Hydrodynamical behaviour in chemotaxis

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Chemotaxis

Chemotaxis is the phenomenon in which a population of single-cell or multicellular organisms rearrange its structure (-taxis) according to certain chemicals (chemo-) in their environment.

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- bacteria (E. coli)
- immune or infection processes
- angiogenesis (blood vessels)
- metastasis...

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- immune or infection processes
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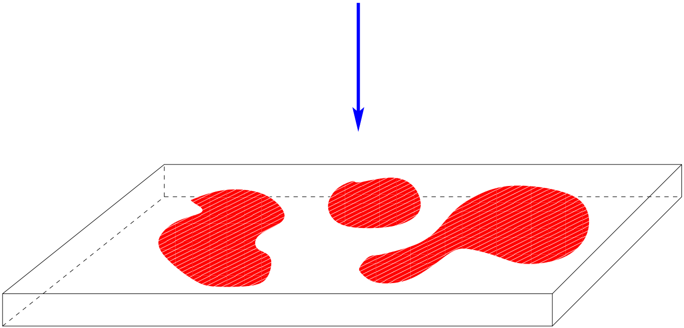
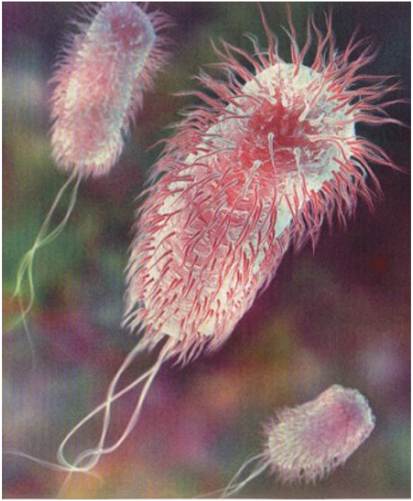
Common features: aggregation, networks

Experiments

View of a Petri box (thanks Institut Curie, Paris)



How E. Coli measures concentrations



- runs
- tumbles

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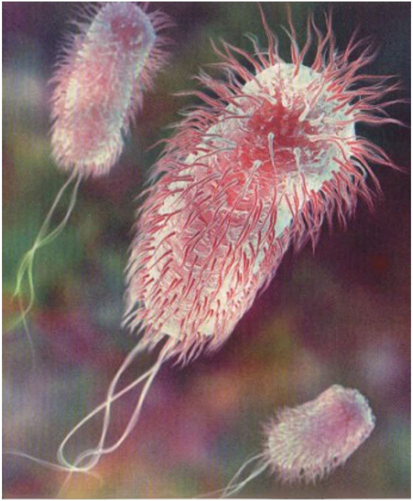

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  along straight lines
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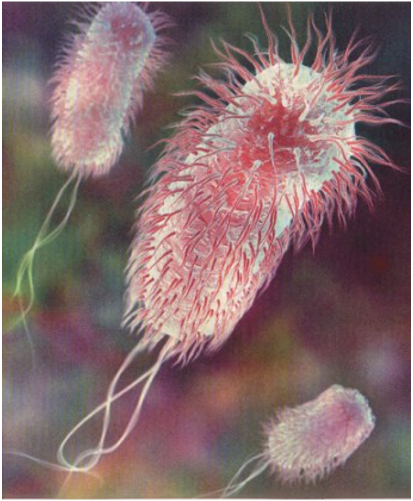
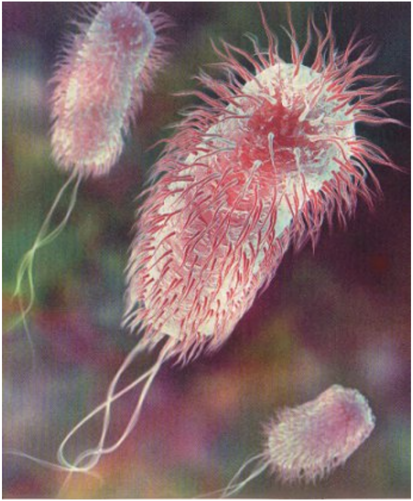


- runs
  along straight lines
  the better is life, the longer
  is time swimming
- tumbles

How E. Coli measures concentrations



- runs
- tumbles
  forget about direction, and choose another one

Kinetic model

Density of cells $f(t, x, v)$ at time t, position $x \in \mathbb{R}^d$, velocity v

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swimming = tumbling

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$$\partial_t f + v \cdot \nabla_x f = \text{tumbling}$$

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T is the tumbling kernel
actually involves the chemoattractant

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where φ gives the rate of changing direction:

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$$\varphi(z) = \varphi_0 \begin{cases} 1+\lambda & \text{if } z \le -\alpha \\ \text{smooth} & \text{if } |z| \le \alpha \\ 1-\lambda & \text{if } z \ge \alpha \end{cases} \qquad \begin{array}{l} \varphi_0 > 0 \\ \alpha \ge 0 \\ 0 < \lambda < 1 \end{array}$$

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corresponds to positive chemotaxis

- For "large" bacteria:

$$T[S](v, v') = \varphi(v \cdot \nabla_x S)$$

Final coupling

$$\partial_t f + v \cdot \nabla_x f = \int_{|v|=c} \left[T[S](v', v) f(v') - T[S](v, v') f(v) \right] dv'$$

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Stroock (74), Alt, Othmer (80-s)
Chalub, Perthame, Markowich, Schmeiser,.... (00-s)
Existence of weak solutions in $L^p(\mathbb{R}^2_x; L^1_v)$ [Bournaveas, Calvez, Gutiérrez, Perthame (08)]
L^∞ blow-up in infinite time [Vauchelet (10)]

Towards hydrodynamical behaviour

Bacteria "time sampling" $\varepsilon \approx 1/(\varphi_0 v_0^d)$

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Weak taxis

diffusive scalings – drift-diffusion models (e.g. Keller-Segel)

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Dominating taxis

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\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon &= \frac{1}{\varepsilon} \int_{|v|=c} \big[T[S_\varepsilon](v', v) f_\varepsilon(v') - T[S_\varepsilon](v, v') f_\varepsilon(v) \big] \, dv' \\
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Mass conservation

$$\partial_t \rho_\varepsilon + \operatorname{div} J_\varepsilon = 0, \quad J_\varepsilon = \int v f_\varepsilon \, dv.$$

Formal limit

In 1-d

$$\begin{aligned} \partial_t f_\varepsilon + v\partial_x f_\varepsilon &= \frac{1}{\varepsilon}\left[\varphi(-v\partial_x S_\varepsilon) f_\varepsilon(-v) - \varphi(v\partial_x S_\varepsilon) f_\varepsilon(v)\right] \\ -\partial_{xx} S_\varepsilon + S_\varepsilon &= \rho_\varepsilon \end{aligned}$$

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$$\varphi(-c\partial_x S_0)f_0(-c) = \varphi(c\partial_x S_0)f_0(c)$$

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- existence

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- existence
- uniqueness
- $\rho_\varepsilon \to \rho$
- proved in $2-$d for fixed smooth S: Dolak-Schmeiser

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Measure-valued solutions – Dirac masses: aggregates

Another limit

Set $$a(\partial_x S_\varepsilon) = \frac{\int v\varphi(v\partial_x S_\varepsilon)\,dv}{\int \varphi(v\partial_x S_\varepsilon)\,dv}$$

Estimates on kinetic model imply

$$J_\varepsilon - a(\partial_x S_\varepsilon)\rho_\varepsilon \longrightarrow 0.$$

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Limit flux $\quad J = -\partial_x A(\partial_x S) + a(\partial_x S) S$

Where is the velocity?

Conservative advection

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Conservative advection

$$-\partial_{xx} S + S = \rho$$

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1. diffusion holds in the weak sense

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(ρ, S) is a duality solution if

there exists $b \in L^\infty$ satisfying $\partial_x b \le \alpha \in L^1_{loc}(0, T)$ such that

1. diffusion holds in the weak sense
2. $$b = a(\partial_x S) \quad a.e.$$

Conservative advection

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1. diffusion holds in the weak sense
2. $$b = a(\partial_x S) \quad a.e.$$
3. for all $0 < t_1 < t_2 < T$

$$\partial_t \rho + \partial_x (b\rho) = 0 \quad \text{in the sense of duality on }]t_1, t_2[,$$

Duality solutions

$$\partial_t \rho + \partial_x \big(a(t,x)\rho \big) = 0$$
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$$\partial_t \rho + \partial_x \big(a(t,x)\rho\big) = 0$$
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$$\partial_t p + a(t,x)\partial_x p = 0$$
$$p(T,.) = p^T$$

Duality solutions

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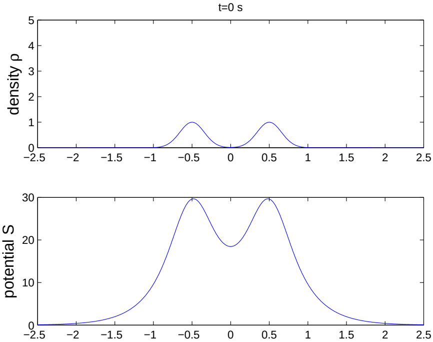
- PIC-type method using aggregate dynamics

Two aggregates



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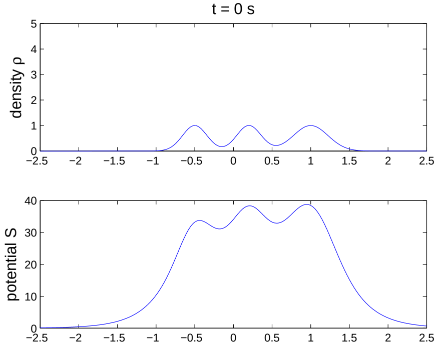

That's all, folks !