Entropy-stable discontinuous Galerkin finite element method with streamline diffusion and shock-capturing

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Seminar for Applied Mathematics
ETH Zurich
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HYP2012, Padova, 25.–29.6.12

Goal

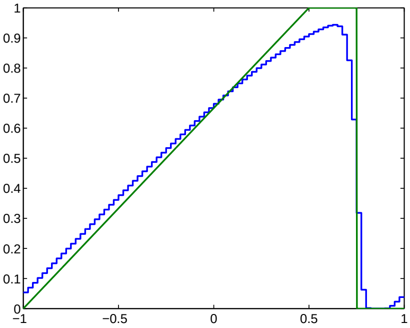
Find a numerical scheme for conservation laws ($\mathbf{U}(x,t) \in \mathbb{R}^m$)

$$\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{U})_{x_k} = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+ \qquad \text{(CL)}$$

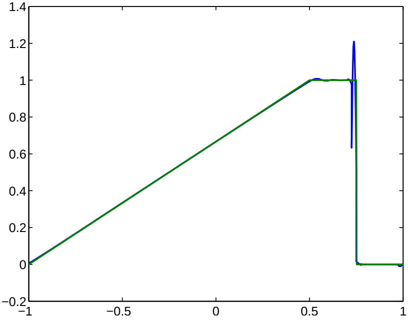
which

Avoid oscillations and too much diffusion

neither

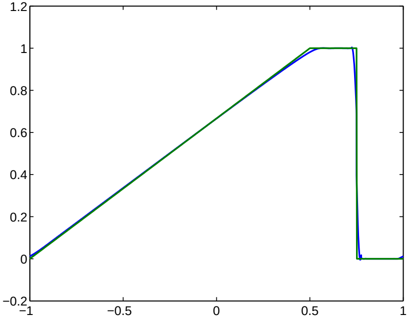


nor



Avoid oscillations and too much diffusion II

but rather



Goal

Find a numerical scheme for conservation laws ($\mathbf{U}(x,t) \in \mathbb{R}^m$)

$$\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{U})_{x_k} = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+ \qquad \text{(CL)}$$

which (among others)

- is arbitrarily high-order accurate
- is **entropy-stable**
- converges to **measure valued solutions** for systems of conservation laws

Our scheme

based on:

- Discontinuous Galerkin (DG)
- plus streamline diffusion (SD)
- plus shock-capturing (SC)

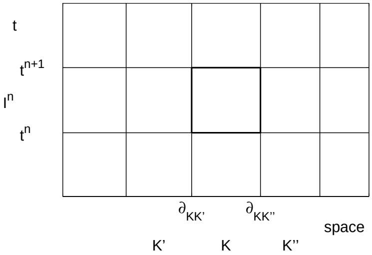
Outline

Derivation of the entropy stable DG FEM I

1. Start with the conservation law
2. Multiply with a test function $\mathbf{W}$ (smooth)
3. Integrate over all elements
4. Integrate by parts
5. Replace the fluxes at the boundary by numerical fluxes that depend on states on both sides of the boundary

Derivation of the entropy stable DG FEM II

Leads to:



$$0 = \sum_{n,K}\Bigg(- \int_{I^n}\int_K \Big(\langle \mathbf{U}, \mathbf{W}_t \rangle + \sum_{k=1}^{d} \Big\langle \mathbf{F}^k(\mathbf{U}), \mathbf{W}_{x_k} \Big\rangle \Big) dxdt$$

$$+ \int_K \langle \mathbb{U}(\mathbf{U}_{n+1,-}, \mathbf{U}_{n+1,+}), \mathbf{W}_{n+1,-} \rangle dx - \int_K \langle \mathbb{U}(\mathbf{U}_{n,-}, \mathbf{U}_{n,+}), \mathbf{W}_{n,+} \rangle dx$$

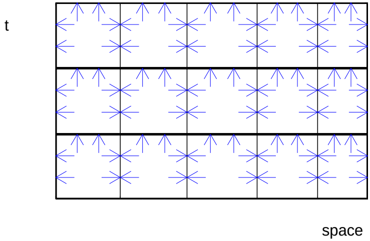
$$+ \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbb{F}(\mathbf{U}_{K,-}, \mathbf{U}_{K,+}, \nu_{KK'}), \mathbf{W}_{K,-} \rangle \, d\sigma(x) dt \Bigg)$$

Numerical fluxes – temporal direction

For the numerical flux $\mathbb{U}$ we use the upwind flux,

$$\mathbb{U}\big(\mathbf{U}(t_-^n), \mathbf{U}(t_+^n)\big) = \mathbf{U}(t_-^n)$$

only this allows to do marching in time



Entropy stability I

Choose entropy function $S(\mathbf{U})$ and associated fluxes $Q^k(\mathbf{U})$
Want a discrete analogue of the entropy inequality

$$S_t + \sum_{k=1}^{d} Q^k_{x_k} \leq 0$$

1. Variable transformation: (entropy symmetrisation)

$$\mathbf{U} = \mathbf{U}(\mathbf{V})$$

where $\mathbf{V} = S_{\mathbf{U}}$ are the entropy variables.
Discretize $\mathbf{V}$ instead of $\mathbf{U}$.

Entropy stability II

2. Entropy stable numerical flux:

$$\mathbb{F}(\boldsymbol{a}, \boldsymbol{b}, \nu) = \sum_{k=1}^{d} \boldsymbol{F}^{k,*}(\boldsymbol{a}, \boldsymbol{b})\nu^k - \frac{1}{2}\boldsymbol{D}(\boldsymbol{b} - \boldsymbol{a})$$

$\boldsymbol{F}^{k,*}$: entropy conservative fluxes [Tadmor, 1987].
$\boldsymbol{D}$: Diffusion matrix. Positive semidefinite.
Mostly Rusanov.

DG: Complete description

Choose the space of ansatz functions $\mathcal{V}_p$ (piecewise polynomials). Then the description is complete:

Find $\mathbf{V}$ in $\mathcal{V}_p$, such that for all $\mathbf{W}$ in $\mathcal{V}_p$:

$$0 = \sum_{n,K} \Bigg(- \int_{I^n} \int_K \Big(\langle \mathbf{U}(\mathbf{V}), \mathbf{W}_t \rangle + \sum_{k=1}^{d} \Big\langle \mathbf{F}^k(\mathbf{V}), \mathbf{W}_{x_k} \Big\rangle \Big) dxdt$$

$$+ \int_K \langle \mathbf{U}(\mathbf{V}_{n+1,-}), \mathbf{W}_{n+1,-} \rangle \, dx - \int_K \langle \mathbf{U}(\mathbf{V}_{n,-}), \mathbf{W}_{n,+} \rangle \, dx$$

$$+ \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \sum_{k=1}^{d} \Big\langle \mathbb{F}^{k,*}(\mathbf{V}_{K,-}, \mathbf{V}_{K,+}), \mathbf{W}_{K,-} \Big\rangle \nu^k_{KK'} d\sigma(x) dt$$

$$- \frac{1}{2} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{W}_{K,-}, \mathbf{D}(\mathbf{V}_{K,+} - \mathbf{V}_{K,-}) \rangle \, d\sigma(x) dt \Bigg)$$

DG: Complete description

Choose the space of ansatz functions $\mathcal{V}_p$ (piecewise polynomials). Then the description is complete:

More compactly:

Find $\mathbf{V}$ in $\mathcal{V}_p$, such that for all $\mathbf{W}$ in $\mathcal{V}_p$:

$$\mathcal{B}_{DG}(\mathbf{V}, \mathbf{W}) = 0$$

Properties

This leads to entropy stability (use $\mathbf{W} = \mathbf{V}$).

However, at discontinuities (shocks) this still leads to oscillations. That is why we introduce the streamline diffusion / shock capturing.

Streamline Diffusion (SD)

[Johnson and Szepessy, 1987] [Johnson et al., 1990]
Add the term

$$\mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K \left\langle \left(\mathbf{U}_{\mathbf{V}}(\mathbf{V})\mathbf{W}_t + \sum_{k=1}^{d} \mathbf{F}^k_{\mathbf{V}}(\mathbf{V})\mathbf{W}_{x_k} \right), \mathbf{D}^{SD}\mathrm{Res} \right\rangle dxdt$$

with intra-element residual:

$$\mathrm{Res} = \mathbf{U}(\mathbf{V})_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{V})_{x_k},$$

and the scaling matrix is chosen as

$$\mathbf{D}^{SD} = C^{SD}\Delta x \mathbf{ID}, \quad C^{SD} > 0$$

Streamline Diffusion (SD)

[Johnson and Szepessy, 1987] [Johnson et al., 1990]
Add the term

$$\mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K \left\langle \left(\mathbf{U_V}(\mathbf{V})\mathbf{W}_t + \sum_{k=1}^{d} \mathbf{F}_{\mathbf{V}}^k(\mathbf{V})\mathbf{W}_{x_k} \right), \mathbf{D}^{SD}\mathrm{Res} \right\rangle dxdt$$

Leads to additional diffusion proportional to

$$\left\langle \left(\mathbf{U_V}(\mathbf{V})\mathbf{V}_t + \sum_{k=1}^{d} \mathbf{F}_{\mathbf{V}}^k(\mathbf{V})\mathbf{V}_{x_k} \right), \mathbf{D}^{SD}\mathrm{Res} \right\rangle$$

$$= \left\langle \mathbf{U}(\mathbf{V})_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{V}))_{x_k}, \mathbf{D}^{SD}\mathrm{Res} \right\rangle$$

$$= \left\langle \mathrm{Res}, \mathbf{D}^{SD}\mathrm{Res} \right\rangle$$

Streamline Diffusion (SD)

[Johnson and Szepessy, 1987] [Johnson et al., 1990]

Add the term

$$\mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K \left\langle \left(\mathbf{U_V}(\mathbf{V})\mathbf{W}_t + \sum_{k=1}^{d} \mathbf{F}^k_{\mathbf{V}}(\mathbf{V})\mathbf{W}_{x_k} \right), \mathbf{D}^{SD}\mathrm{Res} \right\rangle dxdt$$

Leads to control on the residual.

Shock-capturing (SC)

[Johnson and Szepessy, 1987] [Johnson et al., 1990] [Barth]
Idea: at shocks the residual is big
add (homogeneous) diffusion proportional to the residual

$$\mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K D^{SC}_{n,K} \left(\left\langle \mathbf{W}_t, \mathbf{U}_\mathbf{V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{W}_{x_k}, \mathbf{U}_\mathbf{V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k} \right\rangle \right) dxdt$$

$$D^{SC}_{n,K} = \frac{(\Delta x)^{1-\alpha} C^{SC} \overline{\mathrm{Res}}_{n,K} + (\Delta x)^{\frac{1}{2}-\alpha} \bar{C}^{SC} \overline{\mathrm{BRes}}_{n,K}}{\sqrt{\int_{I^n}\int_K \left(\left\langle \mathbf{V}_t, \mathbf{U}_\mathbf{V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{V}_{x_k}, \mathbf{U}_\mathbf{V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k} \right\rangle \right) dxdt} + \Delta x^{\theta}}$$

Parameters: $C^{SC} > 0,\ \bar{C}^{SC} > 0,\ \alpha \geq 0,\ \theta \geq \alpha + d/2$

Shock-capturing (SC)

[Johnson and Szepessy, 1987] [Johnson et al., 1990] [Barth]
Idea: at shocks the residual is big
add (homogeneous) diffusion proportional to the residual

$$\mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K D^{SC}_{n,K} \left(\left\langle \mathbf{W}_t, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{W}_{x_k}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k} \right\rangle \right) dxdt$$

Leads to control on the gradients.

Full scheme:
Find $\mathbf{V}$ in $\mathcal{V}_p$, such that for all $\mathbf{W}$ in $\mathcal{V}_p$:

$$\mathcal{B}_{DG}(\mathbf{V},\mathbf{W}) + \mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) + \mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) = 0 \qquad \text{(S)}$$

Properties - Goals revisited

- formally arbitrarily high-order accurate
- entropy-stable (without/with SD or SC)
- convergence to measure valued solutions for systems of conservation laws?

Measure valued solutions

[DiPerna, 1985]
Consider:

$$\boldsymbol{\mu} : (x,t) \in \Omega \times \mathbb{R}_+ \mapsto Prob(\mathbb{R}^m),$$

$\boldsymbol{\mu}$ is defined as a *measure valued solution* of the system (CL) if

$$\int_\Omega \int_{\mathbb{R}_+} \left(\langle \langle \mathbf{U}, \boldsymbol{\mu}_{x,t} \rangle, \varphi_t \rangle + \sum_{k=1}^{d} \langle \langle \mathbf{F}^k, \boldsymbol{\mu}_{x,t} \rangle, \varphi_{x_k} \rangle \right) dxdt = 0,$$

for all test functions $\varphi \in (C_c^\infty(\Omega \times (0,\infty)))^m$. Here,

$$\langle \mathbf{g}, \boldsymbol{\mu}_{x,t} \rangle = \int_{\mathbb{R}^m} \mathbf{g}(\lambda) d\boldsymbol{\mu}_{x,t}(\lambda).$$

Entropy measure valued solution

μ is defined to be an *entropy* measure valued solution of (CL) if

1. it is a measure valued solution of (CL) and
2. if

$$\int_{\Omega}\int_{\mathbb{R}_+}\left(\langle S,\mu_{x,t}\rangle\varphi_t+\sum_{k=1}^{d}\langle Q^k,\mu_{x,t}\rangle\varphi_{x_k}\right)dxdt\geq 0,$$

for all non-negative test functions $0\leq\varphi\in C_c^{\infty}(\Omega\times(0,\infty))$

Convergence to a measure valued solution

Under the assumption that the approximate solutions $\mathbf{V}^{\Delta x}$ satisfy the uniform L^∞ bound,

$$\|\mathbf{V}^{\Delta x}\|_{L^\infty(\Omega \times \mathbb{R}_+)} \leq C, \tag{UB}$$

the approximate solutions converge to a measure valued solution of the conservation law (CL).

Convergence to an entropy measure valued solution

Under the same uniform bound (UB) and under

$$\alpha > 0$$

the limit measure valued solution μ satisfies the entropy condition.

Shock-capturing (SC)

[Johnson and Szepessy, 1987] [Johnson et al., 1990] [Barth]
Idea: at shocks the residual is big
add (homogeneous) diffusion proportional to the residual

$$\mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K D_{n,K}^{SC} \left(\left\langle \mathbf{W}_t, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{W}_{x_k}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k} \right\rangle \right) dxdt$$

$$D_{n,K}^{SC} = \frac{(\Delta x)^{1-\alpha} C^{SC} \overline{\mathrm{Res}}_{n,K} + (\Delta x)^{\frac{1}{2}-\alpha} \bar{C}^{SC} \overline{\mathrm{BRes}}_{n,K}}{\sqrt{\int_{I^n}\int_K \left(\left\langle \mathbf{V}_t, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{V}_{x_k}, \mathbf{U_V}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_{x_k} \right\rangle \right) dxdt} + \Delta x^{\theta}}$$

Parameters: $C^{SC} > 0$, $\bar{C}^{SC} > 0$, $\alpha \geq 0$, $\theta \geq \alpha + d/2$

Convergence to an entropy measure valued solution

Under the same uniform bound (UB) and under

$$\alpha > 0$$

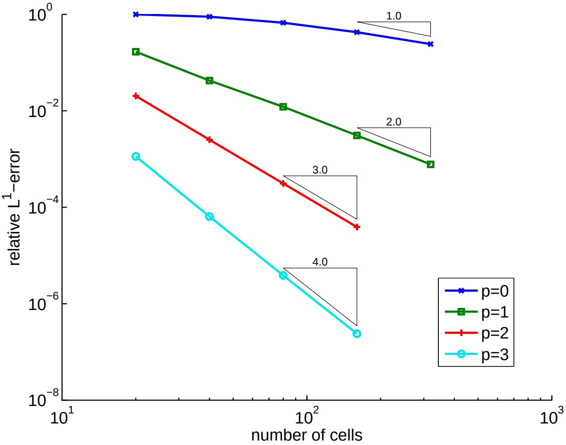
the limit measure valued solution μ satisfies the entropy condition.

Implementation

- Currently in MATLAB
- Quite slow
- For each time interval we have to solve a non-linear system for the dofs associated to it.
- Currently: mostly by a damped Newton method ($\Rightarrow$ we have to compute the Jacobian)
- Planned: Newton-Krylov method ($\Rightarrow$ we have to compute only the multiplication with the Jacobian)
  Preconditioner?

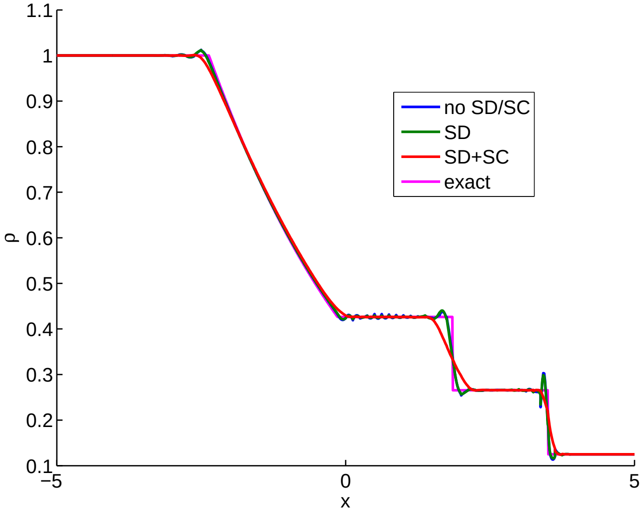
Wave equation (smooth initial data)

$$h_t + cm_x = 0$$
$$m_t + ch_x = 0$$



Euler equations - Sod shock tube

$N_x = 80,\ p = 2$



Pressure scaling

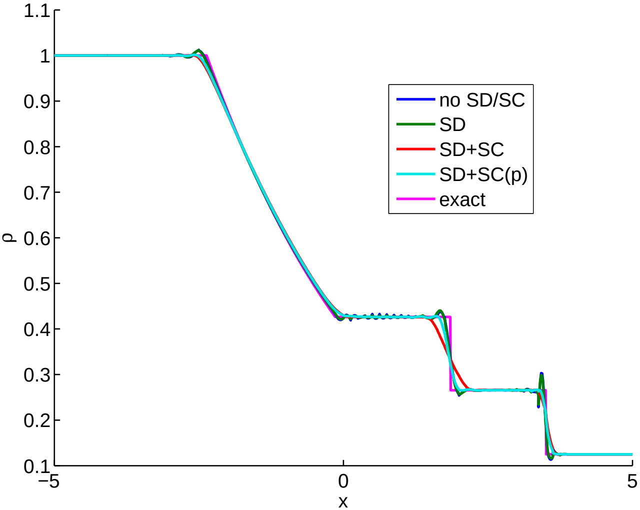
To resolve contact discontinuities better:
use pressure as an indicator

$$D_{n,K}^{SC} = \frac{D_{n,K}^{p}\left((\Delta x)^{1-\alpha} C^{SC}\overline{\mathrm{Res}}_{n,K} + \Delta x^{\frac{1}{2}-\alpha}\bar{C}^{SC}\overline{\mathrm{BRes}}_{n,K}\right)}{\sqrt{\int\limits_{I^n}\int\limits_{K}\left(\left\langle \mathbf{V}_t, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_t\right\rangle + \sum_{k=1}^{d}\left\langle \mathbf{V}_{x_k}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_{x_k}\right\rangle\right) dxdt + \Delta x^{\theta}}}$$

$$D_{n,K}^{p} = \Delta x^2 \frac{\frac{1}{\Delta t^n}\frac{1}{|K|}\int_{I^n}\int_K \sqrt{\sum_{k=1}^{d} p_{x_k x_k}^2}\, dxdt}{\frac{1}{\Delta t^n}\frac{1}{|K|}\int_{I^n}\int_K p\, dxdt}$$

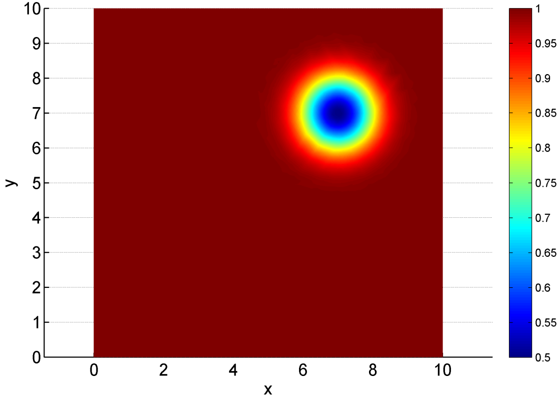
Euler equations - Sod shock tube

$N_x = 80, \; p = 2$

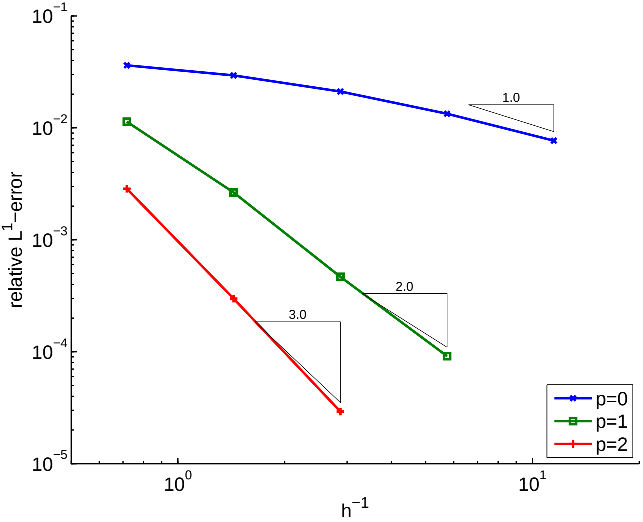


Vortex-Advection

$N_c = 828, \ p = 2$

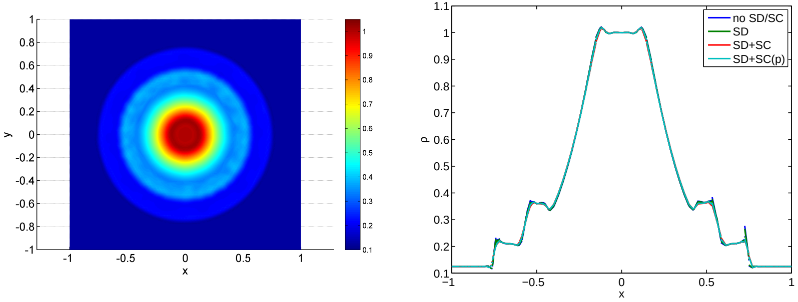


Vortex-Advection



Radial shock tube

$N_c = 13440, \ p = 1$



Conclusions

- We get an entropy-stable DG FE method by:
  -Discretizing entropy variables
  -Using entropy stable numerical fluxes
- The solution is quite oscillatory at discontinuities
- By streamline diffusion and shock-capturing we get a much less oscillatory solution,
  but it is quite diffusive at contact discontinuities
  therefore we introduce a scaling based on pressure
- Method not free of parameters
- Convergence to entropy measure valued solutions (under some assumptions)

Future work

- Implementation: efficient solution of the non-linear systems? Preconditioning?
- Investigate (especially goal-oriented) adaptivity

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Appendix

- Entropy stability
- Scalar conservation laws
- Linear symmetrizable systems
- Scheme in more details
- More experiments

Theorem

Consider the system of conservation laws (CL) *with strictly convex entropy function S and entropy flux functions* $Q^k_{(1\leq k\leq d)}$. *For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain* Ω. *Let the final time be denoted by* t^N_-. *Then, the streamline diffusion-shock capturing-Discontinuous Galerkin scheme* (S) *approximating* (CL) *has the following properties:*

(i.) *The scheme* (S) *is conservative i.e, the approximate solutions* $\mathbf{U}^{\Delta x} = \mathbf{U}(\mathbf{V}^{\Delta x})$ *satisfy*

$$\int_\Omega \mathbf{U}^{\Delta x}(x, t^N_-)dx = \int_\Omega \mathbf{U}^{\Delta x}(x, t^0_-)dx.$$

Theorem

(ii.) *The scheme (S) is entropy stable i.e, the approximate solutions satisfy,*

$$\int_\Omega S(\mathbf{U}^*(t_-^0))dx \leq \int_\Omega S(\mathbf{U}^{\Delta x}(x, t_-^N))dx \leq \int_\Omega S(\mathbf{U}^{\Delta x}(x, t_-^0))dx,$$

with $\mathbf{U}^*$ *being the domain average:*

$$\mathbf{U}^*(t_-^0) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \mathbf{U}(\mathbf{V}(x, t_-^0))dx.$$

Theorem

(iii.) *We obtain the following weak "BV" estimate:*

$$\sum_{n,K}\int_K \|\mathbf{V}^{\Delta x}_{n,-} - \mathbf{V}^{\Delta x}_{n,+}\|^2 dx + \sum_{n,K}\sum_{K'\in\mathcal{N}(K)}\int_{I^n}\int_{\partial_{KK'}}\left\langle \mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-}, \mathbf{D}(\mathbf{V}^{\Delta x}_{K,+} - \mathbf{V}^{\Delta x}_{K,-})\right\rangle d\sigma(x)dt$$

$$+ \Delta x \sum_{n,K}\int_{I^n}\int_K \|\mathbf{U}^{\Delta x}_t + \sum_{k=1}^{d}\mathbf{F}^k(\mathbf{V}^{\Delta x})_{x_k}\|^2 dxdt$$

$$+ (\Delta x)^{1-\alpha}\sum_{n,K}\overline{\mathrm{Res}}_{n,K}\left(\int_{I^n}\int_K \|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2 dxdt\right)^{\frac{1}{2}}$$

$$+ (\Delta x)^{\frac{1}{2}-\alpha}\sum_{n,K}\overline{\mathrm{BRes}}_{n,K}\left(\int_{I^n}\int_K \|\nabla_{xt}\mathbf{V}^{\Delta x}\|^2 dxdt\right)^{\frac{1}{2}} \leq C.$$

For some constant C, depending on the initial data and with the spacetime gradient defined by,

$$\nabla_{xt}\mathbf{V}^{\Delta x} = \left(\mathbf{V}^{\Delta x}_t, \mathbf{V}^{\Delta x}_{x_1}, \mathbf{V}^{\Delta x}_{x_2}, \cdots, \mathbf{V}^{\Delta x}_{x_d}\right).$$

Scalar conservation laws

$$S(U) = \frac{1}{2}U^2, \quad V = U \tag{2}$$

$$\mathbb{F}^{k,*}(a,b) = \frac{F^k(a) + F^k(b)}{2}, \tag{3}$$

Theorem

Assume that the initial data $U_0(x)$ for the scalar conservation law $\mathbf{U} = U$ in (CL) satisfy the bound,

$$a < U_0(x) < b, \quad \forall x \in \Omega,$$

for constants $a, b \in \mathbb{R}$. Let $U^{\Delta x}$ be the approximate solutions generated by the numerical scheme (S) with numerical flux (3) and numerical diffusion operator corresponding to the Godunov scheme, then the approximate solutions converge to the entropy solution of the underlying scalar conservation law, i.e, (CL) with $m = 1$.

Linear symmetrizable systems I

$$\mathbf{U}_t + \sum_{k=1}^{d} A^k \mathbf{U}_{x_k} = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+. \tag{4}$$

Here, $A^k \in \mathbb{R}^{m \times m}$ are constant matrices (for simplicity).
Furthermore, we assume that there exists $B \in \mathbb{R}^{m \times m}$ such that

(a). B is symmetric, (strictly) positive definite.

(b). For all $1 \leq k \leq d$, the matrix BA^k is symmetric.

Linear symmetrizable systems II

Theorem

Consider the linear symmetrizable system (4) *with symmetrizer* B*. Let* $\mathbf{U}^{\Delta x} = \mathbf{U}(\mathbf{V}^{\Delta x})$ *be the approximate solutions generated by the streamline diffusion-shock capturing DG scheme* (S) *with numerical flux* (ECF)*. Then, the approximate solutions satisfy the following energy bounds,*

$$\|\mathbf{U}^{\Delta x}(., t_-^n)\|_{L^2(\Omega)} \leq C\|\mathbf{U}^{\Delta x}(, .t_-^0)\|_{L^2(\Omega)}, \tag{5}$$

for all discrete time levels t^n*.*
Furthermore, the approximate solutions $\mathbf{U}^{\Delta x} \rightharpoonup \mathbf{U}$ *in* $L^2(\Omega \times [0, T])$ *and* $\mathbf{U}$ *is the unique weak solution of the system* (4)*.*

Entropy stability I

Choose entropy function $S(\mathbf{U})$ and associated fluxes $Q^k(\mathbf{U})$
Want a discrete analogue of the entropy inequality

$$S_t + \sum_{k=1}^{d} Q^k_{x_k} \leq 0$$

1. Variable transformation: (entropy symmetrisation)

$$\mathbf{U} = \mathbf{U}(\mathbf{V})$$

where $\mathbf{V} = S_{\mathbf{U}}$ are the entropy variables.
Discretize $\mathbf{V}$ instead of $\mathbf{U}$.

Entropy stability II

2. Entropy stable numerical flux:

$$\mathbb{F}(\boldsymbol{a}, \boldsymbol{b}, \nu) = \sum_{k=1}^{d} \boldsymbol{F}^{k,*}(\boldsymbol{a}, \boldsymbol{b})\nu^k - \tfrac{1}{2}\boldsymbol{D}(\boldsymbol{b} - \boldsymbol{a})$$

$\boldsymbol{F}^{k,*}$: entropy conservative fluxes [Tadmor, 1987].

$$\langle \boldsymbol{b} - \boldsymbol{a}, \mathbb{F}^{k,*}(\boldsymbol{a}, \boldsymbol{b}) \rangle = \Psi^k(\boldsymbol{b}) - \Psi^k(\boldsymbol{a}) \qquad \text{(ECF)}$$

where $\Psi^k = \langle \mathbf{V}, \mathbf{F}^k \rangle - Q^k$ entropy potential

$\boldsymbol{D}$: Diffusion matrix. Positive semidefinite.
Mostly Rusanov:

$$\mathbf{D}(\boldsymbol{a}, \boldsymbol{b}; \nu) = \max_{\mathbf{V} \in \{\boldsymbol{a}, \boldsymbol{b}\}} \lambda_{max}(\mathbf{U}(\mathbf{V}); \nu)\ \mathbf{U}_{\mathbf{V}}\left(\tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\right)$$

where $\lambda_{max}(\mathbf{U}; \nu)$: maximal wave speed in direction of ν at state $\mathbf{U}$

Properties

This leads to entropy stability (use $\mathbf{W} = \mathbf{V}$, suitable boundary conditions):

$$\begin{aligned}
&\int_\Omega S(\mathbf{U}(\mathbf{V}(x, t_-^N)))dx \\
&\leq \int_\Omega S(\mathbf{U}(\mathbf{V}(x, t_-^0)))dx \\
&\quad - \sum_{n,K} \lambda_1 \int_K \|\mathbf{V}_{n,-} - \mathbf{V}_{n,+}\|^2 dx \\
&\quad - \frac{1}{4} \sum_{n,K} \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbf{V}_{K,+} - \mathbf{V}_{K,-}, \mathbf{D}(\mathbf{V}_{K,+} - \mathbf{V}_{K,-}) \rangle \, d\sigma(x) dt
\end{aligned}$$

However, at discontinuities (shocks) this still leads to oscillations. That is why we introduce the streamline diffusion / shock capturing.

Shock-capturing (SC) I

[Johnson and Szepessy, 1987] [Johnson et al., 1990] [Barth]
Idea: at shocks the residual is big
add (homogeneous) diffusion proportional to the residual

$$\mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) =$$

$$\sum_{n,K} \int_{I^n} \int_K D^{SC}_{n,K} \left(\left\langle \mathbf{W}_t, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{W}_{x_k}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_{x_k} \right\rangle \right) dxdt$$

$$D^{SC}_{n,K} = \frac{(\Delta x)^{1-\alpha} C^{SC} \overline{\mathrm{Res}}_{n,K} + (\Delta x)^{\frac{1}{2}-\alpha} \bar{C}^{SC} \overline{\mathrm{BRes}}_{n,K}}{\sqrt{\int_{I^n}\int_K \left(\left\langle \mathbf{V}_t, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_t \right\rangle + \sum_{k=1}^{d} \left\langle \mathbf{V}_{x_k}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K})\mathbf{V}_{x_k} \right\rangle \right) dxdt} + \Delta x^{\theta}}$$

Parameters: $C^{SC} > 0$, $\bar{C}^{SC} > 0$, $\alpha \geq 0$, $\theta \geq \alpha + d/2$

Shock-capturing (SC) II

Interior residual: $\overline{\mathrm{Res}}_{n,K} = \sqrt{\int_{I^n}\int_K \left\langle \mathrm{Res}, \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V})\mathrm{Res}\right\rangle dxdt}.$

Boundary residual:

$$\overline{\mathrm{BRes}}_{n,K} = \Bigg(\int_K \|\mathbf{U}(\mathbf{V}_{n,-}) - \mathbf{U}(\mathbf{V}_{n,+})\|^2 dx$$

$$+ \int_{I^n}\int_{\partial_{KK'}} \Bigg(\sum_{k=1}^{d} \| \left(\mathbb{F}^{k,*}(\mathbf{V}_{K,-}, \mathbf{V}_{K,+}) - \mathbf{F}^k(\mathbf{V}_{K,-})\right) \nu_{KK'}^k \|^2$$

$$+ \|\frac{1}{2}\mathbf{D}(\mathbf{V}_{K,+} - \mathbf{V}_{K,-})\|^2 \Bigg) d\sigma(x)dt \Bigg)^{\frac{1}{2}}$$

Element average:

$$\tilde{\mathbf{V}}_{n,K} = \frac{1}{\mathrm{meas}(I^n \times K)} \int_{I^n}\int_K \mathbf{V}(x,t)dxdt$$

Shock-capturing (SC) III

Shock capturing leads to control on the gradients.

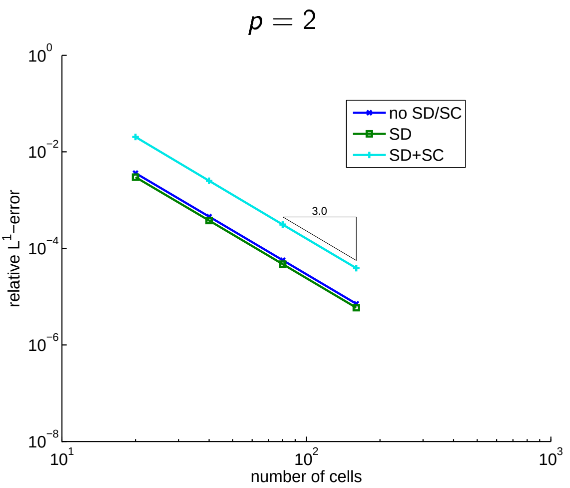
Full scheme:

Find $\mathbf{V}$ in $\mathcal{V}_p$, such that for all $\mathbf{W}$ in $\mathcal{V}_p$:

$$\mathcal{B}_{DG}(\mathbf{V},\mathbf{W}) + \mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) + \mathcal{B}_{SC}(\mathbf{V},\mathbf{W}) = 0 \qquad \text{(S)}$$

Wave equation (smooth initial data)

$$h_t + cm_x = 0$$
$$m_t + ch_x = 0$$



Euler equations - Sod shock tube

$N_x = 80$