Hyperbolic Equations on Networks

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Scope: modeling, analysis and control of network phenomena on the example of gas flow

- Many applications have inherent network and transport structure as for example traffic flow, gas or water transportation networks, telecommunication, blood flow or production systems
- Network as directed[1] graph with arcs j and vertices v
- Transport phenomena along each arc described by a *spatial 1–d* hyperbolic equation
- Physical coupling conditions at each vertex described by an *algebraic condition* [2]

Mathematical description as coupled systems of hyperbolic balance or conservation laws

[1] cw. S. Canic, Hyperbolic Nets as undirected graph

[2] cw. M. Garavello ODE conditions

Example: Traffic Flow On Road Networks

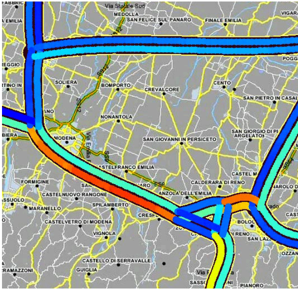
Macroscopic description of traffic flow on one–way road j by density $\rho_j(x, t)$ and average velocity $u_j(x, t9$

- Roads modeled through conservation laws (LWR)

$$\partial_t \rho_j + \partial_x \rho_j u(\rho_j) = 0$$

  or systems (ARZ, Colombo, Goatin . . .)

- Vertices as road intersections modelled by conservation of cars and right–of–way rules
- Incomplete Refs. Colombo, Garavello, Holden, Lebacque, Piccoli, Rascle, . . .



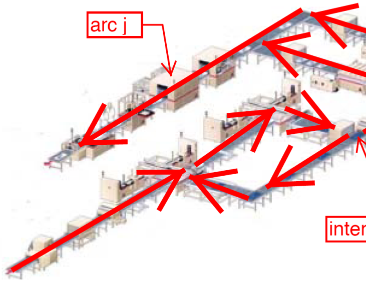
Example: Supply Chain Management

Macroscopic description of large–volume production facilities with density of parts $\rho_j(x,t)$[3]

- Re–entrant machines modelled by conservation laws (possibly non–local fluxes)

$$\partial_t \rho_j + \partial_x \rho_j u(\rho_j) = 0$$

- Vertices as machine–to–machine connections including (sometimes) buffers (leads also to ODE at vertices)
- Incomplete Refs. Armbruster, d'Apice, Degond, Göttlich, Klar, Ringhofer, ...

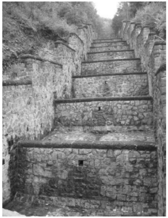


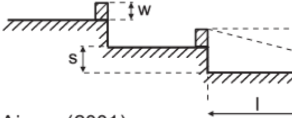
[3] M. Kawski, S. Peng, ...

Example: Water Flow in Open Canals or Pooled Chutes

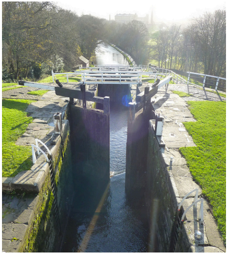
Description of a water level h_j and velocity u_j in open canals j or pooled step cascades j

- Dynamics modelled by shallow–water equations equations with source terms
- Vertices as waterway intersections or (controllable) gates
- Questions of closed and open loop control and stabilization
- Incomplete Refs.
  Andrea-Novel, Bastin, Coron, Gugat, Guerra, Li Tatsien, Leugering, . . .





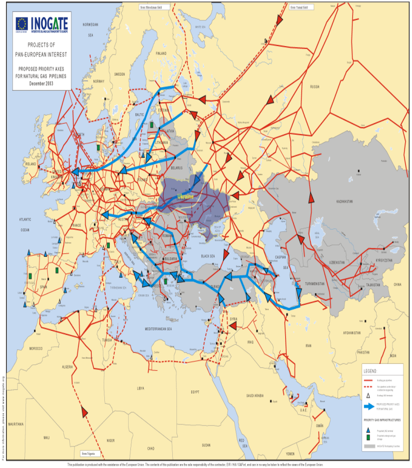
Aigner (2001)
Emiroglu & Baylar (2003)
Chinnarasri & Wongwises (2006



Example of gas transportation networks

Modeling and simulation for high pressure gas transmission systems using the p–system[4]

- Industrial problem:
  Cost–efficient driving of gas through pipe networks
- Major physical effect: Pressure loss along the pipe due to pipewall friction
- Gas supplier operates compressor stations to increase the pressure and fulfill contracts with customers
- Pipe–to–pipe intersections



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[4] In collaboration with Colombo, Guerra, Schleper, Klar, Gugat, Leugering

Common assumptions for high–pressure gas transmission

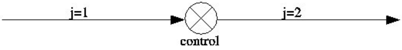
- No influence of temperature gradients
- Horizontal pipes with constant pipe diameter
- Equation of state $p = z\ R\ T\rho$ with constant gas compressibility factor z leads to isothermal Euler equations

$$\partial_t \begin{pmatrix} \rho_j \\ \rho_j u_j \end{pmatrix} + \partial_x \begin{pmatrix} \rho_j u_j \\ p(\rho_j) + \rho_j u_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(\rho_j, u_j) \end{pmatrix}$$

- Pipe wall friction given by $f(\rho_j, u_j) = f_g \frac{\rho_j u_j |\rho_j u_j|}{\rho_j}$
- Theoretical results available for general 2x2 genuine nonlinear hyperbolic balance laws on each arc

Literature in engineering and mathematics since more than 50 years, e.g., Pipeline Simulation Interest Group (www.psig.com)

Modelling of compressor stations

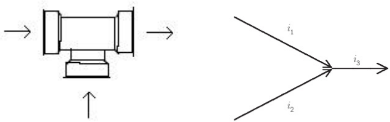


- Compressor station in the 1–d pipe modelled as vertex coupling incoming (in) and outgoing (out) pipes.
- Compressor conserves mass $q_{in} = q_{out}$ with $q = \rho u$
- Pressure at the pipe boundaries enter the equation for a single, idealized compressor

$$P(\cdot) = P(\rho_{in}, \rho_{out}, q) = c\ q\ \left(\left(\frac{p(\rho_{out})}{p(\rho_{in})} \right)^{\kappa} - 1 \right)$$

- $P(\cdot)$ is the energy necessary to increase pressure from $p(\rho_{in})$ to $p(\rho_{out})$

Modelling of a single pipe–to–pipe vertex



- Conservation of mass at vertex located at x_j on arc j, $t > 0$

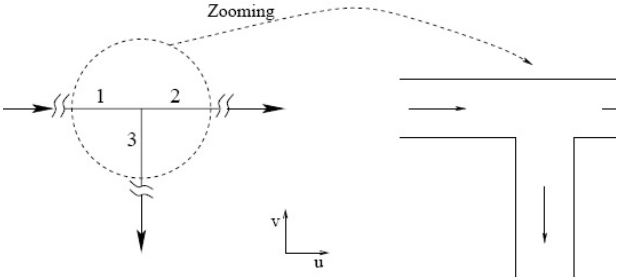
$$\sum_j (\pm_j 1)\,(\rho_j u_j)\,(x_j, t) = 0$$

- Second condition required for example in engineering literature

$$p(\rho_j(x_j, t)) = p(\rho_i(x_i, t))$$

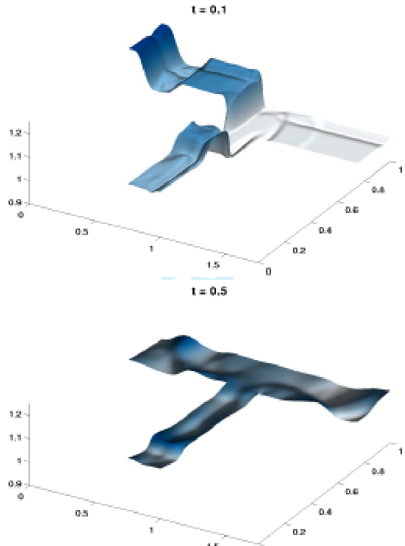
- Variety of other second conditions exists: equal pressure including minor losses depending on geometry and type of gas (tables); equal dynamic momentum; numerically by modeling the domain by a 2–D consideration

Numerical assesment of coupling conditions by $2-D$ simulations

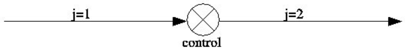


- Vertex is locally a 2d domain and simulate perturbations of constant steady states
- Average to obtain tables to be compared with predictions of 1–d coupling conditions
- Subsonic flow: equal pressure at node as reasonable assumption for tee–shaped pipe–to–pipe intersection

1-d description of pipe networks simplistic as seen for time evolution of the density $\rho(x, y, t)$ for p-system, $\gamma = 1$



Mathematical properties of coupled systems



- Generic situation of a single vertex located at $x = 0$ for each of the n connected arcs, $y(x,t) \in \mathbb{R}^2$
- Control u present at the vertex
- Dynamics on each arc according to a 2×2 nonlinear hyperbolic system of balance laws

$$\partial_t y_j + \partial_x f\left(y_j\right) = g\left(x, y_j\right), \ \Psi\left(y^{(1)}, \dots, y^{(n)}\right) = u\left(t\right)$$

- Coupling conditions for a nonlinear function Ψ and $y^{(i)} = y_i(0,t)$
- *Well-posedness of the problem?*

Result on weak solutions for 2x2 balance laws

Crucial assumption. There exists a constant subsonic state $(\bar{y}, \bar{u})$ fulfilling the coupling condition $\Psi = 0$ and Ψ is locally invertible

Theorem (Colombo, Guerra, H., Schleper)

Let $\mathcal{D}^\delta = \left\{(y,u) \in (\bar{y}, \bar{u}) + L^1\left(\mathbb{R}^+, \Omega \times \mathbb{R}^n\right) : TV\,(y,u) < \delta\right\}$, *where* $\Omega \subset \mathbb{R}^{2n}$ *in a non–empty set. Then, there exist* δ, T *and a semigroup* $\mathcal{E} : [0, T - t_0] \times [0, T] \times \mathcal{D}_{t_0} \to \mathcal{D}^\delta$ *for some* $\mathcal{D}_{t_0} \subset \mathcal{D}^\delta$ *such that* $\mathcal{E}\,(t, t_0, y_0, u)$ *is a solution to the Cauchy problem at the junction with initial condition* $y(t_0, x) = y_0$ *and control function* $u(t)$*. The solution depends Lipschitz-continuous on* Ψ, y_0 *and* u*:*

$$\left\|\mathcal{E}\,(t, t_0, y_0, u) - \mathcal{E}\,(t, t_0, \tilde{y}_0, \tilde{u})\right\| \leq$$
$$L \cdot \left(\|y_0 - \tilde{y}_0\| + \int_{t_0}^{t_0+t} \|u(\tau) - \tilde{u}(\tau)\| d\tau \right)$$

Result on weak solutions ... (cont'd)

$$\partial_t y^{(i)} + \partial_x f\left(y^{(i)}\right) = g\left(x, y^{(i)}\right),\ \Psi\left(y^{(1)}, \ldots, y^{(n)}\right) = u\,(t)$$

- Coupling condition is satisfied for a.e. $t > 0$
- The solution may contain shock waves and its regularity as in the case of the Cauchy problem
- The main assumptions are subsonic initial data and small TV-norm of the initial data and

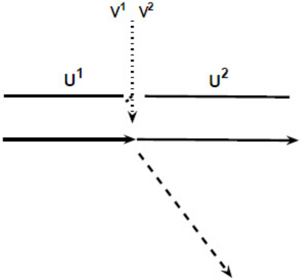
$$det\left(D_1\Psi(\bar{y})r_2^1(\bar{y}_1), \ldots, D_n\Psi(\bar{y})r_2^n(\bar{y}_n)\right) \neq 0$$

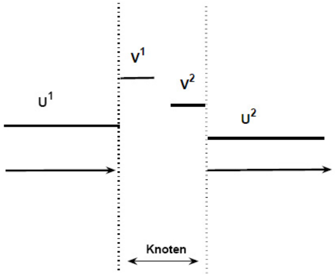
- The solution operator $y = \mathcal{E}(t, t_0, y_0, u)$ enjoys important property

$$\|\mathcal{E}\,(t, t_0, y_0, u) - \mathcal{E}\,(t, t_0, \tilde{y}_0, \tilde{u})\,\| \leq$$
$$L \cdot \left(\|y_0 - \tilde{y}_0\| + \int_{t_0}^{t_0+t} \|u(\tau) - \tilde{u}(\tau)\| d\tau\right)$$

Idea of the proof: Riemann Problems at the Vertex (1/2)

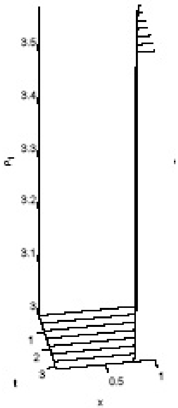
- Consider the situation of piecewise constant initial data in each arc $U^j = y_{j,0}$ – coupling conditions are not necessarily satisfied
- Introduce unknown, artifical states V^j for each arc
- Solve a Riemann problem on each arc j with an artifical state V^j at the node





Choice of V_j (2/2)

- Compute $\Omega_j \in \mathbb{R}^2$, such that for all $V \in \Omega_j$, the self–similar solution $y_j(x,t)$ to a Riemann problem for U^j and V^j consists of waves of non–positive speed (incoming arcs)
- Existence of admissible sets Ω_j due to assumption a subsonic state
- Reduced problem: Find $V_j \in \Omega_j \subset \mathbb{R}^2$, such that the coupling conditions $\Psi = 0$ are fulfilled, uses the assumption on the determinant of Ψ
- A wave – front tracking solution satisfies at the vertex $y_j(0-,t) = V_j \; \forall t > 0$



Application to Gas Transportation Networks

- Gas networks: conditions of conservation of mass and equal pressure is well–posed
- Compressor condition and conservation of mass through compressor is well–posed
- Existence of an open loop control for compressor power control for a single compressor station

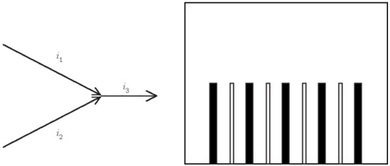
$$\min \mathcal{J} \text{ subj. } \partial_t y^{(i)} + \partial_x f\left(y^{(i)}\right) = g\left(x, y^{(i)}\right), \Psi\left(y^{(1)}, \ldots, y^{(n)}\right) = u(t)$$

Lemma. The cost functional

$$\mathcal{J}(u) = \mathcal{J}_0(u) + \int_0^T \mathcal{J}_1\left(\mathcal{E}\left(t, t_0, y_0, u\right)\right) dt$$

admits a unique minimum on $\left\{u \in \bar{u} + L^1\left([0, T]; \mathbb{R}^n\right) : (y_0, u) \in \mathcal{D}^\delta\right\}$ provided that $\mathcal{J}_{0,1}$ are lower semi–continuous wrt to L^1-norm.

Alternative approach towards coupling conditions via homogenization

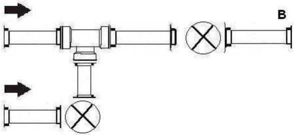


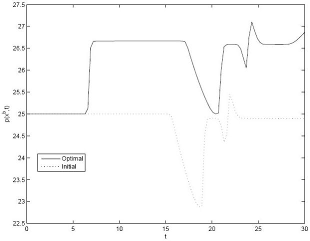
- Every road governed by 2×2 traffic flow model of Aw-Rascle-Zhang (ARZ)
- ARZ $=$ LWR $+$ information traveling with car and influencing speed (e.g., truck or car)
- Vertex introduces a mixture of cars on the outgoing road
- Instead of solving Riemann problems solve an initial-value problem with oscillating initial data on exiting road
- Leads to modified equation on outgoing arc close to vertex

Again: Gas transportation networks

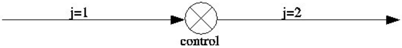
$$\partial_t \begin{pmatrix} \rho_j \\ \rho_j u_j \end{pmatrix} + \partial_x \begin{pmatrix} \rho_j u_j \\ p(\rho_j) + \rho_j u_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(\rho_j, u_j) \end{pmatrix}$$

- Well–posedness for coupling conditions for pipe–to–pipe and compressor vertices
- Existence of open loop (or optimal control) for compressor energy $P(\cdot)$
- *Fulfillment of contracts possible? Stabilization of flow in pipe network by compressor control possible?*





Controllability Result for Compressor Control in Gas Networks



Setting. Two connected pipes – customer ($j = 2$) requires state $y_B(t)$ for $t > t^{**}$ – control is $P(\cdot)$ to be determined.

- Classical subsonic solutions ($\lambda_1(y_i) < 0 < \lambda_2(y_i)$)

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i u_i \end{pmatrix} + A(\rho_i, \rho_i u_i)\partial_x \begin{pmatrix} \rho_i \\ \rho_i u_i \end{pmatrix} = G(t, x, \rho_i, \rho_i u_i) \text{ on } \mathcal{D}_i$$

$$\begin{aligned} \mathcal{D}_1 &= \{(t, x) : t \geq 0, -L \leq x \leq 0\} \\ \mathcal{D}_2 &= \{(t, x) : t \geq 0, \ 0 \leq x \leq L\} \end{aligned}$$

- Theorem (Gugat, H., Schleper): Existence of a control P for suitable y_B but no uniqueness. Assumption on the smallness of C^1-norm of all data.

Equations studied for controllability questions

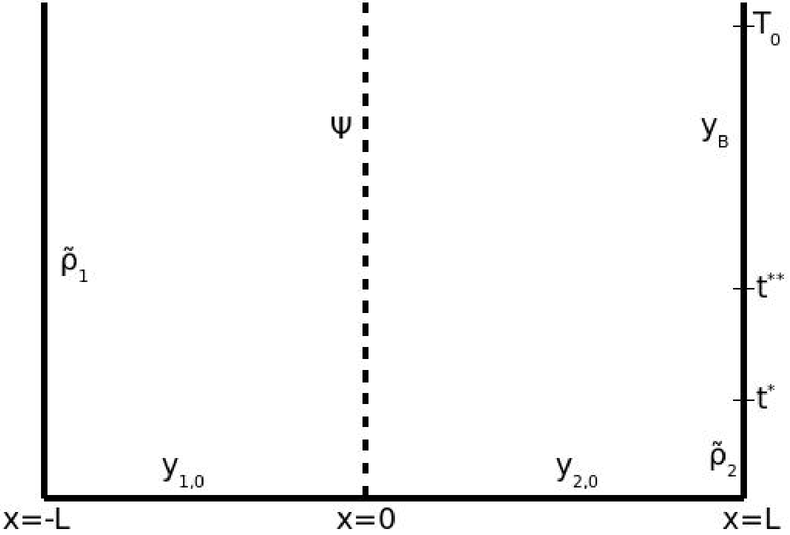
$$\begin{aligned}
\partial_t \begin{pmatrix} \rho_i \\ \rho_i u_i \end{pmatrix} + A(\rho_i, \rho_i u_i)\partial_x \begin{pmatrix} \rho_i \\ \rho_i u_i \end{pmatrix} &= G(t, x, \rho_i, \rho_i u_i) \text{ on } \mathcal{D}_i \\
\Psi(\rho_1, \rho_2, \rho_1 u_1, \rho_2 u_2)(0, t) &= P(t)
\end{aligned}$$

$$\begin{aligned}
\rho_1(t, -L) &= \tilde{\rho}_1(t), \quad \text{for } t > 0 \\
q_2(t, L) &= \tilde{q}_2(t), \quad \text{for } t \in (0, t^*) \\
y_2(t, L) &= y_B(t), \quad \text{for } t > t^{**} > t^*
\end{aligned}$$

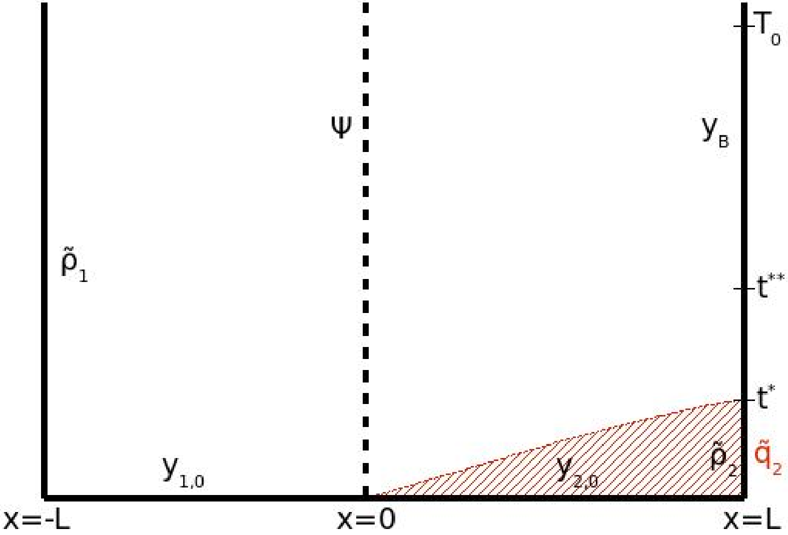
and initial condition $y_i(0, x) = y_{0,i}$.

- Based on existence results for semi–global classical solutions (Li Tatisen et al)
- *Need* smallness of C^1-norm of all data
- Explicit construction of the control $P(t)$ possible

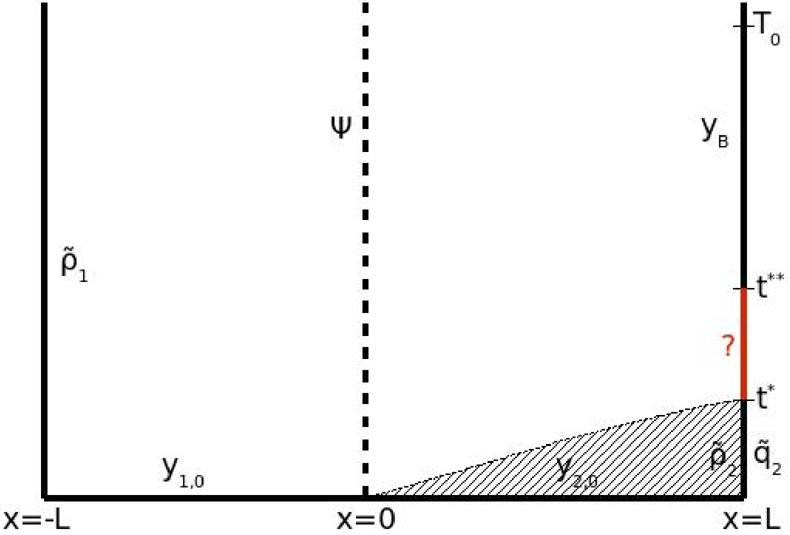
Proof: Construction of control P



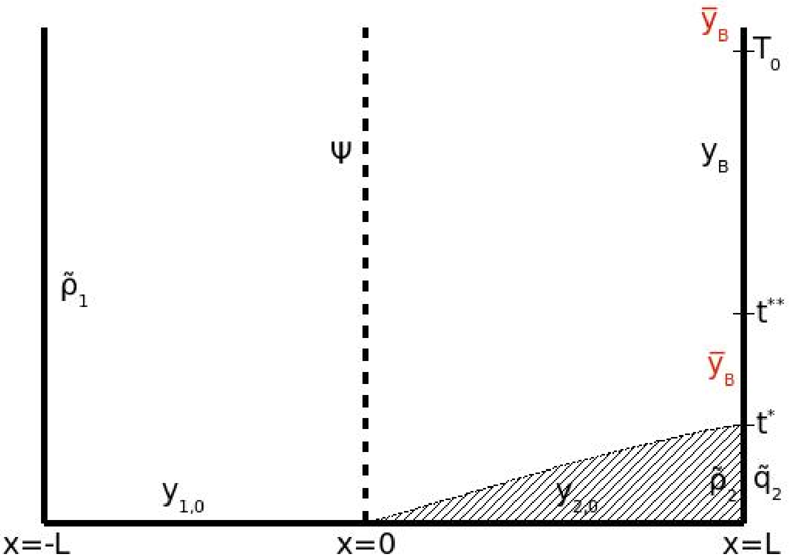
Solution y in red area obtained in particular $\tilde{q}_2$



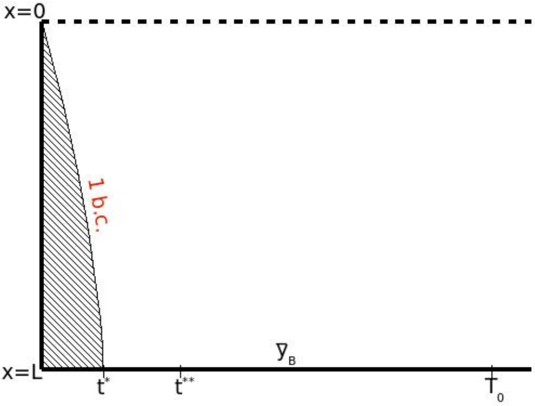
Use any smooth connection of $\tilde{\rho}_2, \tilde{q}_2$ to desired state $\bar{y}_B$



Change meaning of x and t



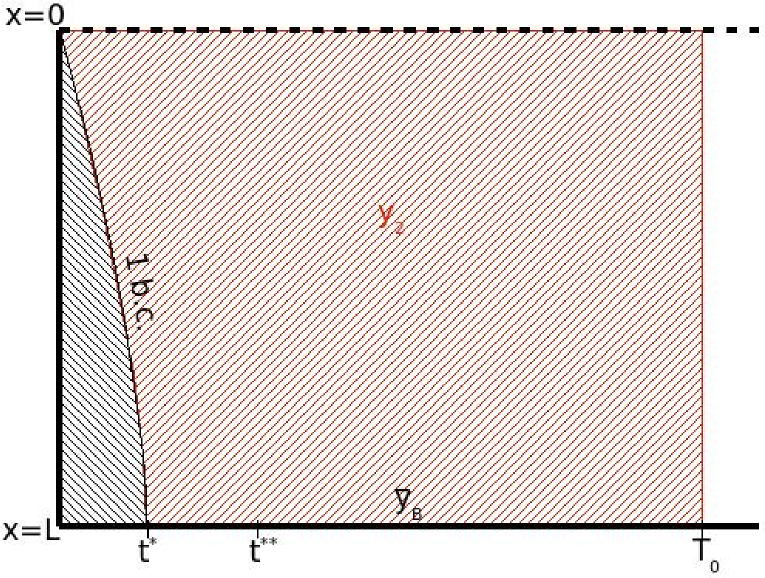
Full solution in second pipe available



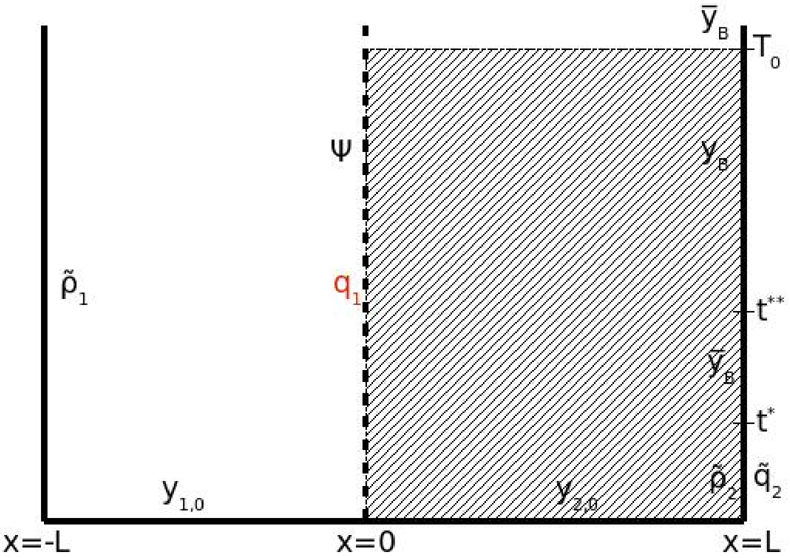
transposed problem:

$$\partial_x y_2 + (A(y_2))^{-1}\, \partial_t y_2 = (A(y_2))^{-1}\, G(t, x, y_2)$$

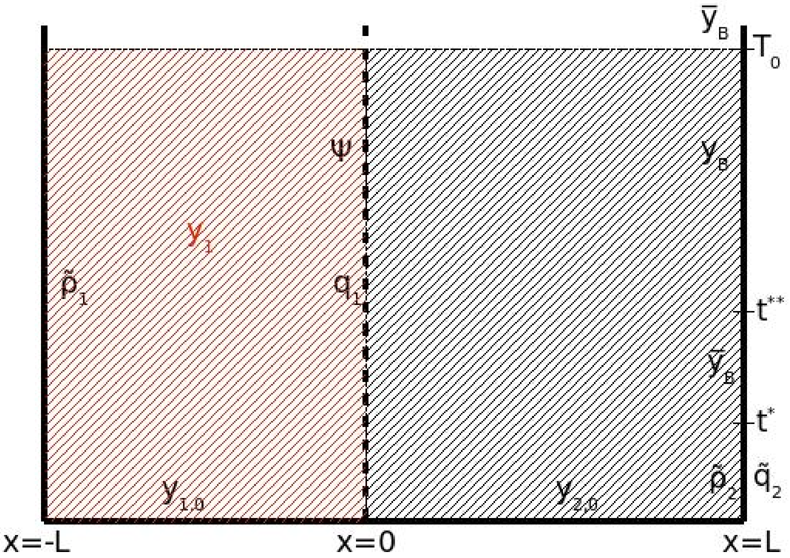
Full solution in second pipe available



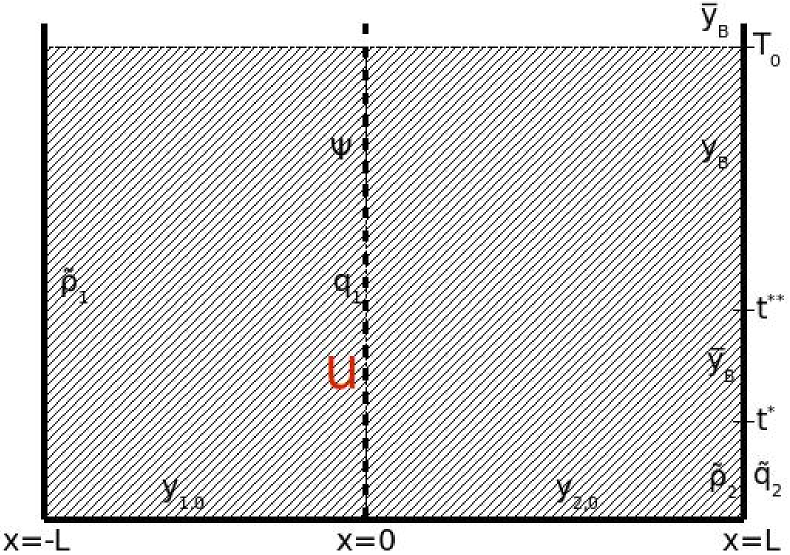
Coupling condition: ρ_2, q_2 gives q_1



Solution in the first pipe available (bc, 2 ic, bc)



Control u due to coupling conditions depending on ρ_1, ρ_2, q_2, q_1



Stabilization of Gas Flow

Stabilization of flow patterns using compressor stations?

- Results for a single pipe and the isothermal Euler equations without source term[5]
- Linearization of the system around constant (also space–dependent possible) stationary states gives a quasi–linear hyperbolic system
- Transformation of the linearized system in Riemann invariants $R_{\pm}$
- Derive boundary conditions (aka compressor) of the linear part of the system to stabilize the flow of the quasi–linear equations
- Proof relies on the design of suitable Lyapunov functions extending previous work by Coron et al

[5] Tree–like networks, compressor stations, pipe–to–pipe intersections also possible

Example of a Lyapunov function for the linearised problem in Riemann invariants

$$\partial_t \begin{pmatrix} R_+ \\ R_- \end{pmatrix} + \begin{pmatrix} \bar\lambda_+ - \frac{R_+ + R_-}{2} & 0 \\ 0 & \bar\lambda_- - \frac{R_+ + R_-}{2} \end{pmatrix} \partial_x \begin{pmatrix} R_+ \\ R_- \end{pmatrix} = 0$$

- Lyapunov function with constants $\mu, A_\pm > 0$

$$L(t) = \int_0^L \frac{A_+}{\bar\lambda_+} \exp(-\frac{\mu}{\lambda_+} x) R_+^2(t,x) dx + \int_0^L \frac{A_-}{\|\bar\lambda_-\|} \exp(\frac{\mu}{\|\lambda_-\|} x) R_-^2(t,x) dx$$

- Equivalent to L^2-norm, therefore L^2-stabilization result
- Provided there is a uniform bound on $\|\partial_x R_\pm\| \leq \tau$ one obtains exponential decay of L

$$\frac{d}{dt} L(t) \leq (-\mu + \tau\alpha) L(t)$$

$$\alpha = \frac{3}{2} + \frac{1}{2} \max \left\{ \frac{A_- \bar\lambda_+}{-A_+ \bar\lambda_-} \exp\left(\mu L (\frac{1}{\bar\lambda_+} - \frac{1}{\bar\lambda_-}) \right), \frac{-A_+ \bar\lambda_-}{A_- \bar\lambda_+} \right\}$$

Stabilization Result for Compressor Controlled Gas Networks

Theorem. (Gugat, H.)

- Assumptions: stationary subsonic state, finite terminal time T, constants $k_0, k_L \in (0,1)$
- Then, there exists $\delta_0 > \delta_1 > 0$ such that any initial data $\|u_i^0\|_{C^1} \leq \delta_1$ and such that the compatibility conditions at $x = 0, L$ are satisfied, there is a C^1-solution u_i of the quasi–linear system.
- Then, the Lyapunov function $L(t)$ satisfies

$$L(t) \leq L(0) \exp(-\frac{\mu}{2} t)$$

  and $R_{\pm}$ decays to zero exponentially fast[6]
- μ depends on $k_{0,L}$

[6] Extension to H^1 stabilzation (Gugat)

Numerical discretization of problem

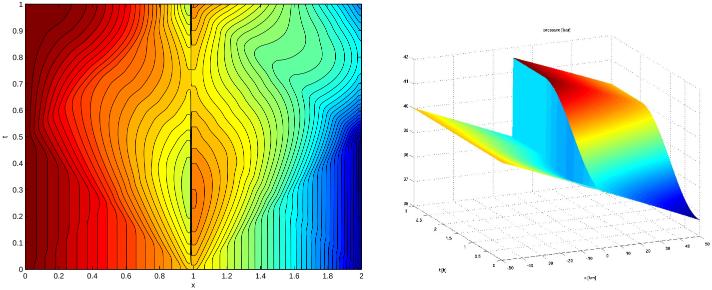
- Discretized version of the continuous Lyapunov function
- Results for a system in diagonal form (i.e. Riemann invariants)
- Assumption on boundness of initial data and spatial derivative necessary in order to prove exponential decay of the discretized Lyapunov functions
- Explicit bounds on decay rate μ
- Decay independent on the grid size

Theoretical expected value of decay rate $\mu = 5.75E-01$.

N_x	$L^{\inf}$	L^2	μ	ν
100	4.32E-03	7.48E-04	5.66E-01	5.69E-
200	2.18E-03	2.67E-04	5.70E-01	5.72E-
400	1.09E-03	9.49E-05	5.73E-01	5.73E-
800	5.48E-04	3.36E-05	5.74E-01	5.74E-
1600	2.74E-05	1.19E-05	5.75E-01	5.75E-

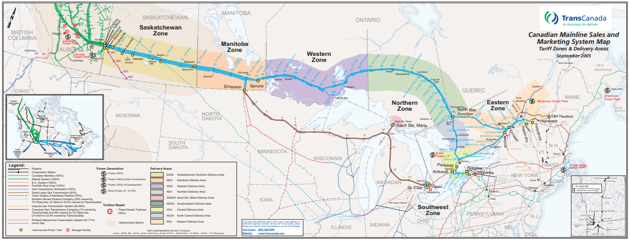
TABLE 1. The number of grid points in the spatial doma
noted by N_x. $L^{\inf}$ denotes the norm $\|(L_n^{exact})_n - (L^n)_n\|$
norm $\|(L_n^{exact})_n - (L^n)_n\|_2$. The values of μ and ν are c
equation (26). The CFL constant is equal to one and $\kappa =$

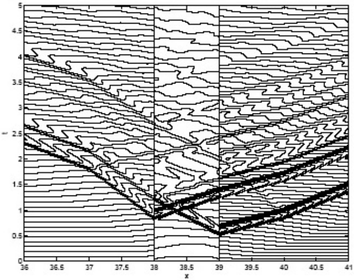
Optimal control vs exact control



Two connected pipes. Depicted is the pressure evolution over both pipes and time. Simulation result using higher–order finite–volume scheme. Left: optimal control, right: exact control for a constant desired pressure.

Simulation of pipe network





Network graph of Candian mainline gas network and Isolines of the pressure selected pipes in Toronto area

Conclusion

- Some ideas on modeling and analysing problems on networks
- Well–posedness theory for nodal control of systems of 2×2 balance laws on network (weak + classical)
- Remarks on existence of optimal controls including shock waves
- Remarks for gas networks on controllability and stabilization using semi–global (classical) solutions

Thank you for your attention.