# Nonlinear fractional equations of mixed hyperbolic parabolic type: Initial theory and numerics.

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Joint work with Nathael Alibaud (Besancon) and Simone Cifani (NTNU).

# Fractional equations in this talk

Fractional/fractal conservation law

(FCL) $$\partial_t u + \nabla \cdot f(u) = -(-\Delta)^{\frac{\alpha}{2}} u \quad \text{in } \mathbb{R}^d \times (0, T) =: Q_T.$$

Fractional degenerate parabolic equation (mixed hyp./par. type)

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$$-(-\Delta)^{\frac{\alpha}{2}} \phi(x) = \int_{|z|>0} [\phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1}] \frac{c_\alpha dz}{|z|^{d+\alpha}},$$

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(iv) $\mathcal{L}$ is the generator of a Levy process, i.e.

$$\mathcal{L}[\phi](x) = \int_{|z|>0} \left[\phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1}\right] \mu(dz),$$

where $\mu$ is a positive Radon measure s.t. $\int_{|z|>0} |z|^2 \wedge 1 \, \mu(dz) < \infty$.

# On $\Delta^{\frac{\alpha}{2}}$ and $\mathcal{L}$

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3. $(-\Delta)^{\frac{\alpha}{2}}$ self-adjoint, but not $\mathcal{L}$ in general.
4. Any pure-jump Levy process has a generator like $\mathcal{L}$:
   (a) Compound Poisson processes:
   $$\mu(dz) = g(z)\, dz \quad \text{for} \quad g \in L^1.$$
   (b) Symmetric $\alpha$-stable processes: $\mathcal{L} = -(-\Delta)^{\alpha/2}$ and
   $$\mu(dz) = c_\alpha \frac{dz}{|z|^{d+\alpha}} \quad \text{for} \quad \alpha \in (0,2).$$
   (c) CGMY or Variance Gamma processes in Finance:
   $$\mu(dz) = C \begin{cases} e^{-G|z|} \frac{dz}{|z|^{1+Y}}, & z < 0 \\ e^{-M|z|} \frac{dz}{|z|^{1+Y}}, & z > 0, \end{cases} \qquad C, G, M \geq 0, \quad Y \in (0,2) \text{ or } Y = 0.$$

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Gas detonations, semiconductor growth, radiation hydrodynamics, ...
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4) Porous medium equations: $f \equiv 0$; $A(u) = u^m$, $m \geq 1$,

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5) New degenerate equations, e.g. nonlocal versions of degenerate local eq'ns

$$\partial_t u + \nabla \cdot f(u) = \Delta A(u).$$

Local equations: Sedimentation, reservoir simulations, traffic flow...

# Some background

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- Solution concept for conservation laws ($A \equiv 0$).
- Alibaud adapts ideas from viscosity solution theory, e.g. split into singular and non-singular part $\mathcal{L} = \mathcal{L}_r + \mathcal{L}^r$
- $u$ entropy solution of (FDE) if the entropy conditions

$$\eta(u)_t + \nabla \cdot q(u) \leq \mathcal{L}_r[\eta(A(u))] + \eta'(u)\mathcal{L}^r[A(u)] \quad \text{in} \quad \mathcal{D}'$$

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**$L^1$-contraction:**

$$\int |u(x,t) - v(x,t)| dx \leq \int |u_0(x) - v_0(x)| dx.$$

Entropy solutions $u, v$ typically satisfy $L^1$-contraction and are unique.

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$$u \in L^\infty \cap L^1, \quad A(u) \in L^2(0,T;H^1(\mathbb{R}^N)).$$

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6) Fractional degenerate parabolic equations, general case (FDE):
   [Cifani-Jakobsen 11]: $L^1$-contraction + well-posedness in $L^\infty \cap L^1$

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9) Convergence as $\alpha \to 2$ to Carrillo entropy solution of

$$\partial_t u + \nabla \cdot f(u) = \Delta A(u).$$

# 1) Motivation and key observation

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3. A key (Kato type) inequality:

$$\mathrm{sgn}(u-k)\mathcal{L}_r[A(u)] \leq \mathcal{L}_r[|A(u)-A(k)|].$$

Proof: Simple calculus + $\mathrm{sgn}(u-k)(A(u)-A(k)) = |A(u)-A(k)|$

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$\mathcal{L}_r^*$ adjoint of $\mathcal{L}_r$; equation makes sense for $u(\cdot,t) \in L^\infty \cap L^1$ $(r > 1)$

# 1) The good entropy formulation

Kruzkov entropies: $\eta_k(u) = |u-k| \ / \ q_k(u) = \eta_k'(u)(f(u)-f(k))$.

Write $\mathcal{L}[\phi] = \mathcal{L}_r[\phi] + \mathcal{L}^r[\phi] + \mathrm{div}\,(b_r\,\phi)$ where

$$\mathcal{L}_r[\phi](x) = \int_{0<|z|\leq r} \phi(x+z) - \phi(x) - z\cdot D\phi(x)\,\mathbf{1}_{|z|\leq 1}\,\mu(dz),$$

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**Definition:** $u$ is entropy solution of (FDE) if

*i)* $u \in L^\infty(Q_T)\cap C(0,T;L^1(\mathbb{R}^d))$;

*ii)* for all $k\in\mathbb{R}$, all $r>0$, and all test functions $0\le\varphi\in C_c^\infty(Q_T)$,

$$\iint_{Q_T} \eta_k(u)\partial_t\varphi + (q_k(u)+b_r)\cdot\nabla\varphi + |A(u)-A(k)|\,\mathcal{L}_r^*[\varphi] + \eta_k'(u)\,\mathcal{L}^r[A(u)]\,\varphi \ge 0;$$

*iii)* $\lim_{t\to 0}\|u(\cdot,t)-u_0(\cdot)\|_{L^1(\mathbb{R}^d)} = 0$.

# 2) $L^1$ contraction and uniqueness

**Theorem:** $u$ and $v$ entropy solutions of (FDE):

$$\partial_t u + \nabla \cdot f(u) = \mathcal{L}[A(u)] \quad \text{in } \mathbb{R}^d \times (0, T) =: Q_T.$$

Then for $t \in [0, T)$,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|u(\cdot, 0) - v(\cdot, 0)\|_{L^1(\mathbb{R}^d)}.$$

# 2) $L^1$ contraction and uniqueness

**Theorem:** $u$ and $v$ entropy solutions of (FDE):

$$\partial_t u + \nabla \cdot f(u) = \mathcal{L}[A(u)] \quad \text{in } \mathbb{R}^d \times (0,T) =: Q_T.$$

Then for $t \in [0,T)$,

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u(\cdot,0) - v(\cdot,0)\|_{L^1(\mathbb{R}^d)}.$$

**Remarks:**

- Uniqueness follows.
- Remember $u \in L^\infty(Q_T) \cap L^\infty(0,T; L^1(\mathbb{R}^d))$.
- No a priori assumptions on $A(u)$!
- Kruzkov type doubling of variables proof!

# 2) Ideas of the proof

**Idea for $L^1$-estimates:**

- $I := \frac{d}{dt}\int |(u-v)(t)|dx = \int \operatorname{sgn}(u-v)(u_t - v_t)(t)dx$

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**$L^1$-contraction:** Send $r \to 0$, then $\varepsilon, \delta \to 0 \quad \Rightarrow \quad I = \lim I_{\varepsilon,\delta} \leq 0$

# 3) Continuous dependence estimates

Let $u_0, v_0 \in L^1 \cap L^\infty \cap BV$ and $u, v$ be entropy solutions of

$$\partial_t u + \nabla \cdot f(u) = \mathcal{L}_\mu[A(u)] \qquad \text{and} \qquad \partial_t v + \nabla \cdot g(v) = \mathcal{L}_\nu[B(v)].$$

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**Theorem:** Continuous dependence estimate for $BV$-solutions:

$$\|(u-v)(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + t\,|u_0|_{BV(\mathbb{R}^d)}\|f' - g'\|_{L^\infty}$$

$$+ \begin{cases} C\, t \vee t^{1/2} \sqrt{\|A' - B'\|_{L^\infty(\mathbb{R})} + \int_{|z|>0} |z|^2 \wedge 1 \, d|\mu - \nu|(z)} \\ \quad \text{or} \\ C\, t \left( \|A' - B'\|_{L^\infty(\mathbb{R})} + \int_{|z|>0} |z| \wedge 1 \, d|\mu - \nu|(z) \right) \end{cases}$$

# 3) Consequences of continuous dependence

**Corollaries:**

1. $A \equiv 0 \equiv B \quad \Rightarrow \quad$ Standard results for conservataion laws.
2. $A(u) = u$, $\mathcal{L}$ symmetric $\quad \Rightarrow \quad$ non-local part of [Karlsen-Ulusoy 11]
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**The nonlocal vanishing viscosity method:**

$$\partial_t u + \nabla \cdot f(u) = 0 \quad \text{and} \quad \partial_t v + \nabla \cdot g(v) = -\varepsilon(-\Delta)A(v).$$

For $\alpha \in (1,2)$, BV entropy solutions $u$ and $v$ satisfy

$$\|(u-v)(t)\|_{L^1} \leq C_T \varepsilon^{\frac{1}{\alpha}}.$$

**Proof:** Use a more refined version of the cont. dependence estimate.

# 4) Numerical method – discretization of $\mathcal{L}$

**The Levy operator:**

$$\mathcal{L}[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot D\phi(x) 1_{|z|<1}\ \mu(dz)$$

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$$\hat{\mu}(dz) = \mathbf{1}_{|z|>\Delta x} \, \mu(dz) \qquad \text{and} \qquad b_{\Delta x} = \int_{|z|>0} z \, 1_{|z|<1} \, \hat{\mu}(dz).$$

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- Upwind difference approximation of "drift":

$$b_{\Delta x,i} D_i \phi \approx b_{\Delta x,i} \hat{D}_i^b \phi = b_{\Delta x,i} \begin{cases} \frac{\phi(x+\Delta x e_i) - \phi(x)}{\Delta x} & \text{when } b_{\Delta x,i} \geq 0, \\ \frac{\phi(x) - \phi(x - \Delta x e_i)}{\Delta x} & \text{when } b_{\Delta x,i} < 0. \end{cases}$$

# 4) Numerical method – approximation of (FDE)

**Approximate equation:**

1. Rectangular subdivisions of space: $R_\alpha = x_\alpha + \Delta x\,(0,1)^d$, $x_\alpha \in \Delta x\,\mathbb{Z}^d$

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**Finite volume approximation:**

Find piecewise constant weak solutions

$$U(x,t) = \sum_{\beta \in \mathbb{Z}^d} U_\beta(t)\,\mathbf{1}_{R_\beta}(x),$$

of the approximate equation.

# 4) Numerical method for (FDE)

**Semidiscrete scheme:**

$$\begin{cases} \partial_t U_\alpha = -(D^- \cdot F)(U_\alpha, U_{\alpha+\cdot}) + \sum_{\beta \neq 0} A(U_\beta) \hat{\mathcal{L}}_\beta^\alpha, \\ U_\alpha(0) = \frac{1}{\Delta x} \int_{R_\alpha} u_0(x)\ dx, \end{cases}$$

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**Remarks:**

- By construction: Monotone (CFL), consistent, conservative methods
- The limit of any converging sequence $\{U^{\Delta x}\}_{\Delta x > 0}$ is an entropy solution of (FDE) (cell entropy inequality).

# 4) Numerical scheme – convergence

1) **A priori estimates on numerical solution:**

*i)* $\|U^{\Delta x}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$,

*ii)* $\|U^{\Delta x}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}$,

*iii)* $|U^{\Delta x}(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)}$,

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4) **Uniqueness of (FDE):**

   The whole sequence $\{U^{\Delta x}\}$ converge.

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Corollary to compactness and convergence of the scheme.

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The limit of the numerical scheme on the prev. slide.

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**Existence in $L^1 \cap L^\infty$:**

1. Take $\{u_0^\varepsilon\}_{\varepsilon>0}$ such that $BV \cap L^1 \cap L^\infty \ni u_0^\varepsilon \to u_0$ in $L^1$.
2. Let $u^\varepsilon \in BV$ be sol'n of (FDE) s.t. $u^\varepsilon(x,0) = u_0^\varepsilon(x)$.
3. $\|u^{\varepsilon_1} - u^{\varepsilon_2}\|_{L^1} \leq \|u_0^{\varepsilon_1} - u_0^{\varepsilon_2}\|_{L^1} \Rightarrow \{u^\varepsilon\}_{\varepsilon>0}$ Cauchy in $L^1$.
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# 6) Error estimates for the numerical method

When $\mathcal{L} = -(-\Delta)^{\alpha/2}$, $u$ BV entropy sol'n of (FDE), and $U$ solution of implicit method:

$$\|(u-U)(t)\|_{L^1} \leq C \begin{cases} \Delta x^{\frac{1}{2}} & \alpha \in (0,1), \\ \Delta x^{\frac{1}{2}} \log(\Delta x) & \alpha = 1, \\ \Delta x^{\frac{2-\alpha}{2}} & \alpha \in (1,2). \end{cases}$$

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- Kuznetsov type results and proof
- General convergence theory for degenerate problems of order in $[0, 2)$

# 6) Numerical methods – some history

**In general few and only recent results.**

Results/schemes that can handle non-smooth entropy solutions:

1. Dedner and Rhode 2004. Finite volume scheme for radiation hydrodynamics equation ("$\alpha = 0$").
2. Droniou 2010. Finite difference schemes for fractional conservation laws.
3. Cifani, Jakobsen, and Karlsen 2011. Discontinuous Galerkin method for fractional conservation laws ($\alpha < 1$).
4. Cifani and Jakobsen, to appear in MCOMP. Spectral vanishing viscosity methods for for fractional conservation laws.
5. Cifani and Jakobsen, submitted. Finite volume type method for fractional degenerate parabolic equations.

Construction of schemes, convergence results, and except for Droniou, also error estimates.

# Publications:

Results of this talk can be found in:


1. S. Cifani and E. R. Jakobsen.
   Entropy solution theory for fractional degenerate convection-diffusion equations.
   *Annales de l'Institut H. Poincare - Analyse non lineaire* 28(3): 413-441, 2011.
2. N. Alibaud, S. Cifani, and E. R. Jakobsen.
   Continuous dependence estimates for nonlinear fractional convection-diffusion equations.
   *SIAM Journal of Mathematical Analysis*, 2012.
3. S. Cifani and E. R. Jakobsen.
   On numerical methods for degenerate fractional convection-diffusion equations.
   Submitted, preprint on arXiv.
4. N. Alibaud, S. Cifani, and E. R. Jakobsen.
   Optimal continuous dependence estimates for fractional degenerate parabolic equations.
   Almost finished.


Preprint at `www.math.ntnu.no/∼erj` or `www.arXiv.org`

Other selected publications:


4. N. Alibaud: Entropy formulation for fractal conservation laws.
   *J. Evol. Equ.*, 7(1):145-175, 2007.
5. S. N. Kružkov: First order quasi-linear equations in several independent variables.
   *Math. USSR Sbornik*, 10(2):217–243, 1970.
6. D. Applebaum: *Lévy processes and stochastic calculus. Second edition.*
   Cambridge University Press, Cambridge, 2009.