Braunschweig, 1992

## For Kerr-Debye Model

The Shizuta-Kawashima [SK] condition does not hold: the linearized system around the null constant equilibrium writes:

$$\partial_t \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = 0,$$

$$\partial_t \chi = -\chi,$$

i.e the dissipative variable $\chi$ and the variable $(E, H)$ are completely uncoupled.

- All known results around [SK] condition do not applied

**Y. Shizuta and S. Kawashima**, *Systems of equations of hyperbolic-parabolic type with application to the discrete Boltzmann equation.* Hokkaido Math.J. **14** (1985)

# Global existence in 3D

## Kerr-Debye Model

$$
\begin{cases}
\partial_t D - \mathrm{curl}\ H = 0, \\
\partial_t H + \mathrm{curl}\ E = 0, \\
\partial_t \chi = |E|^2 - \chi, \\
D = (1+\chi)E, \\
\mathrm{div}\ D = \mathrm{div}\ H = 0.
\end{cases}
$$

The dispersion of the Maxwell equations in the 3-D case + The partial dissipative character of the Kerr-Debye model $\Longrightarrow$ **Global small solution:**

# Global existence in 3D

## Kerr-Debye Model

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \\ \partial_t \chi = |E|^2 - \chi, \\ D = (1+\chi)E, \\ \text{div } D = \text{div } H = 0. \end{cases}$$

The dispersion of the Maxwell equations in the 3-D case + The partial dissipative character of the Kerr-Debye model $\Longrightarrow$ **Global small solution:**

## Theorem 3.1

*There exist an integer $s \geq 7$ and a $\delta > 0$ such that the following holds: if the initial data $V^0 = (E^0, H^0, \chi^0)$ satisfies*

$$\|V^0\|_{s,2} + \|V^0\|_{s,\frac{6}{5}} < \delta, \textit{ with } \chi^0 \geq 0 \textit{ and div } H^0 = \textit{div } [(1+\chi^0)E^0] = 0,$$

*then there exists a unique solution $V$ for the Cauchy problem of the KD model, with:*

$$V = (E, H, \chi) \in \mathcal{C}^0([0,\infty), W^{s,2}) \cap \mathcal{C}^1([0,\infty), W^{s-1,2}).$$

# Sketch of the proof

## Kerr-Debye Model

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \\ \partial_t \chi = |E|^2 - \chi, \\ D = (1+\chi)E, \\ \text{div } D = \text{div } H = 0. \end{cases}$$

The dispersion of the Maxwell equations in the 3-D case + The partial dissipative character of the Kerr-Debye model $\Longrightarrow$ **Global small solution:**

**The two principal steps of the proof**, cf: Klainerman-Ponce (1983), O. Liess (1989) or R. Racke (1992):

- **High energy estimate** by using variational methods
- **a weighted a priori estimate** based on $L^p - L^q$ decay estimates for the linear wave equation

# Sketch of the proof

## Kerr-Debye Model

$$\begin{cases} \partial_t D - \text{curl } H = 0, \\ \partial_t H + \text{curl } E = 0, \\ \partial_t \chi = |E|^2 - \chi, \\ D = (1+\chi)E, \\ \text{div } D = \text{div } H = 0. \end{cases}$$

The dispersion of the Maxwell equations in the 3-D case + The partial dissipative character of the Kerr-Debye model $\Longrightarrow$ **Global small solution**

**Main difficulty:** degree of vanishing of the nonlinearity near zero is not great enough

**New idea:** we split the model in two parts:

- Maxwell's equations
- Ordinary differential equation satisfied by $\chi$

# High energy estimate

We denote by $T^*$ the lifespan of the local solution

We define

$$M_{s_1}(t) = \max_{0 \leq \tau \leq t} (1+\tau)^{2/3} \|(E,H)(\tau)\|_{s_1,6}$$

## Proposition 4.1

*Let $s, s_1 \in \mathbb{N}; 2 \leq s_1 \leq s-1$. Then there exists a constant $c = c(s, s_1)$ such that if the initial data $V^0 = (E^0, H^0, \chi^0)$ satisfies*

$$\|V^0\|_{s,2} \leq 1/2, \text{ and } \chi^0 \geq 0,$$

*then*

$$\|V(t)\|_{s,2} \leq c\|V^0\|_{s,2}\left[(1 + M_{s_1}^2(t))exp\{cM_{s_1}^2(t)\}\right] \;\; \forall\, t \in [0, \bar{T}],$$

*where $\bar{T} > 0$ is defined by*

$$\bar{T} := max\{T < T^\star \text{ such that } \|V\|_{L^\infty(0,T;W^{s,2})} \leq 1\}.$$

# High energy estimate

## Idea of the proof

- We use the variable $(E, H, \chi)$:

$$\begin{cases} (1+\chi)\partial_t E + (\partial_t \chi)E - \text{curl } H = 0 \\ \partial_t H + \text{curl } E = 0 \\ \partial_t \chi = |E|^2 - \chi \end{cases}$$

- Classical variationnal estimates on the Maxwell part

- We solve the ODE to estimate $\chi$:

$$\chi(t) = \chi^0 e^{-t} + \int_0^t e^{(s-t)} |E(s)|^2 ds$$

- The introduction of the weight $(1+t)^{2/3}$ is necessary to control $\chi$

# Weighted a priori estimate

## Proposition 4.2

*Let* $s,\ s_1 \in \mathbb{N}; 3 \leq s_1 \leq s-4$. *Then for all* $M_0 > 0$ *there exists a* $0 < \delta(M_0) \leq 1$ *such that if:*

$$\|V^0\|_{s,2} + \|V^0\|_{s_1+3,\frac{6}{5}} \leq \delta \text{ and } \|\chi^0\|_{s,\frac{3}{2}} \leq \delta,$$

$$\text{with } \chi^0 \geq 0 \text{ and } \mathrm{div}\, H^0 = \mathrm{div}\,[(1+\chi^0)E^0] = 0,$$

*then*

$$M_{s_1}(t) \leq M_0 \qquad \forall\, t \in [0, \tilde{T}],$$

*where*

$$\tilde{T} := \max\{T < T^\star \text{ such that } \|V\|_{L^\infty(0,T;W^{s,2})} \leq \delta(M_0)\}.$$

# Weighted a priori estimate

**Idea of the proof**

- We use the variable $(D, H, \chi)$:

$$\begin{cases} \partial_t D - \text{curl } H = 0 \\ \partial_t H + \text{curl } D = \text{curl } (\chi E) \end{cases}$$

# Weighted a priori estimate

## Idea of the proof

- We use the variable $(D, H, \chi)$:

$$\begin{cases} \partial_t D - \text{curl } H = 0 \\ \partial_t H + \text{curl } D = \text{curl } (\chi E) \end{cases}$$

- Representation of Duhamel : $\Lambda$ is the linear Maxwell operator, $U = (D, H)$ and $f(t) = {}^t(0, \text{curl } (\chi E)))$

$$U(t) = e^{t\Lambda}U^0 + \int_0^t e^{(t-\tau)\Lambda} f(\tau)d\tau, \quad 0 \leq t \leq \tilde{T},$$

# Weighted a priori estimate

## Idea of the proof

- We use the variable $(D, H, \chi)$:

$$\begin{cases} \partial_t D - \text{curl } H = 0 \\ \partial_t H + \text{curl } D = \text{curl } (\chi E) \end{cases}$$

- Representation of Duhamel : $\Lambda$ is the linear Maxwell operator, $U = (D, H)$ and $f(t) = {}^t(0, \text{curl } (\chi E)))$

$$U(t) = e^{t\Lambda} U^0 + \int_0^t e^{(t-\tau)\Lambda} f(\tau) d\tau, \quad 0 \le t \le \tilde{T},$$

- Semi group estimate: for $1 < p \le 2 \le q < \infty, \; 1/p + 1/q = 1$ and $N_q > 3(1 - 2/q)$,

$$\|e^{t\Lambda} U^0\|_q \le c(1+t)^{-(1-\frac{2}{q})} \|U^0\|_{N_q, p}$$

# Weighted a priori estimate

## Idea of the proof

- We use the variable $(D, H, \chi)$:

$$\begin{cases} \partial_t D - \text{curl } H = 0 \\ \partial_t H + \text{curl } D = \text{curl } (\chi E) \end{cases}$$

- Representation of Duhamel : $\Lambda$ is the linear Maxwell operator, $U = (D, H)$ and $f(t) = {}^t(0, \text{curl } (\chi E)))$

$$U(t) = e^{t\Lambda} U^0 + \int_0^t e^{(t-\tau)\Lambda} f(\tau) d\tau, \quad 0 \leq t \leq \tilde{T},$$

- Semi group estimate: for $1 < p \leq 2 \leq q < \infty, \ 1/p + 1/q = 1$ and $N_q > 3(1 - 2/q)$,

$$\| e^{t\Lambda} U^0 \|_q \leq c(1+t)^{-(1-\frac{2}{q})} \| U^0 \|_{N_q, p}$$

1. $L^2 - L^2$ estimate from classical result on Maxwell equations
2. $L^\infty - W^{3,1}$ estimate by the Kirchoff representation formula on linear wave equations

# Weighted a priori estimate

## Idea of the proof

- We use the variable $(D, H, \chi)$:

$$\begin{cases} \partial_t D - \text{curl } H = 0 \\ \partial_t H + \text{curl } D = \text{curl } (\chi E) \end{cases}$$

- Representation of Duhamel : $\Lambda$ is the linear Maxwell operator, $U = (D, H)$ and $f(t) = {}^t(0, \text{curl } (\chi E)))$

$$U(t) = e^{t\Lambda} U^0 + \int_0^t e^{(t-\tau)\Lambda} f(\tau) d\tau, \quad 0 \leq t \leq \tilde{T},$$

- Semi group estimate: for $1 < p \leq 2 \leq q < \infty$, $1/p + 1/q = 1$ and $N_q > 3(1 - 2/q)$,

$$\|e^{t\Lambda} U^0\|_q \leq c(1+t)^{-(1-\frac{2}{q})} \|U^0\|_{N_q,p}$$

  1. $L^2 - L^2$ estimate from classical result on Maxwell equations
  2. $L^\infty - W^{3,1}$ estimate by the Kirchoff representation formula on linear wave equations

- Estimate of the nonlinear term $f$

$$\|f(t)\|_{s_1+3,6/5} \leq c\left[\|\chi^0\|_{s,3/2} e^{-t} M_{s_1}(t) + \left(e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)\right) \|U(t)\|_{s,2}\right]$$

# Weighted a priori estimate

## Idea of the proof

We obtain

$$M_{s_1}(t) \leq c\delta\big(1 + M_{s_1}(t) + (1 + M_{s_1}^2(t))^2 exp\{cM_{s_1}^2(t)\}\big), \ \ 0 \leq t \leq \tilde{T}.$$

We introduce $x = M_{s_1}(t)$

We study the function

$$\varphi(x) \ := c\delta(1 + x + (1 + x^2)^2 e^{cx^2}) - x,$$

using

$$\varphi(M_{s_1}(t)) \geq 0, \ \ 0 \leq t \leq \tilde{T}.$$

# A priori bound

## Proposition 4.3

*There exist $c > 0$, an integer $s \geq 7$ and a $\delta > 0$ sufficiently small such that if*

$$\|V^0\|_{s,2} + \|V^0\|_{s,\frac{6}{5}} \leq \delta/2\,,\ \|\chi^0\|_{s,\frac{3}{2}} \leq \delta/2, \text{ with } \chi^0 \geq 0 \text{ and } \mathrm{div}\, H^0 = \mathrm{div}\,[(1+\chi^0)E^0] = 0,$$

*then $V = (E, H, \chi)$ satisfies*

$$\|V(t)\|_{s,2} \leq c\|V^0\|_{s,2}\left[(1 + M_0^2)exp\{cM_0^2\}\right] \quad \forall\, t \in [0, \tilde{T}],$$

*where*

$$\tilde{T} = max\left\{T < T^\star \text{ such that } \|V\|_{L^\infty(0,T;W^{s,2})} \leq \delta\right\}.$$

.

**Asymptotic behavior**

$$\|V(t)\|_\infty + \|V(t)\|_6 = O(t^{-2/3}),$$

$$\|V(t)\|_{s,2} = O(1) \quad as\ t \to \infty.$$

# Conclusion and perspectives

## Global small solution in 3D

For the Cauchy problem:

- Kerr : Racke 1992
- **Kerr-Debye**

For the IBVP:

- **Kerr** with Dirichlet condition (Recently)

## Global solution in 3D without smallness condition ?

## Global solution in 2D Transverse magnetic case ?

Thank you