Global existence of strong solution for the compressible Navier-Stokes equations with large initial data on the irrotational part

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Let us recall the compressible Navier-Stokes equations:

- Mass equation :

$$\partial_t \rho + \operatorname{div} \rho \mathbf{u} = 0,$$

- Momentum equation :

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) D(u)) - \nabla(\lambda(\rho)\operatorname{div} u) + \nabla P(\rho) = 0,$$

- Initial data :

$$(\rho, u)_{/t=0} = (\rho_0, u_0).$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ the strain tensor.
We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ (such a condition ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$).
In the sequel we shall only consider the shallow-water system which corresponds to:

$$\mu(\rho) = \mu\rho \text{ with } \mu > 0 \quad \text{and} \quad \lambda(\rho) = 0.$$

We supplement the problem with initial condition (ρ_0, u_0).

We aim at solving the system in the case where the data (ρ_0, u_0) have *critical* regularity. By critical, we mean that we want to solve our system in functional spaces with invariant norm for the scaling of the equations. It is easy to see that the following transformations:

$$(\rho_0(x), u_0(x)) \longrightarrow (\rho_0(lx), lu_0(lx)), \quad \forall\, l \in \mathbb{R}.$$
$$(\rho(t,x), u(t,x), P(t,x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx), l^2 P(l^2 t, lx)).$$

have this property of invariance, provided that the pressure term has been changed in $l^2 P$. A good candidate corresponds to the space $H^{\frac{N}{2}} \times (H^{\frac{N}{2}-1})^N$.

Some results of global strong solutions

- A. Matsumura and T. Nishida [80s], Global strong solutions with small initial data.
- D. Hoff [90s,07], Global weak-strong solution with small initial data with discontinuous initial density $\rho_0 \in L^\infty$.
- R. Danchin [2000], Global strong solutions with small initial data in critical space for the scaling of the equations with $u_0 \in B^{\frac{N}{2}-1}_{2,1}$, $(\rho_0 - 1) \in B^{\frac{N}{2}}_{2,1}$.
- Z. Chen et al, F. Charve and R. Danchin, BH [2010,2011], Global strong solutions with small initial data with $u_0 \in B^{\frac{N}{p}-1}_{p,1}$, $(\rho_0 - 1) \in B^{\frac{N}{p}-1}_{p,1} \cap B^{\frac{N}{p}}_{p,1}$ with $1 \leq p \leq N$. In particular we have large initial oscillary initial data in L^N,

  $$u_0(x) = \phi(x) sin(\varepsilon^{-1} x \cdot \omega) n,$$

  with ω and n some unit vectors and $\phi \in C_0^\infty(\mathbb{R}^N)$.

How to obtain some global results for a family of large initial data?

To do this we are going to consider some irrotational data. More precisely we are interested in constructing irrotational solution. It is natural to deal with such solution, indeed let us mention the case of the so-called compressible Euler system with quantum pressure which is transformed in a non linear Schrd̈ingier equations via the famous transformation of Madelung.

More precisely when $P(\rho)=0$ we can check that:

$$(\rho_1, -\mu\nabla\ln\rho_1),$$

is a particular solution of our system (with initial data $(\rho_0, -\mu\nabla\ln\rho_0)$) if ρ_1 verifies an heat equation:

$$\begin{cases} \partial_t\rho_1 - \mu\Delta\rho_1 = 0, \\ (\rho_1)_{/t=0} = (\rho_1)_0. \end{cases}$$

We are going to call $(\rho_1, -\mu\nabla\ln\rho_1)$ a *quasi-solution*. The idea in the sequel will consist in working around these profiles.

Remark

- *In order to ensure the irrotational form of the solution, we need that the term $\nabla\ln\rho\cdot Du$ be a gradient, which implies the structure in $\nabla\ln\rho$ for the velocity.*
- *Very surprisingly we have some regular effects on the density which is a priori governed by a transport equation.*

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let (φ,ψ) a couple of regular functions such that φ has support in $\mathcal{C}=\{\xi\in\mathbb{R}^N/\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$, and ψ in $B(0,\frac{4}{3})$ with:

$$\psi(\xi)+\sum_{l\in\mathbb{N}}\varphi(2^{-l}\xi)=1.$$

Let us note $h=\mathcal{F}^{-1}\varphi$, we can define the dyadic blocks by:

$$\Delta_l u=\varphi(2^{-l}D)u=2^{lN}\int_{\mathbb{R}^N}h(2^l y)u(x-y)dy \;\text{ if }\; l\in\mathbb{Z}$$

$$S_l u=\sum_{k\leq l-1}\Delta_k u\,.$$

We have then modulo the polynoms in $\mathcal{S}^{'}(\mathbb{R}^N)$:

$$u=\sum_{k\in\mathbb{Z}}\Delta_k u.$$

Definition

The Besov space $B^s_{p,r}$ corresponds to the set of temperated distributions $u\in\mathcal{S}^{'}_h$ such that :

$$\|u\|_{B^s_{p,r}}=\Big(\sum_{l\in\mathbb{Z}}2^{lsr}\|\Delta_l u\|^r_{L^p}\Big)^{\frac{1}{r}}<+\infty.$$

Definition

Let $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ *and* $s_1 \in \mathbb{R}$. *We define the space* $\widetilde{L}^{\rho}_{T}(B^{s_1}_{p,r})$ *as follows,* u *is in* $\widetilde{L}^{\rho}_{T}(B^{s_1}_{p,r})$ *if:*

$$\|u\|_{\widetilde{L}^{\rho}_{T}(B^{s_1}_{p,r})} = \Big(\sum_{l\in\mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|^r_{L^{\rho}_{T}(L^p)}\Big)^{\frac{1}{r}}.$$

Proposition

Let $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$. *Assume that* $u_0 \in B^s_{p,r}$ *and* $f \in \widetilde{L}^1_T(B^s_{p,r})$. *Let* u *be a solution of:*

$$\begin{cases} \partial_t u - \mu\Delta u = f \\ u_{t=0} = u_0 \,. \end{cases}$$

Then there exists $C > 0$ *depending only on* N, μ, ρ_1 *and* ρ_2 *such that:*

$$\|u\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} \leq C\big(\|u_0\|_{B^s_{p,r}} + \|f\|_{\widetilde{L}^1_T(B^s_{p,r})}\big)\,.$$

If in addition r *is finite then* u *belongs to* $C([0,T], B^s_{p,r})$.

We now search some solutions written under the following form $\ln\rho = \ln\rho_1 + h^2$ ($\rho = \rho_1 e^{h^2}$, with $\rho_1 = 1 + q^1$) and $u = -\mu\nabla\ln\rho_1 + u^2$, we have then modulo that the density does not admit vacuum:

$$\begin{cases} \partial_t \ln\rho + u\cdot\nabla\ln\rho + \mathrm{div}u = 0, \\ \partial_t u + u\cdot\nabla - \mu\Delta u - \mu\nabla\ln\rho\cdot Du + \nabla F(\rho) = 0, \\ (\ln\rho, u)_{/t=0} = (\ln\rho_0, u_0). \end{cases} \tag{1}$$

By using the fact that $(\rho^1, u^1) = (\rho^1, -\mu\nabla\ln\rho^1)$ is a *quasi solution* with:

$$\begin{cases} \partial_t\rho^1 - \mu\Delta\rho^1 = 0. \\ \rho^1(0,\cdot) = \rho^1_0. \end{cases}$$

we can rewrite the system under the following form:

$$\begin{cases} \partial_t h^2 + u\cdot\nabla h^2 + \mathrm{div}u^2 = -u^2\cdot\nabla\ln\rho^1, \\ \partial_t u^2 + u\cdot\nabla u^2 - \mu\Delta u^2 + a\nabla h^2 = -a\nabla\ln\rho^1 - u_2\cdot\nabla u^1 + \mu\nabla\ln\rho^1\cdot Du^2 \\ \qquad\qquad +\mu\nabla h^2\cdot Du^1 + \mu\nabla h^2\cdot Du^2, \\ (h^2, u^2)_{/t=0} = (h^2_0, u^2_0). \end{cases} \tag{2}$$

where we have assumed to simplify that $P(\rho) = a\rho$. We are going to solve the system (2) with small initial data (h^2_0, u^2_0) and with large ρ_1.

Theorem

Let $\rho_0 = (\rho_1)_0 e^{h_0^2}$ *and* $u_0 = -\mu\nabla \ln(\rho_1)_0 + u_0^2$. *Moreover we assume that* $(\rho_1)_0 \geq c > 0$, $(\rho_1 - 1)_0 \in B_{2,1}^{\frac{N}{2}-2} \cap B_{2,1}^{\frac{N}{2}}$, $h_0^2 \in B_{2,1}^{\frac{N}{2}}$ *and* $u_0^2 \in B_{2,1}^{\frac{N}{2}-1}$. *Then it exists* $\varepsilon > 0$, $C > 0$ *and* $l > 0$ *large enough (depending on* $\|(\rho_1 - 1)_0\|_{B_{2,1}^{\frac{N}{2}}}$ *such that if:*

$$\|\sum_{k\geq l} \Delta_k(\rho_1 - 1)_0\|_{B_{2,1}^{\frac{N}{2}-2}} \leq \varepsilon \ \text{ and } \ \|h_0^2\|_{B_{2,1}^{\frac{N}{2}}} + \|u_0^2\|_{B_{2,1}^{\frac{N}{2}-1}} \leq \varepsilon, \tag{3}$$

then it exists a global strong solution (ρ, u) *of the shallow-water system under the following form:* $\rho = \rho_1 e^{h^2}$ *and* $u = -\mu\nabla \ln \rho_1 + u^2$ *with:*

$$\begin{cases} \partial_t \rho_1 - \mu\Delta\rho_1 = 0, \\ (\rho_1)_{t=0} = (\rho_1)_0. \end{cases} \tag{4}$$

and such that:

$$h^2 \in \widetilde{C}(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}}) \cap L^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}})$$
$$\text{and } u^2 \in \widetilde{C}(\mathbb{R}^+; B_{2,1}^{\frac{N}{2}-1}) \cap L^1(\mathbb{R}+, B_{2,1}^{\frac{N}{2}+1}).$$

Remark

We have a super-critical condition of smallness for the initial density, more precisely we can write this condition under the following form:

$$\|(\rho_1 - 1)_0\|_{B_{2,1}^{\frac{N}{2}-2}} \leq C exp(-C\|(\rho_1 - 1)_0\|_{B_{2,1}^{\frac{N}{2}}}).$$

We can choose initial data with α *and* β *well-chosen of the form:*

$$q_0^1(x) = \frac{1}{\varepsilon^\beta} e^{i\frac{x_3}{\varepsilon}} f(x_1, \frac{x_2}{\varepsilon^\gamma}, x_3),$$

with $(\rho_1)_0 = \frac{1}{\varepsilon^\beta} + q_0^1$ *and* $f \in \mathcal{S}(\mathbb{R}^N)$.
In particular we allows initial density with large L^∞ *norm.*

Remark

In particular we can obtain global strong solution in dimension $N = 2$ *with large initial data in the energy spaces. We also refer to a remarkable work of Kazhikhov and Waigant for other viscosity coefficients.*

Remark

In a very surprising way the density is decomposed in a small part h^2 *and in a regular part* ρ^1. *This regularizing effect remains quite surprising.*

In the sequel we are going to show the global existence for (h^2, u^2).

First step: Construction of approximate solutions

We regularize the initial data:

$$(h_2)_0^n = S_n h_0^2,\ \ (u_2)_0^n = S_n u_0^2.$$

We obtain then a solution (h_2^n, u_2^n) on the interval $(0, T_n)$ such that:

$$h_2^n \in \widetilde{C}([0,T_n], B^{N}_{2,1} \cap B^{\frac{N}{2}-1}_{2,1})\ \ u_2^n \in \widetilde{C}([0,T_n], B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1([0,T_n], B^{\frac{N}{2}+1}_{2,1}).$$

Second step: Uniform estimates

We are going to obtain uniform estimates on (h_2^n, u_2^n) in order to show that $T_n = +\infty$. To do this, we need to distinguish the behavior in low and high frequencies.

We need to study the following system:

$$\begin{cases} \partial_t h^n + u^n \cdot \nabla h^n + \mathrm{div} u_2^n = F_1^n, \\ \partial_t u_2^n - \mu \Delta u_2^n + a \nabla h^n = G_1^n, \\ h_0^n = h_0,\ (u_2^n)_{/t=0} = (u_2)_0^n. \end{cases} \tag{5}$$

Let us mention that it is necessary to include in the study the convection term in order to not lose derivative on the density.

Estimate on the linearized system in Besov spaces:

Proposition

Let $s \in \mathbb{R}$. Let (h^2, u^2) a solution of the previous system. It exists a constant C such that:

$$\|(h^2,u^2)(t)\|_{(B^{s-1}_{2,1}\cap B^{s}_{2,1})\times B^{s-1}_{2,1}} + \int_0^t \|(h^2,u^2)(s)\|_{B^{s}_{2,1}\times B^{s+1}_{2,1}} ds$$
$$\leq C\big(\|(h_0^2,u_0^2)\|_{(B^{s-1}_{2,1}\cap B^{s}_{2,1})\times B^{s-1}_{2,1}} + \int_0^t e^{-V(s)}\|(F_1,G_1)(s)\|_{(B^{s-1}_{2,1}\cap B^{s}_{2,1})\times B^{s-1}_{2,1}} ds\big).$$

with $V(T) = \int_0^T \|\nabla u(s)\|_{L^\infty} ds$.

To prove this proposition we need to distinguish the behavior in low frequencies and in high frequencies.
In high frequencies, we introduce a *effective velocity* v^2 which diagonalize the system:

$$v_2 = u^2 - a(\Delta)^{-1}\nabla h^2.$$

We can rewrite the system as follows:

$$\begin{cases} \partial_t h^2 + u\cdot\nabla h^2 + ah^2 + \mathrm{div} v^2 = F_1, \\ \partial_t v^2 - \mu\Delta v^2 = G_1 + f, \end{cases} \tag{6}$$

with f a low regularity order term which can be absorbed in high frequencies.

It means that roughly speaking we have only to deal with a heat equation on v^2 and a transport equation with a damping term for h^2.
In low frequencies we are directly working with the unknowns (h^2, u^2) and we are going to take advantage of the regularizing effect on the density h^2 (in low frequencies). In particular it allows us to consider the convection term $u \cdot \nabla h^2$ as a remainder term, we have to study then the following system:

$$\begin{cases} \partial_t h^2 + \mathrm{div} u^2 = F_1 - u \cdot \nabla h^2, \\ \partial_t u^2 - \mu \Delta u^2 + a \nabla h^2 = G_1, \end{cases} \tag{7}$$

It remains to study the Green kernel associated to this linear system by using the Fourier transform. It concludes the proof of the proposition 2.1.
We are going to use this proposition in order to obtain uniforms bound estimates on (h_2^n, u_2^n). Let us define the following norm:

$$E(h,u) = \|h\|_{\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}})} + \|u\|_{\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1})} + \|u\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})},$$

When we apply the proposition 2.1 to our system and by using paraproduct laws for the Besov spaces in order to treat the remainder term, we obtain the following inequality:

$$E(h_2^n, u_2^n) \leq C e^{\|(\rho_1, -\mu \nabla \ln \rho_1 + u_2^n)\|_E} \big(\|h_0^2\|_{B_{2,1}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}}} + \|u_0^2\|_{B_{2,1}^{\frac{N}{2}-1}}$$
$$+ a \|\nabla \ln \rho_1\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} + E(h_2^n, u_2^n) \big).$$

At this level we can apply a classical bootstrap argument in order to obtain global estimate for (h_2^n, u_2^n).

Let us observe that to study the system under his eulerian form we need to control the vacuum, this is done because ρ_1 verifies an heat equation, we can apply a maximum principle in order to prove that $\rho_1 \geq c > 0$.

Let us point out now that here as ρ^1 verifies an heat equation and that we control the vacuum on ρ_1, we have:

$$\|\nabla \ln \rho_1\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \leq C\|(\rho_0)_1\|_{B_{2,1}^{\frac{N}{2}-2}}.$$

We observe then that we have a smallness condition on ρ_1 which is supercritical. More precisely we have a condition of the form:

$$\|(\rho_1-1)_0\|_{B_{2,1}^{\frac{N}{2}-2}} \leq Cexp(-C\|(\rho_1-1)_0\|_{B_{2,1}^{\frac{N}{2}}}).$$

We can choose initial data with α and β well-chosen of the form:

$$q_0^1(x) = \frac{1}{\varepsilon^\beta} e^{i\frac{x_3}{\varepsilon}} f(x_1, \frac{x_2}{\varepsilon^\gamma}, x_3),$$

with $(\rho_1)_0 = \frac{1}{\varepsilon^\beta} + q_0^1$ and $f \in \mathcal{S}(\mathbb{R}^N)$. In particular we allows initial density with large L^∞ norm.

Third step: Existence of solutions
Finally by using compactness arguments involving the Ascoli theorem, we show that (h_2^n, u_2^n) converges weakly in E to (h^2, u^2) a solution.

Fourth step: Uniqueness
The uniqueness is classical (we refer to Danchin).

THANK YOU FOR YOUR ATTENTION!