On the asymptotics of solutions to resonator equations

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Padova, 28. June 2012

Microresonators

- Resonators are oscillatory systems or devices, which naturally oscillate at some frequencies (resonant frequencies).
- Microresonators are resonators, whose size are in the micrometer area.
- Microresonators have high sensitivity at room temperature.
- **Thermoelastic damping** is one of the reasons for the dissipation or loss of energy from the system to its surroundings.

Preliminaries

We consider the plate equation and a modified heat equation.

$$\begin{aligned} a\Delta^2 u + \Delta\theta + u_{tt} &= F, & (1) \\ \Delta\theta - m\theta + d\Delta\hat{u}_t &= c\hat{\theta}_t + G, & (2) \end{aligned}$$

where

$$\hat{f} = f + \tau f_t.$$

$d, c, a, \tau > 0$ and $m \geq 0$.

Preliminaries ($m = 0$)

We get with the heat flux q and $m = 0$ the following system

$$\begin{aligned} a\Delta^2 u + \Delta\theta + u_{tt} &= 0, & (3)\\ c\theta_t + \nabla' q - d\Delta u_t &= 0, & (4)\\ \tau q_t + q + \nabla\theta &= 0. & (5) \end{aligned}$$

$\tau = 0$ (relaxation parameter): linear thermoelastic plate equations.

System in a bounded domain

We consider the equations (1) and (2) in $B \subset \mathbb{R}^n$ to the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), && (6) \\ \theta(x,0) &= \theta_0(x), \quad \theta_t(x,0) = \theta_1(x) && (7) \end{aligned}$$

and the boundary conditions

$$u(x,t) = \Delta u(x,t) = \theta(x,t) = 0, \quad x \in \partial B, \quad t \in [0,\infty) \qquad (8)$$

or

$$u(x,t) = \nabla u(x,t) \cdot n(x) = \theta(x,t) = 0, \quad x \in \partial B, \quad t \in [0,\infty). \qquad (9)$$

Non-Exponential Stability

We consider the system(1), (2) to the boundary conditions (9). Defining

$$V := (\hat{u}, \hat{u}_t, \theta, \theta_t)'$$

we obtain

$$V_t = A_f V + \tilde{F} \quad \text{and} \quad V_0(x) := (\hat{u}, \hat{u}_t, \theta, \theta_t)'(x, 0). \tag{10}$$

with the (formal) differential operator A_f und $\tilde{F}$ given by the symbol

$$A_f := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -a\Delta^2 & 0 & -\Delta & -\tau\Delta \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d}{c\tau}\Delta & \frac{1}{c\tau}(\Delta - m) & -\frac{1}{\tau} \end{pmatrix}, \quad \tilde{F} := (0, \hat{F}, 0, -\frac{1}{c\tau}G)'.$$

Non-Exponential Stability

One can show for the spectral bound

$$S(A) := \sup \{Re\lambda \mid \lambda \in \mathbb{S}\} = 0,$$

where λ is an eigenvalue of the problem.

Theorem

The associated semigroup $\left\{e^{tA}\right\}_{t\geq 0}$ is not exponentially stable.

Next, we introduce a damping term

$$a\Delta^2 u + \Delta\theta + u_{tt} + \gamma u_t = 0, \quad (11)$$
$$\Delta\theta - m\theta + d\Delta\hat{u}_t = c\hat{\theta}_t. \quad (12)$$

We define a Lyapunov-Functional

$$F(t) :=$$
$$NE(t) + \int_B \left(\frac{1}{2}(|\nabla\beta|^2 + m\beta^2) + \tau\nabla\beta\nabla\beta_t + m\tau\beta\beta_t + d\hat{u}\hat{u}_t + \frac{\gamma d}{2}\hat{u}^2 \right) dV,$$

where $\beta(x,t) = \int_0^t \theta(x,s)ds + Q(x)$ and

$$E(t) = \int_B (d\hat{u}_t^2 + ad|\Delta\hat{u}|^2 + c\hat{\theta}^2 + \tau(|\nabla\theta|^2 + m\theta^2))dV.$$

Theorem

$$\exists c_1, c_2 > 0, \forall t \geq 0 : \qquad E(t) \leq c_1 E(0) e^{-c_2 t}$$

Asymptotic properties of the roots

We consider the system in the whole space. Application of the Fourier transform $(Fu(x,t))(\xi) =: v(\xi,t)$ and $(F\theta(x,t))(\xi) =: w(\xi,t)$ implies

$$
\begin{aligned}
a\rho^2 v(\xi,t) - \rho w(\xi,t) + v_{tt}(\xi,t) &= 0, && (13)\\
-\rho w(\xi,t) - m w(\xi,t) - d\rho \hat{v}_t(\xi,t) &= c\hat{w}_t(\xi,t), && (14)
\end{aligned}
$$

where $\rho := |\xi|^2$. We obtain for v and w

$$v_{tttt} + \frac{1}{\tau} v_{ttt} + \frac{1}{c\tau}\left((ac+d)\tau\rho^2 + m + \rho\right) v_{tt} + \frac{1}{c\tau}\rho^2 (ac+d)\, v_t + \frac{a}{c\tau}\rho^2(\rho+m)v = 0.$$

The characteristic polynomial is given by

$$P_\rho(\lambda) = \lambda^4 + \frac{1}{\tau}\lambda^3 + \frac{1}{c\tau}(\rho + m + \tau(ac+d)\rho^2)\lambda^2 + \frac{1}{c\tau}(ac+d)\rho^2\lambda + \frac{a}{c\tau}(\rho^3 + m\rho^2). \qquad (15)$$

Asymptotic properties of the roots

Theorem

The roots of the characteristic polynomial have the following properties for $\xi \to 0$ and $\rho := |\xi|^2$:

$$\begin{aligned}
\lambda_1 &= -\frac{d}{2m}\rho^2 + \mathcal{O}(\rho^3) + i\sqrt{a}\rho + i\mathcal{O}(\rho^2), \\
\lambda_2 &= -\frac{d}{2m}\rho^2 + \mathcal{O}(\rho^3) - i\sqrt{a}\rho + i\mathcal{O}(\rho^2), \\
\lambda_3 &= -\frac{1}{2\tau} + \frac{1}{2}\left(\frac{1}{\tau^2} - \frac{4m}{c\tau}\right)^{1/2} - \frac{1}{c\tau\left(\frac{1}{\tau^2} - \frac{4m}{c\tau}\right)^{1/2}}\rho + \mathcal{O}(\rho^2), \\
\lambda_4 &= -\frac{1}{2\tau} - \frac{1}{2}\left(\frac{1}{\tau^2} - \frac{4m}{c\tau}\right)^{1/2} + \frac{1}{c\tau\left(\frac{1}{\tau^2} - \frac{4m}{c\tau}\right)^{1/2}}\rho + \mathcal{O}(\rho^2).
\end{aligned}$$

Asymptotic properties of the roots

Theorem

The roots of the characteristic polynomial have the following properties for $\xi \to \infty$ *and* $\rho := |\xi|^2$. *There are positive constants* $c_1, c_2, c_3, c_4 > 0$ *such that*

$$\lambda_{1,2} = -c_1 \frac{1}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right) \pm c_2 \rho i + i\mathcal{O}(\rho^2),$$
$$\lambda_{3,4} = -c_3 + \mathcal{O}\left(\frac{1}{\rho}\right) \pm c_4 \sqrt{\rho} i + i\mathcal{O}(\rho).$$

The solution has the form

$$v(t,\xi) = a_1(\xi)e^{\lambda_1 t} + a_2(\xi)e^{\lambda_2 t} + a_3(\xi)e^{\lambda_3 t} + a_4(\xi)e^{\lambda_4 t},$$

where $a_j(\xi)$ for $j = 1,2,3,4$ are explicitly given.

The coefficients are explicitly given by

$$a_j(\xi) = \sum_{k=0}^{3} b_j^k v_k \qquad \text{for} \quad j = 1, 2, 3, 4,$$

where

$$b_j^0 = \frac{-\prod_{l \neq j} \lambda_l}{\prod_{l \neq j} \lambda_j - \lambda_l}, \qquad b_j^1 = \frac{\sum_{l \neq j \neq i} \lambda_i \lambda_l}{\prod_{l \neq j} \lambda_j - \lambda_l},$$

$$b_j^2 = \frac{-\sum_{l \neq j} \lambda_l}{\prod_{l \neq j} \lambda_j - \lambda_l}, \qquad b_j^3 = \frac{1}{\prod_{l \neq j} \lambda_j - \lambda_l}.$$

Decay rates

Let u be the inverse Fourier transformed of v, so we obtain

$$\begin{aligned} |u_t| &= C\Big|\int\limits_{\mathbb{R}^n} e^{ix\xi} v_t(\xi,t)d\xi\Big| \\ &\leq C \int\limits_{|\xi|\leq R_1} |v_t(\xi,t)|d\xi + C \int\limits_{R_1\leq|\xi|\leq R_2} |v_t(\xi,t)|d\xi + C \int\limits_{|\xi|\geq R_2} |v_t(\xi,t)|d\xi \end{aligned}$$

We obtain

$$|u_t| \leqslant C(1+t)^{-n/4}\left(||u_0||_{3n+3,1} + ||u_1||_{3n+3,1} + ||u_2||_{3n+3,1} + ||u_3||_{3n+3,1}\right).$$

Dissipativity implies L^2-L^2-estimate. Using interpolation we obtain the following theorem.

Decay rates

Theorem

Let $2 \leqslant q \leqslant \infty$, $1/p + 1/q = 1$, $m \neq 0$, $N_p > (1 - 2/q)(3n + 3)$. *Then* $\exists c = c(n, q) > 0 \;\; \forall V_0 \in W^{N_p, p}(\mathbb{R}^n) \;\; \forall t \geq 0$*:*

$$||V_t(t)||_q \leqslant c(1 + t)^{-\frac{n}{4}(1 - \frac{2}{q})} ||V_0||_{N_p, p},$$

where $V(t) = (\hat{u}(t), \hat{u}_t(t), \theta(t), \theta_t(t))$.

For $n \geq 3$ one can obtain

$$||V(t)||_q \leqslant c(1 + t)^{-\frac{n-2}{4}(1 - \frac{2}{q})} ||V_0||_{N_p, p}.$$

Decay rates

The asymptotic behaviour of the roots changes for $m = 0$. We get analogously:

Theorem

Let $2 \leqslant q \leqslant \infty$, $1/p + 1/q = 1$, $m = 0$, $N_p > (1 - 2/q)(3n + 5)$. Then $\exists c = c(n, q) > 0 \;\; \forall V_0 \in W^{N_p,p}(\mathbb{R}^n) \;\; \forall t \geq 0$:

$$||V_{tt}(t)||_q \leqslant c(1 + t)^{-\frac{n}{2}(1-\frac{2}{q})}||V_0||_{N_p,p},$$

where $V(t) = (\hat{u}(t), \hat{u}_t(t), \theta(t), \theta_t(t))$.

For $n \geq 5$ one can obtain

$$||V(t)||_q \leqslant c(1 + t)^{-\frac{n-4}{2}(1-\frac{2}{q})}||V_0||_{N_p,p}.$$