Asymptotic behavior of solutions for Damped wave equations with non-convex convection term on the half line

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Joint work with
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Contents

1. Preliminaries

Damped wave equation.

$$\begin{cases} u_{tt} - u_{xx} + u_t + f(u)_x = 0, & x > 0, \ t > 0, \\ u(0,t) = u_-, & t > 0, \\ \lim_{x\to\infty} u(x,t) = u_+, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x > 0. \end{cases} \tag{1}$$

- **Orive and Zuazua (2003),**
  Initial problem with the convection term $f(u) = |u|^{\gamma-1}u$ $(\gamma \geq 2)$.
  (Global existance)

· **Ueda-Kawashima (2007)**

Generalized to the case $|f'(0)| < 1$ and shows (1) can be well approximated by the viscous conservation law for $t \to \infty$:

$$u_t + f(u)_x = \big(\mu(u)u_x\big)_x, \quad \text{(viscous conservation law,)} \tag{2}$$

$$\text{where } \mu(u) = 1 - (f'(u))^2.$$

· Chapman-Enskog expansion (Second order expansion)

· **Purpose of this talk :** Solution of (1) $\sim$ Solution of (2).

Now we want to review some properties of the solution of viscous conservation law

viscous conservation law

$$\begin{cases} u_t + f(u)_x = u_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0, & \lim_{x \to \pm\infty} u_0(x) = u_\pm. \end{cases}$$

$$\Updownarrow$$

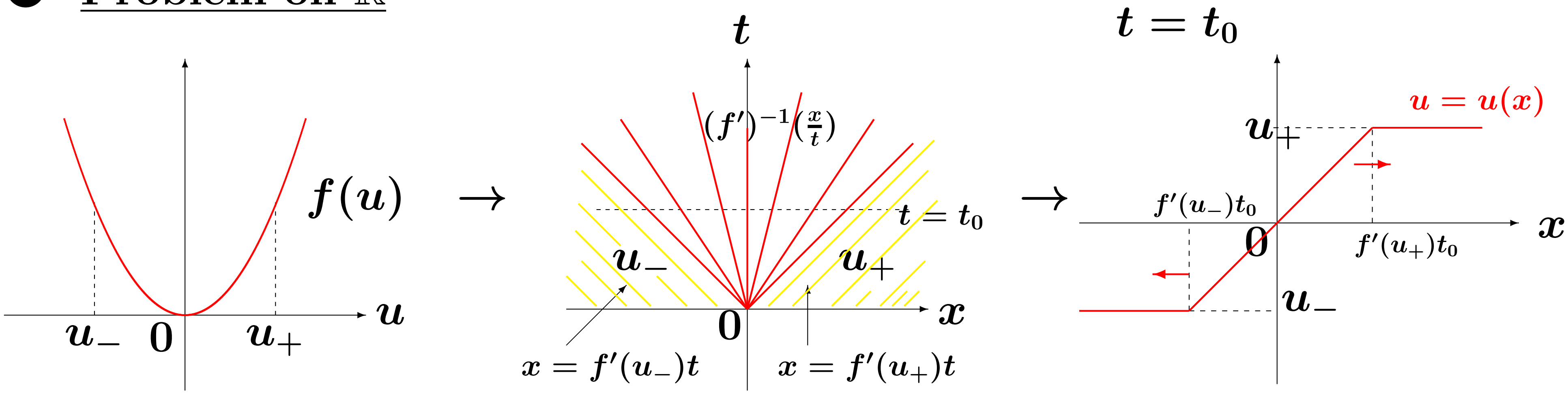
Riemann problem

$$\begin{cases} u_t + f(u)_x = 0, & x \in R, t > 0, \\ u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0. \end{cases} \end{cases}$$

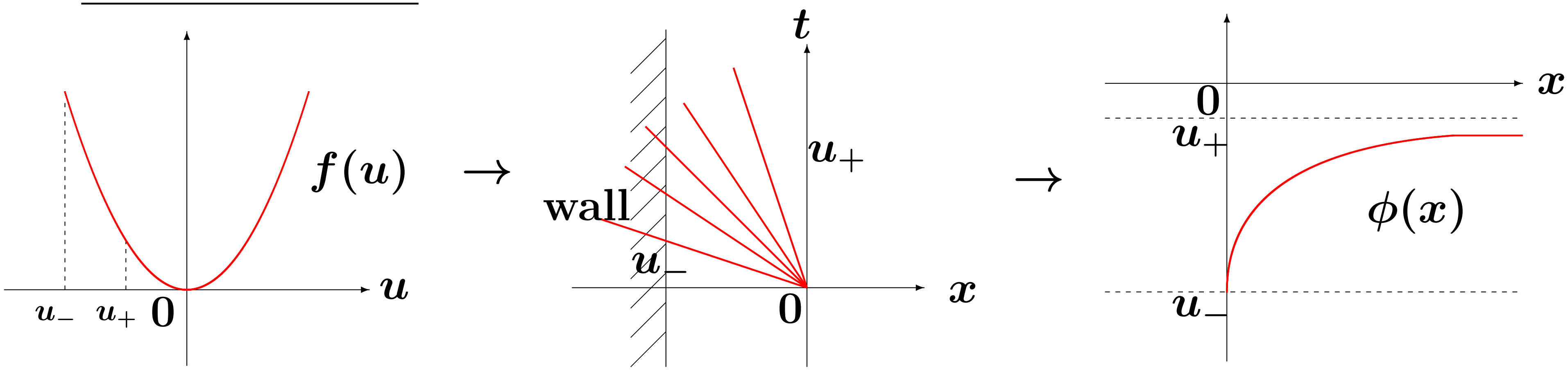
- The large time behavior of solution of viscous conservation law is well characterized by the corresponding Riemann Problem for the hyperbolic part. Now we review the property of the behavior of Rieman solution.

$f'' > 0, \quad u_- < u_+$ (Characteristic curve and Riemann solution)

● Problem on $\mathbb{R}$

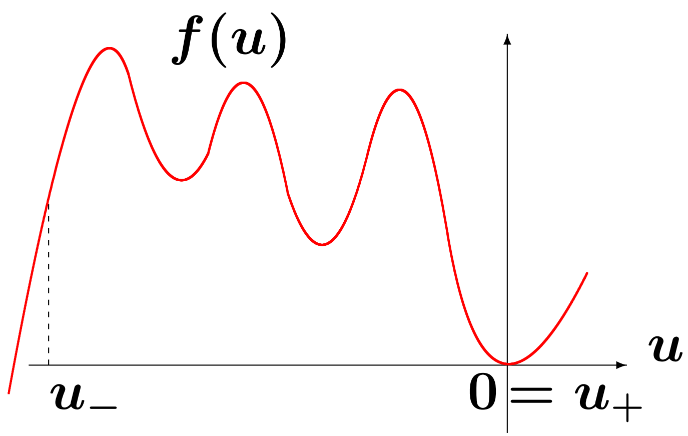


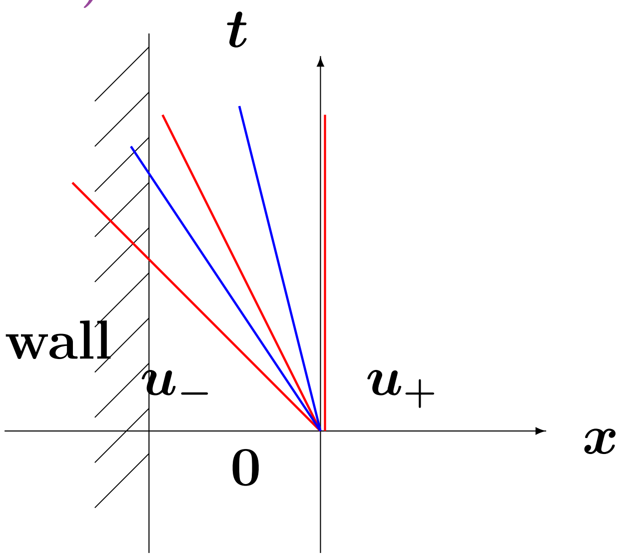
● Problem on $\mathbb{R}_+$



Half line problem (Non-convex flux)

(Riemann solution have some shock in general)



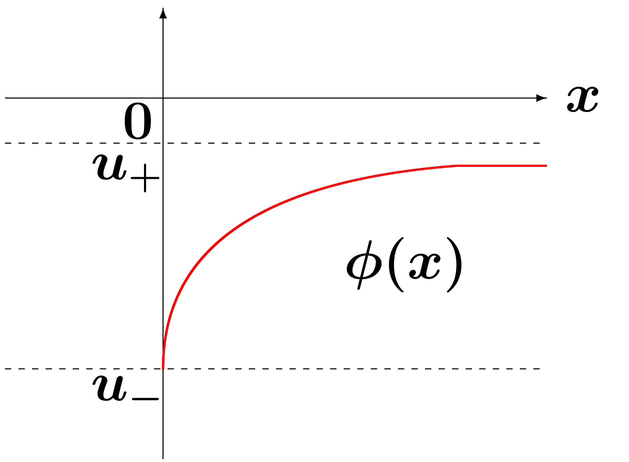


$\Downarrow$

Stationary wave

Stationary solution:

$$(s):\ \begin{cases} f(\phi)_x = \phi_{xx}, & x > 0, \\ \phi(0) = u_-, & \lim_{x\to\infty} \phi(x) = 0. \end{cases}$$



Known result (viscous conservation law with non-convex flux)

- **Liu-Nishihara (1997),** (Small data in H^2)

$$\lim_{t\to\infty}\sup_{R_+}|u(t,x)-\phi(x)|=0, \qquad x\in\mathbb{R}_+.$$

- **Freistühler-Serre (2001),** (Large data)

$$\lim_{t\to\infty}\|u-\phi\|_{L^1}=0, \qquad x\in\mathbb{R}_+.$$

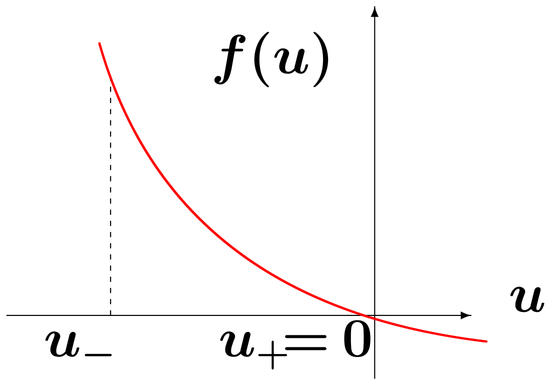
- **Matsumura-H (2007),** (Small data in H^1)

$$\lim_{t\to\infty}\sup_{R_+}|u(t,x)-\phi(x)|=0, \qquad x\in\mathbb{R}_+.$$

Original Motivation:
We want to know whether we can show the same asymptotic stability of stationary wave for the damped wave equation with any convection term.

First of all, I want to show the known result of Danped wave equation for this direction.

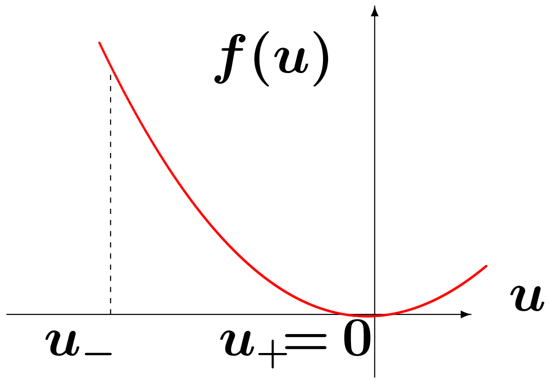
Known results



- **Ueda (2008)**

Non-degenerate stationary wave

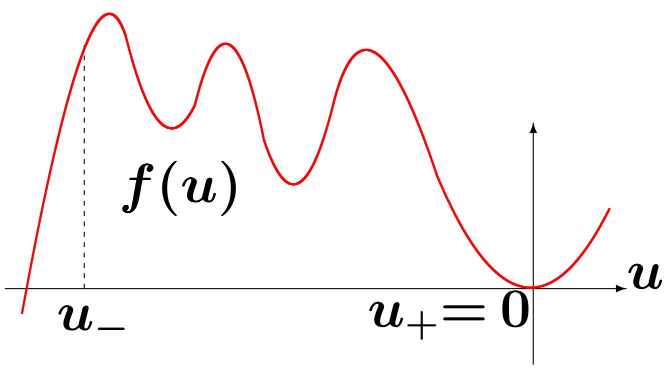
$$f''(u) > 0, \quad \underline{0 < |f'(u)| < 1},$$
$$\Downarrow$$
$$\lim_{t\to\infty} \|u(t) - \phi\|_{L^\infty} = 0$$



- **Ueda-Nakamura-Kawashima (2010)**

Degenerate stationary wave

$$f''(u) > 0, \quad f'(0) = 0, \quad \underline{0 < |f'(u)| < 1},$$
$$\Downarrow$$
$$\lim_{t\to\infty} \|u(t) - \phi\|_{L^\infty} = 0$$



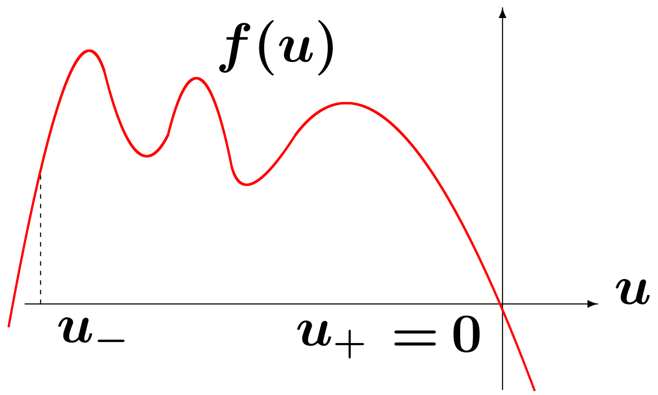
- **Ueda-H (2011)**

Degenerate stationary wave

$$f''(0) > 0, \quad f'(0) = 0,$$
$$\Downarrow$$
$$\lim_{t\to\infty} \|u(t) - \phi\|_{L^\infty} = 0$$

Purpose of today's talk

To cover the any types of convection term, we consider the asymptotic stability of non-degenerate stationary wave with non-convex convection term. So we consider the initial boundary value problem of damped wave equation with the following convection term:



$$\begin{aligned} & u_- < u_+ = 0, \\ & f(u) > 0 \text{ for } u \in [u_-, 0), \\ & f(0) = 0, \\ & 0 < |f'(0)| < 1 \end{aligned} \qquad \cdots \ (*)$$

Method

- **Asymptotic stability: Anti-derivative method, Weighted energy method**
- **Decay rate: Space-time weighted energy method**

2. Main Result

Asymptotic stability of stationary wave.

$$
\begin{aligned}
&u_0 - \phi \in H^1 \cap L^1, \quad u_1 \in L^2 \cap L^1, \\
&z_0 := -\int_x^\infty (u_0(y) - \phi(y))dy \in L^2,\ z_1 := -\int_x^\infty u_1(y)dy \in L^2.
\end{aligned} \tag{3}
$$

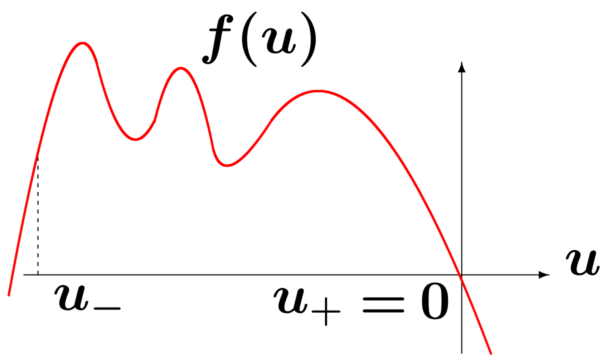
Theorem 1.

Suppose that $(*)$ and (3).
Then, $\exists \epsilon > 0$ s.t. if $\|z_0\|_{H^2} + \|z_1\|_{H^1} \le \epsilon$, the problem (1) has a unique global solution in time u satisfying

$$u - \phi \in C^0([0,\infty); H^1_0) \cap C^1([0,\infty); L^2) \cap L^2(0,\infty; L^2),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0.$$



$$
\begin{aligned}
&u_- < u_+ = 0, \\
&f(u) > 0 \text{ for } u \in [u_-, 0), \\
&f(0) = 0, \\
&0 < |f'(0)| < 1
\end{aligned} \quad \cdots (*)
$$

Theorem 2. (Decay rate.)

Assume the same condition in Theorem 1.
u: global solution to the problem (1) .

· Polynomial Decay

Then, if $z_0 \in H^2_\alpha$ and $z_1 \in H^1_\alpha$ for $\alpha \geq 0$, we have

$$\|u(t) - \phi\|_{H^1} \leq C(1+t)^{-\alpha/2} \quad t \geq 0,$$

· Exponential Decay

Then, if $z_0 \in H^2_{\alpha,exp}$ and $u_1 \in H^1_{\alpha,exp}$ for $\alpha \geq 0$, we have

$$\|u(t) - \phi\|_{H^1} \leq Ce^{-\beta t} \quad t \geq 0,$$

where $\beta = \beta(\alpha) > 0$.

- $\|f\|^2_{L^2_\alpha} := \int_0^\infty (1+x)^\alpha |f(x)|^2 dx, \qquad \alpha > 0.$
- $\|f\|^2_{L^2_{\alpha,exp}} := \int_0^\infty e^{\alpha x} |f(x)|^2 dx, \qquad \alpha > 0.$

Stability of stationary wave

Proof of Theorem 1 (Stability of stationary wave)

We first consider the properties of the stationary solution ϕ which is given by the solution to the boundary value problem for the ordinary differential equation:

$$\begin{cases} f(\phi)_x = \mu\phi_{xx}, \\ \phi(0) = u_-, \quad \lim_{x\to\infty}\phi(x) = 0. \end{cases} \tag{4}$$

This is the ordinary differential equation and we can solve it explicitly. Especially, we have a Lemma as follows.

Lemma. (stationary wave)

Assume $u_- < 0$. Then, the boundary value problem (4) has a unique solution $\phi \in C^3([0,\infty))$ satisfying

$$\begin{cases} u_- < \phi(x) < 0, \quad \text{and} \quad \phi_x(x) > 0, \quad x > 0, \\ |\phi(x)| \leq C(1+x)^{-1}, \quad x \geq 0. \end{cases}$$

Reformulated Problem

Put $v(x,t) = u(x,t) - \phi(x)$.

By using (4), we have the following reformulated problem;

$$\begin{cases} v_{tt} - v_{xx} + v_t + \{f(\phi + v) - f(\phi)\}_x = 0, & x > 0,\ t > 0, \\ v(0,t) = 0, \quad t > 0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x), & x > 0 \\ v_t(x,0) = v_1(x) := u_1(x), & x > 0, \end{cases} \tag{5}$$

$$z(x,t) := -\int_x^\infty v(y,t)dy, \quad \text{then} \quad (5) \Rightarrow$$

$$\begin{cases} z_{tt} - z_{xx} + \{f(\phi + z_x) - f(\phi)\} + z_t = 0, x > 0,\ t > 0, \\ z_x(0,t) = 0, \quad t > 0, \\ z(x,0) = z_0(x) := -\int_x^\infty u_0(y) - \phi(y)dy, \\ z_t(x,0) = z_1(x) := -\int_x^\infty u_1(y)dy,, \end{cases} \tag{6}$$

Then, the theorem we shall prove is:

Theorem 3.

Suppose the same condition as in Theorem 1. Then, $\exists\epsilon > 0$ s.t. if $\|z_0\|_{H^2} + \|z_1\|_{H^1} \le \epsilon$, the problem (6) has a unique global solution in time u satisfying $z \in C^0([0,\infty); H^2) \cap C^1([0,\infty); H^1)$, and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |z(x,t)| = 0.$$

Proposition.(a priori estimate)

Under the condition of Theorem 1, if z is the solution of the problem (6), then it holds

$$||z||^2_{H^2} + \|z_t\|^2_{H^1} + \int_0^t (||\sqrt{\phi_r} z(s)||^2_{L^2} + ||z_x(s)||^2_{H^1} + \|z_t\|^2_{H^1}) ds$$
$$\le C(||z_0||^2_{H^1} + 1), \qquad t \in [0,T].$$

Outline of proof

$$\int_0^\infty (6) \times w(\phi) \cdot (z + 2z_t) dx \Rightarrow$$

(w: weight function which is defined later.)

$$\frac{d}{dt}\int_0^\infty E dx + \int_0^\infty D dx + F = \int_0^\infty R dx \qquad \cdots \qquad (E)$$

$$E = w(\phi)\left(\frac{1}{2}z^2 + z_t^2 + z_x^2 + z z_x\right),$$

$$D = w(\phi)(z_x^2 + z_t^2) + 2\big(fw\big)'(\phi) z_x z_t - \frac{1}{2}\big(fw\big)''(\phi)\phi_x z^2,$$

$$F = -\frac{1}{2}(fw)'(u_-)\ z^2(0,t),$$

$$R = O(|z| + |z_t|) z_x^2.$$

Lemma (Weight function and Discriminant)

Let $(fw)(u) := -e^{Au} + 1 \ \ for\ u \in [u_-, 0]$.

Then, $\exists A \gg 1$. s.t.

(i) $w(u) > 0$ (ii) $\big((fw)'(u)\big)^2 < w^2(u)$,

(iii) $(fw)''(u) < 0$, (iv) $(fw)'(u_-) < 0$.

$for\ \ u \in [u_-, 0]$.

$$\frac{d}{dt}\int_0^\infty E dx + \int_0^\infty D dx + F = \int_0^\infty R dx \qquad \cdots \qquad (E)$$

$$E = w(\phi)\left(\frac{1}{2}z^2 + z_t^2 + z_x^2 + zz_x\right) \geq c(z^2 + z_x^2 + z_t^2),$$

$$D = w(\phi)(z_x^2 + z_t^2) + 2\big(fw\big)'(\phi)z_x z_t - \frac{1}{2}\big(fw\big)''(\phi)\phi_x z^2,$$

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$$\geq c(z_x^2 + z_t^2 + \phi_x z^2),$$

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$$D = w(\phi)(z_x^2 + z_t^2) + 2\big(fw\big)'(\phi)z_x z_t - \frac{1}{2}\big(fw\big)''(\phi)\phi_x z^2,$$

$$\geq c(z_x^2 + z_t^2 + \phi_x z^2),$$

$$F = -\frac{1}{2}(fw)'(u_-)\ z^2(0,t) \geq cz^2(0,t),$$

$$R = O(|z| + |z_t|)z_x^2.$$

Lemma (Weight function and Discriminant)

Let $(fw)(u) := -e^{Au} + 1 \ \ for\ u \in [u_-, 0]$.

Then, $\exists A \gg 1$. s.t.

(i) $w(u) > 0$ (ii) $\big((fw)'(u)\big)^2 < w^2(u)$,

(iii) $(fw)''(u) < 0$, (iv) $(fw)'(u_-) < 0$.

$for\ \ u \in [u_-, 0]$.

Decay estimate

(Space-time weighted energy method by Kawashima and Matsumura '94)

Let $\gamma, \beta > 0$ and multiply (E) by $(1+t)^{\gamma}(1+x)^{\beta}$ and integrate the resultant equality over $(0,\infty)\times(0,t)$ then we have

$$(1+t)^{\gamma}\|(z,z_t,z_x)(t)\|^2_{L^2_\beta} + \int_0^t (1+\tau)^{\gamma}\big(\|(z_t,z_x,\sqrt{\phi_x}z)(\tau)\|^2_{L^2_\beta} + \beta\|z(\tau)\|^2_{L^2_{\beta-1}}\big)d\tau$$
$$\leq CE_\beta^2 + \gamma C\int_0^t (1+\tau)^{\gamma-1}\|(z,z_t,z_x)(\tau)\|^2_{L^2_\beta}d\tau + \beta C\int_0^t (1+\tau)^{\gamma}\|(z_t,z_x)(\tau)\|^2_{L^2_{\beta-1}}d\tau, \tag{7}$$

Applying the induction argument with respect to γ and β to (7), we have

$$(1+t)^{\gamma}\|(z,z_t,z_x)(t)\|^2_{L^2} + \int_0^t (1+\tau)^{\gamma}\|(z_t,z_x,\sqrt{\phi_x}z)(\tau)\|^2_{L^2}d\tau$$
$$\leq CE^2(1+t)^{\gamma-\alpha}, \tag{8}$$

for $\alpha < \gamma$. Multiply (8) by $(1+t)^{-\gamma}$, we have the desired estimate:

$$\|(z,z_t,z_x)(t)\|^2_{L^2} \leq CE_\alpha^2(1+t)^{-\alpha}.$$

We also estimate the higher order derivative in the same way and can derive the desired estimate. □

Exponential decay estimate

Let α, $\beta > 0$. Multiplying (E) by $e^{\beta t}e^{\alpha x}$ and integrate the resultant equality over $(0,\infty) \times (0,t)$, we obtain

$$
\begin{aligned}
& e^{\beta t}\|(z, z_t, z_x)(t)\|^2_{H^1_{\alpha,exp}} + \alpha \int_0^t e^{\beta t}\|z(\tau)\|^2_{H^1_{\alpha,exp}} d\tau \\
& + \int_0^t e^{\beta t}\left(\|(z_t, z_x)(\tau)\|^2_{H^1_{\alpha,exp}} + \|\sqrt{\phi_x} z(\tau)\|^2_{L^2_{\alpha,exp}} + \|\sqrt{\phi_x} z_x(\tau)\|^2_{L^2_{\alpha,exp}}\right) d\tau \\
& \leq C E^2_{\alpha,exp} + \beta C_0 \int_0^t e^{\beta t}\|z(\tau)\|^2_{H^1_{\alpha,exp}} d\tau + (\alpha + \beta) C_1 \int_0^t e^{\beta t}\|(z_t, z_x)(\tau)\|^2_{H^1_{\alpha,exp}} d\tau,
\end{aligned}
$$

Letting $\alpha > 0$ and $\beta > 0$ suitably small such that $\beta C_0 \leq \alpha$ and $(\alpha+\beta)C_1 \leq 1$, we arrive at the desired estimate. $\square$

Thank you very much for your attention.