Global well-posedness of classical solution to 2D compressible Navier-Stokes equations with large data and vacuum

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• Consider the following 2D compressible and isentropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + \nabla((\mu + \lambda(\rho))\mathrm{div} u), \end{cases} \tag{1}$$

- $x \in \mathbb{T}^2$**: 2-dimensional torus** $[0,1] \times [0,1]$**,** $t \in (0, \infty)$
- $\rho(t,x) \geq 0$**: density;**
- $u(t,x) = (u_1, u_2)(t,x)$ **: velocity;**
- $P(\rho) = A\rho^\gamma$**: pressure,** $\gamma > 1$**. Normalize** $A = 1$ **for simplicity.**

- **viscosities:**

$$\mu = \text{const.} > 0, \quad \lambda(\rho) = \rho^{\beta}, \ \beta > 0,$$

such that the operator

$$\mathcal{L}_{\rho} u := \mu \Delta u + \nabla((\mu + \lambda(\rho))\text{div} u)$$

is strictly elliptic.

- **The initial values are given by**

$$(\rho, u)(t = 0, x) = (\rho_0, u_0)(x), \quad x \in \mathbb{T}^2. \tag{2}$$

- **First proposed and studied by Kazhikhov and Vaigant (1995), global well-posedness of strong solution with large data away from vacuum with $\beta > 3$.**

Goals:

- **Global well-posedness of classical solution with large data and vacuum on periodic domain.**
- **Two-dimensional Cauchy problem (In progress).**

Well-posedness of Multi-Dimensional CNS (N=2,3):

(1) Both the shear and bulk viscosities are positive constants:

- **local well-posedness of classical(or strong) solutions:**
  - **absence of vacuum:**
    Nash(1962), Itaya(1971) and Tani(1977) (3D initial-boundary-value problem and Cauchy problem);
  - **with vacuum:**
    Cho-Kim (2004, 3D), Luo (2012, 2D) (a natural compatibility condition (Cho-Kim, 2004));

- Global well-posedness with small data and absence of vacuum:

  - Matsumura-Nishida (1980):
    Global classical solution to the CNS with small perturbation on a non-vacuum equilibrium state in $H^s(\mathbb{R}^3)$ $(s > \frac{5}{2})$.

  - Hoff (1997):
    Discontinuous data, small energy perturbation.

  - Danchin (2000), Chen-Miao-Zhang (2010):
    Besov spaces.

- **Global existence of weak solution with large data and vacuum:**
  - **P. L. Lions(1998):** $\gamma > \frac{3}{2}, N = 2, \gamma > \frac{9}{5}, N = 3$**;**
  - **Jiang-Zhang(2001):** $\gamma > 1$**, spherically symmetric;**
  - **E. Fiereisl(2004):** $\gamma > \frac{N}{2}, (N = 2, 3)$**;**
- **Regularity and uniqueness of such weak solution?**
  - **Desjardins(1997):** $u \in L^\infty([0, T]; H^1(\mathbb{T}^2))$ **in the case 2D periodic problems under the assumption of uniform bound of the density;**
  - **Hoff(2005): a new type of global weak solution with small total energy, which has extra structure and regularity information (such as Lagrangian structure in the non-vacuum region) compared with the renormalized weak solutions;**

- However, the uniqueness and regularity of these weak solutions remain completely open in general!

- Weak-strong uniqueness: (P. L. Lions, P. Germain, 2011)

How about global well-posedness of classical (or strong) solution in the presence of vacuum?

- Subtle and difficult problem in general!

- **Xin (1998): For Full CNS, finite time blow-up results of classical solution in $C^1([0,T]; H^s(\mathbb{R}^N))$ with $s > [\frac{N}{2}]+2$ if the initial density has compact support and without heat conduction; 1D isentropic CNS;**
- **Rozanova (2008): Non-compact support but rapidly decreasing (at far fields) initial density;**
- **Xin-Yan (2012): Removed the assumptions of compact support and finite total energy.**
- **Huang-Li-Xin (2012) : Proved the global well-posedness of classical solutions with small energy but large oscillations which can contain vacuums to 3D isentropic compressible Navier-Stokes equations.**

(2) Density-dependent viscosities: Both the shear and bulk viscosities depend on the density.

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = \mathbf{div}(h(\rho)D(u)) + \nabla(g(\rho)\mathbf{div} u), \end{cases}$$

where

$$D(u) = \frac{\nabla u + (\nabla u)^t}{2}$$

- **Derived from Boltzmann equation by the Chapman-Enskog expansions (Liu-Xin-Yang, 1998);**
- **Shallow water model (**$h(\rho) = \rho, g(\rho) = 0, P(\rho) = \rho^2$**);**

R. Danchin, B. Didier, B. Desjardins, Z. H. Guo, S. Jiang, H.L. Li, J. Li, C.K. Lin, T. P. Liu, A. Mellet, Š.Matušu-Nečasová, Makino, M.Okada, I. Straskraba, A. Vasseur, Z.P.Xin, T. Yang, Z. A. Yao, P.Zhang, T. Zhang, D.Y.Fang, H. J. Zhao , C. J. Zhu, A. Zlotnik, $\cdots\cdots$

1D case:

- **Li-Li-Xin (2008): vanishing vacuum phenomena(IBVP or periodic domain);**
- **J.-Xin (2008): Cauchy problem, constants $\bar{\rho} \geq 0$ at far fields (vacuum or nonvacuum);**
- **J.-Wang-Xin(2011,2012): Cauchy problem, stability of rarefaction wave, far field vacuum will not vanish.**
- **Ding-Wen-Zhu (2011), Ding-Wen-Yao-Zhu(2012), global classical solution with large data and vacuum, under assumption that the viscosity has a lower bound: $h(\rho), g(\rho) \geq \mu_0 > 0$.**

Multi-D case:

- Bresch-Desjardins (2004), Bresch-Desjardins-Lin (2003):Entropy estimates (provide higher regularity of the density), global existence of weak solutions with extra friction terms;
- Mellet-Vasseur(2007):L^1 stability, the compactness framework for the weak solution, under some restrictions of the viscosity:$g(\rho) = \rho h'(\rho) - h(\rho)$.
- Guo-J.-Xin (2008): Existence of global weak solution for spherically symmetric flow;
- Guo-Li-Xin (2012): Lagrange structure and dynamics for weak solutions to the spherically symmetric flow;
- In general, the global existence of the weak solutions remains open! The key is to construct the smooth approximate solutions satisfying estimates required in L^1 stability.

(3) Density-dependent viscosities: the shear viscosity is a constant and the bulk viscosity depends on the density.

- **Vaigant-Kazhikhov (1995, 2D):**

$$\mu = \text{const.} > 0, \quad \lambda(\rho) = \rho^{\beta}, \ \beta > 3,$$

  Large data, away from vacuum.

- **Perepelitsa (2006): the global existence and large time behavior of weak solutions, uniform lower and upper bounds of the density.**

Main results:

- **Theorem 1.1.** **Assume that**

$$0 \le (\rho_0(x), P(\rho_0)(x)) \in W^{2,q}(\mathbb{T}^2)\times W^{2,q}(\mathbb{T}^2), u_0(x) \in H^2(\mathbb{T}^2), \int_{\mathbb{T}^2} \rho_0(x)dx > 0$$

for some $q > 2$ **and the compatibility condition**

$$\mathcal{L}_{\rho_0} u_0 - \nabla P(\rho_0) = \sqrt{\rho}_0 g(x)$$

with some $g \in L^2(\mathbb{T}^2)$**. Then there exists a unique global classical solution** $(\rho, u)(t, x)$ **to the CNS (1) with initial values (2) and periodic boundary conditions, satisfying**

$$\begin{aligned}
&0 \le \rho(t, x) \le C, \quad (\rho, P(\rho))(t, x) \in C([0, T]; W^{2,q}(\mathbb{T}^2)),\\
&u \in C([0, T]; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)), \quad \sqrt{t}u \in L^\infty(0, T; H^3(\mathbb{T}^2)),\\
&tu \in L^\infty(0, T; W^{3,q}(\mathbb{T}^2)), \quad u_t \in L^2(0, T; H^1(\mathbb{T}^2))\\
&\sqrt{t}u_t \in L^2(0, T; H^2(\mathbb{T}^2)) \cap L^\infty(0, T; H^1(\mathbb{T}^2)), \quad tu_t \in L^\infty(0, T; H^2(\mathbb{T}^2)),\\
&\sqrt{t}\sqrt{\rho}u_{tt} \in L^2(0, T; L^2(\mathbb{T}^2)), \quad t\sqrt{\rho}u_{tt} \in L^\infty(0, T; L^2(\mathbb{T}^2)),\\
&t\nabla u_{tt} \in L^2(0, T; L^2(\mathbb{T}^2)).
\end{aligned}$$

If the initial values are more regular, then we can prove

- **Theorem 1.2. Assume that**

$$0 \le (\rho_0(x), P(\rho_0)(x)) \in H^3(\mathbb{T}^2)\times H^3(\mathbb{T}^2), u_0(x) \in H^3(\mathbb{T}^2), \int_{\mathbb{T}^2} \rho_0(x)dx > 0$$

and the compatibility condition

$$\mathcal{L}_{\rho_0} u_0 - \nabla P(\rho_0) = \sqrt{\rho}_0 g(x)$$

with some $g \in L^2(\mathbb{T}^2)$. Then the results of Theorem 1.1 old true with any $2 < q < \infty$. Furthermore, it holds that

$$u \in L^2(0,T; H^4(\mathbb{T}^2)), \quad (\rho, P(\rho)) \in C([0,T]; H^3(\mathbb{T}^2)),$$
$$\rho u \in C([0,T]; H^3(\mathbb{T}^2)).$$

- **Remark 1** **In Theorem 1.2, it is not clear whether or not** $u \in C([0,T]; H^3(\mathbb{T}^2))$ **even though one has** $\rho u \in C([0,T]; H^3(\mathbb{T}^2))$**.**

- **Remark 2** **If the initial data contains vacuum, then the compatibility condition is a necessary one to the classical solution.**

- **Denote the effective viscous flux by**

$$F = (2\mu + \lambda(\rho))\mathrm{div} u - P(\rho),$$

and the vorticity by

$$\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

- **Kazhikhov and Vaigant introduced**

$$H = \frac{1}{\rho}(\mu\omega_{x_1} + F_{x_2}), \qquad L = \frac{1}{\rho}(-\mu\omega_{x_2} + F_{x_1}).$$

Then the momentum equation can be rewritten as

$$\begin{cases} u_{1t} + u \cdot \nabla u_1 = \frac{1}{\rho}(-\mu\omega_{x_2} + F_{x_1}) = L, \\ u_{2t} + u \cdot \nabla u_2 = \frac{1}{\rho}(\mu\omega_{x_1} + F_{x_2}) = H, \end{cases}$$

that is, the material derivative of u (introduced by Hoff)

$$\dot{u} = (\dot{u}_1, \dot{u}_2)^t = (L, H)^t.$$

Kazhikhov and Vaigant's Formulation

- **Then the effective viscous flux F and the vorticity ω solve the following system:**

$$\begin{cases} \omega_t + u\cdot\nabla\omega + \omega \mathrm{div} u = H_{x_1} - L_{x_2}, \\ F_t + u\cdot\nabla F - \rho(2\mu+\lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)}{2\mu+\lambda(\rho)})']\mathrm{div} u \\ +(2\mu+\lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2] = (2\mu+\lambda(\rho))(H_{x_2}+L_{x_1}). \end{cases}$$

- **Furthermore, the system for $\dot{u} = (L,H)^t$ can be derived as**

$$\begin{cases} \rho H_t + \rho u\cdot\nabla H - \rho H \mathrm{div} u + u_{x_2}\cdot\nabla F + \mu u_{x_1}\cdot\nabla\omega + \mu(\omega \mathrm{div} u)_{x_1} \\ \qquad -\big\{\rho(2\mu+\lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)}{2\mu+\lambda(\rho)})']\mathrm{div} u\big\}_{x_2} \\ \qquad +\big\{(2\mu+\lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2]\big\}_{x_2} \\ \qquad = [(2\mu+\lambda(\rho))(H_{x_2}+L_{x_1})]_{x_2} + \mu(H_{x_1}-L_{x_2})_{x_1}, \\ \rho L_t + \rho u\cdot\nabla L - \rho L \mathrm{div} u + u_{x_1}\cdot\nabla F - \mu u_{x_2}\cdot\nabla\omega - \mu(\omega \mathrm{div} u)_{x_2} \\ \qquad -\big\{\rho(2\mu+\lambda(\rho))[F(\frac{1}{2\mu+\lambda(\rho)})' + (\frac{P(\rho)}{2\mu+\lambda(\rho)})']\mathrm{div} u\big\}_{x_1} \\ \qquad +\big\{(2\mu+\lambda(\rho))[(u_{1x_1})^2 + 2u_{1x_2}u_{2x_1} + (u_{2x_2})^2]\big\}_{x_1} \\ \qquad = [(2\mu+\lambda(\rho))(H_{x_2}+L_{x_1})]_{x_1} - \mu(H_{x_1}-L_{x_2})_{x_2}. \end{cases}$$

To prove the main results, we will utilize the above systems in different steps, which are equivalent to each other for the smooth solution.

The main approach is:

- **Approximate solutions by using Vaigant-Kazhikhov's theory**
- **Uniform a priori estimates**
  - L^p **bound of the density for** $1 < p < \infty$ **large**
  - **First derivative estimates of the velocity**
  - **Second derivative estimates of the velocity**
  - **Upper bound of the density**
  - **Higher derivative estimates of the velocity and density**
- **Passage to the limit, well-posedness of the classical solution**

Approximation solutions

First, the initial density and pressure can be approximated as

$$\rho_0^\delta = \rho_0 + \delta, \qquad P_0^\delta = P(\rho_0) + \delta,$$

for any small positive constant $\delta > 0$.

To approximate the initial velocity, we define u_0^δ to be the unique solution to the following elliptic problem

$$\mathcal{L}_{\rho_0^\delta} u_0^\delta = \nabla P_0^\delta + \sqrt{\rho_0} g$$

with the periodic boundary conditions on $\mathbb{T}^2$ and $\int_{\mathbb{T}^2} u_0^\delta dx = \int_{\mathbb{T}^2} u_0 dx := \bar{u}_0$.

By the elliptic regularity one can get that

$$\|u_0^\delta - u_0\|_{H^2(\mathbb{T}^2)} \to 0, \qquad \text{as} \;\; \delta \to 0.$$

By Vaigant-Kazhikhov's result (1995), there exists a unique global strong solution (ρ^δ, u^δ) such that $c_\delta \le \rho^\delta \le C_\delta$ for some positive constants c_δ, C_δ.

Elementary energy estimates:

$$\sup_{t\in[0,T]}\int(\rho|u|^2+\rho^\gamma)(t,x)dx+\int_0^T\int\left(|\nabla u|^2+(2\mu+\lambda(\rho))|\mathrm{div}u|^2\right)dxdt\le C.$$

L^p Density estimates for $1<p<\infty$ large

Applying the operator div to the momentum equation:

$$[\mathrm{div}(\rho u)]_t+\mathrm{div}[\mathrm{div}(\rho u\otimes u)]=\Delta F.$$

Consider the following two elliptic problems:

$$\Delta\xi=\mathrm{div}(\rho u),\qquad \int\xi dx=0,$$

$$\Delta\eta=\mathrm{div}[\mathrm{div}(\rho u\otimes u)],\qquad \int\eta dx=0,$$

both with the periodic boundary condition on the torus $\mathbb{T}^2$.

Define

$$\theta(\rho) = \int_1^\rho \frac{2\mu + \lambda(s)}{s} ds = 2\mu \ln \rho + \frac{1}{\beta}(\rho^\beta - 1).$$

Then we obtain a new transport equation

$$(\xi + \theta(\rho))_t + u \cdot \nabla(\xi + \theta(\rho)) + P(\rho) + \eta - u \cdot \nabla \xi + \int F(t, x) dx = 0.$$

- **For any** $k \geq 1$, **it holds that**

$$\sup_{t \in [0,T]} \|\rho(t, \cdot)\|_k \leq C k^{\frac{2}{\beta - 1}}.$$

- **one of key point here is Poincare inequality in two dimension:**

$$\|u - \bar{u}\|_{2m} \leq C m^{\frac{1}{2}} \|\nabla u\|_{\frac{2m}{m+1}},$$

where $m > 2$ **and** $\bar{u} = \bar{u}(t) = \int u(t, x) dx.$

First-order derivative estimates of the velocity:

Denote

$$Z(t) = \int (\mu\omega^2 + \frac{F^2}{2\mu + \lambda(\rho)})dx, \varphi(t) = \int_0^T \int \rho(H^2 + L^2)dxdt.$$

Then

$$\frac{1}{2}\frac{d}{dt}(Z^2(t)) + \frac{1}{2}\varphi(t)^2 \leq Cm(\frac{m}{\varepsilon})^{\frac{2}{\beta-1}}(1 + Z(t)^2)^{2+4\varepsilon},$$

with $m \gg 1, \varepsilon = 2^{-m}$**.**

Need

$$Cm^{1+\frac{2}{\beta-1}}2^{-m(1-\frac{2}{\beta-1})} \ll 1,$$

provided that $\beta > 3$**.**

This leads to

$$\sup_{t\in[0,T]} \int (\mu\omega^2 + \frac{F^2}{2\mu + \lambda(\rho)})dx + \int_0^T \int \rho(H^2 + L^2)dxdt \leq C.$$

Second order derivative estimates for the velocity:

$$\sup_{t\in[0,T]}\int \rho(H^2+L^2)dx+\int_0^T\int \mu(H_{x_1}-L_{x_2})^2+(2\mu+\lambda(\rho))(H_{x_2}+L_{x_1})^2dxdt \le C.$$

Here we used the compatibility condition to deal with the initial data. By the approximate compatibility condition, one has

$$\mathcal{L}_{\rho_0^\delta}u_0^\delta - \nabla P_0^\delta = \sqrt{\rho}_0 g, \quad \text{with} \quad g\in L^2(\mathbb{T}^2).$$

On the other hand

$$\mathcal{L}_{\rho_0^\delta}u_0^\delta - \nabla P_0^\delta = \rho_0^\delta(L_0^\delta, H_0^\delta)^t.$$

Therefore

$$\sqrt{\rho_0}g = \rho_0^\delta(L_0^\delta, H_0^\delta)^t.$$

Consequently, it holds that

$$\|\sqrt{\rho_0^\delta}(L_0^\delta, H_0^\delta)\|_2^2 = \|\frac{\sqrt{\rho_0}}{\sqrt{\rho_0^\delta}}g\|_2^2 \le C.$$

Upper bound of the density:

- **It holds that**

$$\int_0^T \|(F,\omega)\|_\infty^3 dt \le C.$$

- **It holds that**

$$\rho(t,x) \le C, \qquad \forall (t,x) \in [0,T]\times \mathbb{T}^2,$$

by using the transport equation:

$$\theta(\rho)_t + u\cdot\nabla\theta(\rho) + P(\rho) + F = 0.$$

Higher order estimates:

It holds that for any $1 < p < \infty$,

- $$\int_0^T \left(\|\mathrm{div} u\|_\infty^3 + \|\nabla(F,\omega)\|_p^2\right) dt \le C.$$

- $$\sup_{t\in[0,T]} \|(\nabla\rho, \nabla P(\rho))(t,\cdot)\|_p + \int_0^T \|\nabla u\|_\infty^2 dt \le C.$$

- $$\sup_{t\in[0,T]} \left[\|u(t,\cdot)\|_\infty + \|\nabla u\|_p + \|(\rho_t, P_t)\|_p + \|(\rho_t, P(\rho)_t)\|_{H^1} + \|(\rho,u)\|_{H^2}\right] + \int_0^T \|u\|_{H^3}^2 dt \le C.$$

It holds that

- $$\sup_{t\in[0,T]} \|\sqrt{\rho}u_t\|_2^2(t) + \int_0^T \|u_t\|_{H^1}^2 dt \le C.$$

- $$\sup_{t\in[0,T]} \|(\rho_t, P(\rho)_t, \lambda(\rho)_t)\|_{H^1}(t) + \int_0^T \|(\rho_{tt}, P(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 dt \le C.$$

- $$\sup_{t\in[0,T]} \Big[t\|u_t\|_{H^1}^2 + t\|u\|_{H^3}^2 + t\|(\rho_{tt}, P(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|(\rho, P(\rho))\|_{W^{2,q}}\Big] + \int_0^T t\Big[\|\sqrt{\rho}u_{tt}\|_2^2(t) + \|u_t\|_{H^2}^2(t) + \|u\|_{H^4}^2\Big] dt \le C.$$

- $$\sup_{t\in[0,T]} \big[t^2\|\sqrt{\rho}u_{tt}\|_2^2(t) + t^2\|u_t\|_{H^2}^2 + t^2\|u\|_{W^{3,q}}^2\big] + \int_0^T t^2\|\nabla u_{tt}\|_2^2(t) dt \le C.$$

Recent progress:

- **Bian-Guo (2012), Two-dimensional compressible MHD, periodic boundary problem, no vacuum.**
- **Huang-Li(2012),** $\beta > 4/3$**, periodic boundary problem.**
- **Jiu-Wang-Xin (2012) 2D Cauchy problem:** $x \in \mathbb{R}^2, t \in [0, \infty)$.

Elementary energy estimates:

$$\sup_{t\in[0,T]} \int (\rho|u|^2 + \rho^\gamma)(t,x)dx + \int_0^T \int \left(|\nabla u|^2 + (2\mu + \lambda(\rho))|\mathrm{div} u|^2\right) dxdt \le C.$$

New difficulties:

- **If** $\rho \to 0$ **as** $|x| \to \infty$**, then there is no any integrability on** u **itself by the elementary energy estimates.**

- **No Poincare type inequality in the whole space.**

Suitable new weighted estimates are necessary.

Lemma (Caffarelli-Kokn-Nirenberg) **For** $h \in C_0^\infty(\mathbb{R}^2)$**, it holds that**

$$\||x|^\kappa h\|_r \le C \||x|^\alpha |\nabla h|\|_p^\theta \ \||x|^\beta h\|_q^{1-\theta}$$

where $1 \le p, q < \infty, 0 < r < \infty, 0 \le \theta \le 1, \frac{1}{p} + \frac{\alpha}{2} > 0, \frac{1}{q} + \frac{\beta}{2} > 0, \frac{1}{r} + \frac{\kappa}{2} > 0$ **and satisfying**

$$\frac{1}{r} + \frac{\kappa}{2} = \theta(\frac{1}{p} + \frac{\alpha - 1}{2}) + (1 - \theta)(\frac{1}{q} + \frac{\beta}{2}),$$

- **Theorem 2.1. Assume that**

  $$0 \le (\rho_0(x), P(\rho_0)(x)) \in W^{2,q}(\mathbb{R}^2) \times W^{2,q}(\mathbb{R}^2), u_0(x) \in D^1 \cap D^2(\mathbb{R}^2),$$
  $$\rho_0(1+|x|^{\alpha_1}) \in L^1(\mathbb{R}^2), \sqrt{\rho_0}u_0(1+|x|^{\frac{\alpha}{2}}) \in L^2(\mathbb{R}^2), \nabla u_0|x|^{\frac{\alpha}{2}} \in L^2(\mathbb{R}^2),$$

  for $q > 2$**,** $0 < \alpha < 2\sqrt{\sqrt{2}-1}$**,** $\alpha < \alpha_1$**, and the compatibility condition**

  $$\mathcal{L}_{\rho_0}u_0 - \nabla P(\rho_0) = \sqrt{\rho}_0 g(x)$$

  with some g **satisfying** $g(1+|x|^{\frac{\alpha}{2}}) \in L^2(\mathbb{R}^2)$**. Then there exists a unique global classical solution** $(\rho, u)(t, x)$ **to the Cauchy**

problem of the CNS (1), satisfying

$$
\begin{aligned}
&0 \le \rho \le C, \quad (\rho, P(\rho))(t,x) \in C([0,T]; W^{2,q}(\mathbb{R}^2)),\\
&\rho(1+|x|^{\alpha_1}) \in C([0,T]; L^1(\mathbb{R}^2)),\\
&\sqrt{\rho}u(1+|x|^{\frac{\alpha}{2}}), \sqrt{\rho}\dot{u}(1+|x|^{\frac{\alpha}{2}}), \nabla u|x|^{\frac{\alpha}{2}} \in C([0,T]; L^2(\mathbb{R}^2)),\\
&u \in C([0,T]; L^{\frac{4}{\alpha}} \cap D^2(\mathbb{R}^2)) \cap L^2(0,T; L^{\frac{4}{\alpha}} \cap D^3(\mathbb{R}^2)),\\
&\sqrt{t}u \in L^\infty(0,T; D^3(\mathbb{R}^2)), \quad tu \in L^\infty(0,T; D^{3,q}(\mathbb{R}^2)),\\
&u_t \in L^2(0,T; L^{\frac{4}{\alpha}}(\mathbb{R}^2) \cap D^1(\mathbb{R}^2)), \quad tu_t \in L^\infty(0,T; D^2(\mathbb{R}^2))\\
&\sqrt{t}u_t \in L^2(0,T; D^2(\mathbb{R}^2)) \cap L^\infty(0,T; L^{\frac{8}{\alpha}} \cap D^1(\mathbb{R}^2)),\\
&\sqrt{t}\sqrt{\rho}u_{tt} \in L^2(0,T; L^2(\mathbb{R}^2)), \quad t\sqrt{\rho}u_{tt} \in L^\infty(0,T; L^2(\mathbb{R}^2)),\\
&t\nabla u_{tt} \in L^2(0,T; L^2(\mathbb{R}^2)),
\end{aligned}
$$

where $\dot{u} = u_t + u \cdot \nabla u$ **is the material derivative of** u**.**

- **Homogenous Sobolev space** $D^{k,p}(\mathbb{R}^2) = \{u \in L^1_{loc}(\mathbb{R}^2) | \|\nabla^k u\|_p < +\infty\}$ **with** $\|u\|_{D^{k,p}} := \|\nabla^k u\|_p$ **and** $D^k(\mathbb{R}^2) := D^{k,2}(\mathbb{R}^2)$**.**

If the initial values are more regular, then

- **Theorem 2.2 If the initial values $(\rho_0, u_0)(x)$ satisfy the conditions in Theorem 2.1 and**

  $$0 \le (\rho_0(x), P(\rho_0)(x)) \in H^3(\mathbb{R}^2) \times H^3(\mathbb{R}^2), \quad u_0(x) \in D^1 \cap D^3(\mathbb{R}^2),$$

  then the results of Theorem 2.1 hold true for any $2 < q < \infty$. Furthermore,

  $$\begin{aligned} &u \in L^2(0,T; D^4(\mathbb{R}^2)), \quad (\rho, P(\rho)) \in C([0,T]; H^3(\mathbb{R}^2)), \\ &\rho u \in C([0,T]; D^1 \cap D^3(\mathbb{R}^2)), \quad \sqrt{\rho}\nabla^3 u \in C([0,T]; L^2(\mathbb{R}^2)). \end{aligned}$$

Remark. The estimates are much more complicated and delicate.

THANK YOU!