High frequency waves and the maximal smoothing effect for nonlinear scalar conservation laws

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I) Lions, Perthame, Tadmor 1994

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot F(u) &= 0 \\ u(0,X) &= u_0(X) \end{aligned}$$

$X \in \mathbb{R}^d, \quad u : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}, \quad F : \mathbb{R} \to \mathbb{R}^d$

Smoothing effect for NONLINEAR flux

$$M > 0, \qquad \exists \quad s = s(F,M) > 0$$

$$\sup_X |u_0(X)| \leq M \quad \Rightarrow \quad u \in W^{s,1}_{loc}(]0,+\infty[\times\mathbb{R}^d) \cap W^{s,1}_{loc}(\mathbb{R}^d), \quad \forall t > 0$$

II) Lions, Perthame, Tadmor 1994

NONLINEAR flux

velocity : $a(v) = F'(v)$: $\exists \alpha > 0$, $\exists C > 0$,

$$\sup_{\tau^2+|\xi|^2=1} \text{measure}\{|v| \leq M,\ |\tau + \xi \cdot a(v)| \leq \delta\} \quad \leq \quad C\delta^\alpha$$

Sobolev exponent : $\forall\ s < \dfrac{\alpha}{2+\alpha}$

Improvement : Tadmor, Tao, 2007 : $\forall\ s < \dfrac{\alpha}{1+2\alpha}$

Conjecture : Lions, Perthame, Tadmor 1994

$s_{sup} = \alpha_{sup}$

$s_{uniform}$

UNIFORM Sobolev bound and boundary layers

$$
\begin{array}{ll}
\text{Let } M > 0, \varepsilon > 0, A > 0, & s < s(F, M), \\
\exists\, C = C(\varepsilon, s, \|u_0\|_\infty, A) & \|u_0\|_{L^\infty} \leq M \\
\|u\|_{W^{s,1}([\varepsilon,A]\times[-A,A]^d)} + & \sup_{t>\varepsilon} \|u(t,.)\|_{W^{s,1}([-A,A]^d)} \leq C
\end{array}
$$

Proof : Kinetic formulation & averaging lemmas

$$\partial_t f + a(v).\nabla_x f = \partial_v m$$

$f(t, x, v)$, $v \in \mathbb{R}$, $m(t, x, v) \geq 0$

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One dimensional case : $d = 1$

Linear case : $a' \equiv 0$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

No smoothing effect :

$$u(t,x) = u_0(x - ct)$$

Propagations of high oscillations with large amplitude :

$$u_\varepsilon^0(x) = u_0\left(\frac{x}{\varepsilon}\right) \quad \Rightarrow \quad u_\varepsilon(t,x) = u_0\left(\frac{x-ct}{\varepsilon}\right),$$

where u_0 is periodic.

The strongest Nonlinear case : strictly convex flux

with $u_0(x+1) \equiv u_0(x) \in L^\infty(\mathbb{R})$ and mean $\overline{u_0} = \int_0^1 u_0(\theta) d\theta$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x}, \qquad \forall\, v \in \mathbb{R},\ a'(v) \neq 0, \qquad f(u) = \frac{u^2}{2}$$

Smoothing effect, O. Oleinik ; P. D. Lax 50'

$$On\ (0,1) \qquad \text{Total variation } u(t,.) \quad \leq \quad \frac{C}{t}.$$

$\Rightarrow$ Decay of the amplitude, $|u(t,x) - \overline{u_0}| \leq \dfrac{C}{t}$

$\Rightarrow$ No propagations of high oscillations with large amplitude :

$$u_\varepsilon^0(x) = u_0\left(\frac{x}{\varepsilon}\right) \quad \Rightarrow \quad |u_\varepsilon(t,x) - \overline{u_0}| \leq \varepsilon \frac{C}{t}$$

Proof : $y = \dfrac{x}{\varepsilon},\ s = \dfrac{t}{\varepsilon}$

Propagations of high oscillations with small amplitude

$$u_\varepsilon^0(x) = \varepsilon u_0\left(\frac{x}{\varepsilon^q}\right)$$

$$u_\varepsilon(t,x) \simeq \begin{cases} \varepsilon u_0\left(\frac{x}{\varepsilon^q}\right) & \text{if } q<1 \quad \textit{linear propagation} \\ \varepsilon U\left(t, \frac{x}{\varepsilon}\right) & \text{if } q=1 \quad \textit{nonlinear propagation} \\ \varepsilon \overline{u_0} & \text{if } q>1 \quad \textit{nonlinear smoothing effect} \end{cases}$$

Critical exponent, $q = 1 = \alpha$,
R. Diperna, A. Majda, C.M.P., 1985,

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial \theta}\frac{U^2}{2} = 0, \qquad U(0,\theta) = u_0(\theta)$$

$$u_\varepsilon(t,x) \;=\; \varepsilon U\left(t, \frac{x}{\varepsilon}\right).$$

Degenerate Nonlinear case

$p \geq 2$, $a(v) = v^p$, $\alpha_{sup} = \dfrac{1}{p} < 1$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^{1+p}}{1+p}\right) = 0$$

$$u_\varepsilon^0(x) = \varepsilon u_0\left(\frac{x}{\varepsilon^q}\right) \quad \Rightarrow \quad u_\varepsilon(t,x) \simeq \begin{cases} \varepsilon u_0\left(\frac{x}{\varepsilon^q}\right) & \text{if } q < p \\ \varepsilon U\left(t, \frac{x}{\varepsilon^q}\right) & \text{if } q = p \\ \varepsilon \overline{u_0} & \text{if } q > p \end{cases}$$

Profile equation for critical exponent, $q = p$,

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial \theta}\frac{U^{1+p}}{1+p} = 0, \qquad U(0,\theta) = u_0(\theta)$$

$$u_\varepsilon(t,x) \quad = \quad \varepsilon U\left(t, \frac{x}{\varepsilon^p}\right)$$

Burgers 2D is not non-linear

$$\begin{aligned} \partial_t u + \partial_x u^2 + \partial_y u^2 &= 0 \\ u(0,x,y) &= u_0\left(\frac{x-y}{\varepsilon^{2012}}\right) \end{aligned}$$

- Panov : $\xi = (1,-1)$, $F = (u^2, u^2)$,

$$\xi \cdot F(u) \equiv 0$$

- Lions-Perthame-Tadmor : $\xi \cdot F'(v) = \xi \cdot a(v) \equiv 0$,

$$\alpha_{sup} = 0$$

- Enguist-E : $\xi \cdot F''(v) \equiv 0$,
- stationary solutions without smoothing effect

$$u(t,x,y) = u_0(x-y)$$

2D genuine nonlinear example

$$\begin{aligned} \partial_t u + \partial_x \frac{u^2}{2} + \partial_y \frac{u^3}{3} &= 0 \\ u(0,x,y) &= 0 + \varepsilon u_0\left(\frac{\phi_0(x,y)}{\varepsilon^2}\right) \end{aligned}$$

- Genuine nonlinear flux : $a(u) = (u, u^2)$

$$\begin{pmatrix} a'(u) \\ a''(u) \end{pmatrix} = \begin{pmatrix} 1 & 2u \\ 0 & 2 \end{pmatrix}$$

- $\phi_0(x,y) = x$ cancellations
- $\phi_0(x,y) = y$ propagations : $(0,1) = \quad \nabla\phi_0 \perp a(\underline{u}=0) \quad = (1,0)$

High frequency waves

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_x . F(u) &= 0 \\ u(0,x) &= \underline{u} + \varepsilon U_0 \left(\frac{\phi_0(x)}{\varepsilon^q} \right) \\ \phi_0(x) &= v \cdot x \end{aligned}$$

1. Propagation : $u^\varepsilon(t,x) = \underline{u} + \varepsilon U\left(t, \varepsilon^{-q}\phi(t,x)\right) + \dots$
2. Cancellation $u^\varepsilon(t,x) = \underline{u} + \varepsilon \overline{U} + \dots$

Chen,J, Rascle (JDE 06), multiphase :
$u(0,x) = \underline{u} + \varepsilon U_0\left(\varepsilon^{-q_1}\phi_1(x), \cdots, \varepsilon^{-q_d}\phi_d(x)\right)$

Propagation of smooth high oscillations

Theorem

$q > 1$ an integer, $\mathbf{F} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$, $U_0 \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $\mathbf{v} \neq (0, \cdots, 0)$,

$$\frac{d^k a}{du^k}(\underline{u}) \,.\, \mathbf{v} = 0, \qquad k = 1, \cdots, q-1 \tag{1}$$

then $\exists\ T_0 > 0$ such that, for all $\varepsilon \in]0, 1]$, u^ε smooth on $[0, T_0] \times \mathbb{R}$:

$$u^\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon U\left(t, \frac{\phi(t, \mathbf{x})}{\varepsilon^q}\right) + \mathcal{O}(\varepsilon^2) \text{ in } C^1([0, T_0] \times \mathbb{R}^d),$$

$$\frac{\partial U}{\partial t} + b\frac{\partial U^{q+1}}{\partial \theta} = 0, \qquad U(0, \theta) = U_0(\theta).$$

$b = \frac{1}{(q+1)!}\left(a^{(q)}(\underline{u}) \,.\, \mathbf{v}\right)$, $\quad \phi(t, \mathbf{x}) = \mathbf{v} \cdot (\mathbf{x} - t\, a(\underline{u}))$.

Proof : WKB expansions

$$u^\varepsilon(t,\mathbf{x}) = \underline{u} + \varepsilon\, U_\varepsilon\left(t, \frac{\phi(t,\mathbf{x})}{\varepsilon^q}\right), \qquad U_\varepsilon(0,\theta) = U_0(\theta)$$

Taylor expansions : $\mathbf{v} = \nabla\phi$

$$\begin{aligned}
\partial_t U_\varepsilon\left(t, \frac{\phi(t,\mathbf{x})}{\varepsilon^q}\right) &= \partial_t U_\varepsilon - \varepsilon^{-q}(a(\underline{u})\cdot\mathbf{v})\partial_\theta U_\varepsilon \\
\mathrm{div}_{\mathbf{x}}\mathbf{F}(u^\varepsilon) &= \sum_{k=0}^{q} \frac{1}{\varepsilon^{q-(k+1)}} \frac{\partial_\theta U_\varepsilon^{k+1}}{(k+1)!} a^{(k)}(\underline{u})\cdot\mathbf{v} + \varepsilon^{q+2}\mathrm{div}_{\mathbf{x}}\mathbf{G}_q^\varepsilon(U_\varepsilon) \\
&= \varepsilon^{1-q}(a(\underline{u})\cdot\mathbf{v})\partial_\theta U_\varepsilon + \varepsilon b\partial_\theta U_\varepsilon^{q+1} + \varepsilon^2\partial_\theta g_q^\varepsilon(U_\varepsilon),
\end{aligned}$$

Simplification and characteristics

$$0 = \partial_t u^\varepsilon + \mathrm{div}_{\mathbf{x}}\mathbf{F}(u^\varepsilon) \;=\; \varepsilon\left(\partial_t U_\varepsilon + b\partial_\theta U_\varepsilon^{q+1} + \varepsilon\partial_\theta g_q^\varepsilon(U_\varepsilon)\right).$$

Uniform Sobolev bounds

sequence bounded in L^∞ for $0 < \varepsilon \leq 1$

$$u^\varepsilon(t,\mathbf{x}) = \underline{u} + \varepsilon U\left(t, \frac{\phi(t,\mathbf{x})}{\varepsilon^{q}}\right) + \dots$$

and if $\frac{d}{d\theta} U_0 \neq 0$ a.e. there exists C > 0, $0 \leq t \leq T_0$

$$\frac{1}{C} \leq \|u^\varepsilon(t,.)\|_{W^{s,1}_{loc}(\mathbb{R}^d)} \leq C, \quad s = \frac{1}{q}$$

For $s > \frac{1}{q}$ the Sobolev norm $W^{s,1}_{loc}(\mathbb{R}^d)$ blows up.

Sobolev estimates

$0 < s < 1$,

$$\frac{d^{``s''}}{dx^{``s''}}\left(\varepsilon V\left(\frac{x}{\varepsilon^q}\right)\right) \sim \frac{\varepsilon}{\varepsilon^{sq}} = \varepsilon^{1-sq}$$

- Upper bound : interpolation between L^1 and $W^{1,1}$

order ε in L^1_{loc} order ε^{1-q} in $W^{1,1}_{loc}$

order $\varepsilon^{(1-s)1+s(1-q)}$ in $W^{s,1}_{loc}$

- Lower bound : intrinsic semi-norm $W^{s,p}(\mathbb{R}^d)$

$$\int\int \frac{|V(x)-V(y)|^p}{|x-y|^{d+sp}}dxdy$$

If the orthogonality condition :

$$\frac{d^k a}{du^k}(\underline{u})\,.\,\mathbf{v} = 0, \qquad k = 1, \cdots, q-1,$$

is violated then

Cancellation of high oscillations, Chen, J, Rascle, 06'

Let **F** belongs to C^{q+1} and $U_0 \in L^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$,

If for some $0 < j < q$

$$\frac{d^j a}{du^j}(\underline{u})\,.\,\mathbf{v} \neq 0$$

then

$$u^\varepsilon(t, \mathbf{x}) = \underline{u} + \varepsilon \overline{U}_0 + o(\varepsilon) \qquad \text{in } L^1_{loc}(]0, +\infty[\times \mathbb{R}^d).$$

proof : compactness with kinetic formulation

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Nonlinear flux

- Lax : d=1, $f'' = a' \neq 0 \Longleftrightarrow$ genuine nonlinear
- Tartar : d=1, not a linear function on any interval
- Engquist-E : independence of functions $\mathbf{F}_1''(v), \cdots, \mathbf{F}_d''(v)$
- Lions-Perthame-Tadmor : related to $a(v) = \mathbf{F}'(v)$: α_{sup}
- Panov : $\forall \xi \neq 0, \quad \xi \cdot \mathbf{F}$ is non constant on any interval
- genuine nonlinearity condition for smooth multi-D flux $\mathbf{F}$:
  $\det(a'(v), a''(v), \cdots, a^{(d)}(v)) \neq 0$ everywhere
  - Chen, J., Rascle 2006,
  - Crippa ; Otto ; Westdickenberg 2008.

Nonlinear Flux

Definition

$\mathbf{F} \in C^\infty(\mathbb{R}, \mathbb{R}^d)$, $I = [-M, M]$,

$$
\begin{aligned}
d_{\mathbf{F}}[u] &= \inf\{k \geq 1, rank\{a'(u), \cdots, a^{(k)}(u)\} = d\} \qquad \geq d, \\
&= \inf\{k \geq 1, rank\{\mathbf{F}''(u), \cdots, \mathbf{F}^{(k+1)}(u)\} = d\},
\end{aligned}
$$

$$
d_{\mathbf{F}} = \sup_{|u| \leq M} d_{\mathbf{F}}[u] \qquad \in \{d, d+1, \cdots\} \cup \{+\infty\}.
$$

1. nonlinear flux **F** on $[-M, M]$ if $d_{\mathbf{F}} < +\infty$
2. genuine nonlinear if $d_{\mathbf{F}} = d$

Sharp measurement of the flux non-linearity

Theorem

$\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$,

the Lions-Perthame-Tadmor measurement of the flux non-linearity : $\alpha_{\sup}$ is given by

$$\alpha_{\sup} = \frac{1}{d_{\mathbf{F}}} \leq \frac{1}{d}.$$

If $\alpha_{\sup} > 0$ there exists $\underline{u} \in [-M, M]$ such that

$$d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}].$$

Sketch of the proof

1. $F(u) = c\dfrac{u^{1+p}}{(1+p)!}$, $a(v) = c\dfrac{v^p}{p!}$, $c \neq 0$

$$d_F = p, \qquad \underline{u} = 0$$

$$(a'(0), \cdots, a^{(p-1)}(0), a^{(p)}(0)) = (0, \cdots, 0, c)$$

$$\alpha = \frac{1}{p}$$

measure$\{|v| \leq M,\ |a(v)| \leq \delta\} = (2|c/p!|^{-\alpha}) \times \delta^{\alpha}$

2. $u \to d_{\mathbf{F}}[u]$ is upper semi-continuous
3. Uniform control of measure$\{|v| \leq M,\ |A(v; \tau, \xi)| \leq \delta\}$ where $A(v; \tau, \xi) = \tau + \xi \cdot a(v)$, $\tau^2 + |\xi|^2 = 1$.

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Maximal uniform Sobolev bound

Theorem

IF : nonlinear flux $\mathbf{F} \in C^\infty([-M, M], \mathbb{R}^d)$.
and : $\alpha_{sup} > 0$: the sharp measurement of the flux non-linearity.
THEN $\exists\ \underline{u} \in [-M, M]$, $\exists\ T_0 > 0$, $\exists\ (u_0^\varepsilon)_{0<\varepsilon<1}$ such that
$\|u_0^\varepsilon - \underline{u}\|_{L^\infty(\mathbb{R}^d)} < \varepsilon$,
and the sequence of entropy solutions $(u^\varepsilon)_{0<\varepsilon<1}$

- for all $s \leq \alpha_{\mathrm{sup}}$, the sequence $(u^\varepsilon)_{0<\varepsilon<1}$ is uniformly bounded in $W^{s,1}_{loc}([0, T_0] \times \mathbb{R}^d) \cap C^0([0, T_0], W^{s,1}_{loc}(\mathbb{R}^d))$,
- for all $s > \alpha_{\mathrm{sup}}$, the sequence $(u^\varepsilon)_{0<\varepsilon<1}$ is **unbounded** in $W^{s,1}_{loc}([0, T_0] \times \mathbb{R}^d)$ and in $C^0([0, T_0], W^{s,1}_{loc}(\mathbb{R}^d))$.

Construction of the supercritical oscillating sequences

1. $\dfrac{1}{\alpha_{sup}} = d_{\mathbf{F}} = d_{\mathbf{F}}[\underline{u}]$
2. Oscillating initial data near $\underline{u}$: with $U_0' \neq 0$ a.e.

$$u^\varepsilon(0,x) \quad = \quad \underline{u} + \varepsilon U_0\left(\frac{\phi_0(x)}{\varepsilon^q}\right)$$

3. The best phase ϕ_0 :

$$\begin{aligned} (0,\cdots,0) \neq \mathbf{v} = \nabla\phi_0 \quad & \perp \quad \{a'(\underline{u}),\cdots,a^{(q-1)}(\underline{u})\} \\ q \quad & = \quad d_{\mathbf{F}} \\ \nabla\phi_0 \cdot a^{(q)}(\underline{u}) \quad & \neq \quad 0 \end{aligned}$$

4. Sobolev uniform optimal bounds for $(u^\varepsilon)_{0<\varepsilon\leq 1}$

$$s = \frac{1}{q} = \frac{1}{d_{\mathbf{F}}} = \alpha_{sup}$$

Highlight of the conjecture

- Optimal oscillating example
  $\exists (u^\varepsilon)_{0<\varepsilon<1}$ uniformly bounded in the best Sobolev spaces conjectured by Lions-Perthame-Tadmor

$$s = \alpha_{sup}$$

- for $s > s_{sup}$ there is no uniform bound in $W^{s,1}$ for solutions with initial data bounded in L^∞

- Upper bound for the maximal Sobolev exponent
  with it *uniform* control of the Sobolev norm

$$s_{sup\ uniform} \leq \alpha_{sup}$$

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Recent Works

1. Optimal smoothing effect $s = \alpha_{sup}$
   for nonlinear degenerate convex flux in one dimension :$d = 1$.
   Smoothing effect with Christian Bourdarias & Marguerite Gisclon
   with fractional BV spaces : $BV^s \simeq W^{s,1/s}$

   $$s_{sup} = \alpha_{sup} \quad \text{for } d = 1 \ \& \ a(v) \nearrow$$

   Examples for optimality with Pierre Castelli
2. $d \geq 1$ bound for the smoothing effect with Pierre Castelli
   $\forall \varepsilon > 0,\ \exists\ |u_0(x)| \leq M$ such that for $t > 0$
   $u(t,.) \in W^{\alpha_{sup}-\varepsilon,1}$ and $u(t,.) \notin W^{\alpha_{sup}+\varepsilon,1}$ i.e. :

   $$s_{sup} \leq \alpha_{sup}$$

Optimality for the 1D case

1. $f(u) = u^3$, $u_0(x) = x \sin\left(\frac{1}{x}\right)$
   Kuo-Shung Cheng.
   The space BV is not enough for hyperbolic conservation laws.
   J.D.E. 1983.
2. For power-law flux and piecewise constant solutions :
   Camillo De Lellis ; Michael, Westdickenberg.
   On the optimality of velocity averaging lemmas.
   Ann. Inst. H. Poincaré Anal. Non Linéaire, 2003.

Optimality for all smooth flux with continuous solutions

joint work with Pierre Castelli

1. space dimension $d = 1$
   For all smooth flux and continuous solutions :
   $u_0(x) = x^\beta \cos\left(\frac{1}{x^\gamma}\right)$
   Pierre Castelli,
   Master's Thesis 2012 and current work.
2. $d > 1$
   Like High frequency waves : construction using the new characterisation of smooth nonlinear flux : slides 20-24

THANK YOU

FOR

YOUR ATTENTION