Exact solutions to the Riemann problem for compressible isothermal Euler equations for two phase flows with and without phase transition

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Outline

Introduction

- Models of Baer-Nunziato type
  - full Euler system to each phase
  - Zein, Hantke, Warnecke. *Modeling phase transition for compressible two-phase flows applied to metastable liquids*, J. Comput. Phys., 229 (2010), pp. 2964-2998.
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Balance equations

Isothermal Euler equations

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + (\rho v^2 + p)_x &= 0 \end{aligned}$$

Jump conditions across discontinuities

$$\begin{aligned} [\![\rho(v - W)]\!] &= 0 \\ \rho(v - W)[\![v]\!] + [\![p]\!] &= 0 \end{aligned}$$

Mass flux across discontinuities

$$Z = -\rho(v - W)$$

with

$$Z = \begin{cases} Q & \text{shock wave} \\ z & \text{phase boundary} \end{cases} \quad \text{and} \quad W = \begin{cases} S & \text{shock wave} \\ w & \text{phase boundary} \end{cases}$$

Equations of state

Euler equations with phase transition

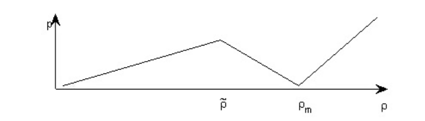
M. Hantke

Ideal gas law

$$p_V = \rho_V \frac{kT_0}{m} \qquad 0 \leq \rho_V \leq \tilde{\rho}$$

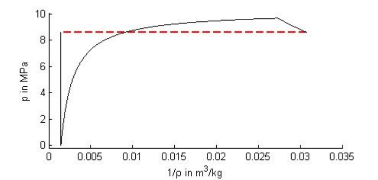
Liquid equation of state

$$p_L = p_0 + K_0 \left(\frac{\rho_L}{\rho_0} - 1 \right) \qquad \rho_L \geq \rho_m$$

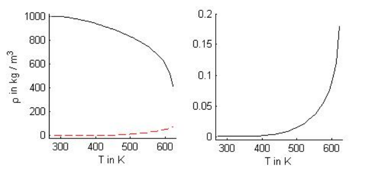


Maxwell condition

$$\int_{1/\rho_0}^{1/\rho_V(p_0)} p(\rho)\, d\frac{1}{\rho} = \left(\frac{1}{\rho_V(p_0)} - \frac{1}{\rho_0}\right)\cdot p_0$$



Equation of state
black: $p(1/\rho)$
for $T_0 = 573.15\,K$
dashed red: Maxwell line



a) red: $\tilde{\rho}(T)$
a) black: $\rho_m(T)$

b) $\tilde{\rho}(T)/\rho_m(T) < \frac{1}{4}$

Kinetic relation

Case 1: trivial case

$$z = 0$$

Case 2

$$\begin{aligned} z &= \frac{p_V}{\sqrt{2\pi}}\left(\frac{m}{kT_0}\right)^{3/2} [\![g + e_{kin}]\!] \\ z &= \frac{p_V}{\sqrt{2\pi}}\left(\frac{m}{kT_0}\right)^{3/2} \left[\frac{K_0}{\rho_0}\ln\frac{\rho_L}{\rho_0} - \frac{kT_0}{m}\ln\frac{p_V}{p_0} \right. \\ &\qquad \left. + \frac{1}{2}(v_L - w)^2 - \frac{1}{2}(v_V - w)^2\right] \end{aligned}$$

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Interface pressure relations I

Case 1: $z = 0$

$$z = 0 \qquad \text{implies} \qquad \begin{array}{c} w = v_V = v_L \\ p_V = p_L \end{array}$$

Case 2

$$[\![p]\!] = -z^2[\![\frac{1}{\rho}]\!]$$

$$z = 0 \iff p_L = p_V \quad (= p_0)$$

$$\text{otherwise} \qquad p_V < p_L$$

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Case 2

$$p_L - p_V = \qquad [\![p]\!] = -z^2 [\![\frac{1}{\rho}]\!] \qquad = -z^2 \left(\frac{1}{\rho_L} - \frac{1}{\rho_V} \right)$$

$$z = 0 \quad \Longleftrightarrow \quad p_L = p_V \quad (= p_0)$$

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Interface pressure relations II

Lemma 1: Uniqueness of liquid interface pressure

Consider the isothermal case with $273.15\,K \leq T_0 \leq 623.15\,K$. Then for given interface pressure p_V^* of the vapor phase with $0 \leq p_V^* \leq \tilde{p}$ the kinetic relation and the corresponding equations of state define the liquid pressure p_L^*, uniquely. Furthermore, by these relations the mass flux z is uniquely defined.

Lemma 2: Monotonicity of liquid interface pressure

For given temperature $273.15\,K \leq T_0 \leq 623.15\,K$ the implicitely defined function $p_L^*(p_V^*)$ is strictly increasing.

Lemma 3: Monotonicity of $z[\![1/\rho]\!]$

For given temperature $273.15\,K \leq T_0 \leq 623.15\,K$ the expression $z[\![\frac{1}{\rho}]\!]$ is strictly increasing in p_V^*, where z depends on the implicitely defined function $p_L^*(p_V^*)$.

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Classical waves

$$\mathbf{A} = \begin{pmatrix} v & \rho \\ \frac{a^2}{\rho} & v \end{pmatrix} \qquad \begin{matrix} \lambda_1 = v - a \\ \lambda_2 = v + a \end{matrix} \qquad \begin{matrix} I_1 = v + a \ln \rho \\ I_2 = v - a \ln \rho \end{matrix}$$

Rarefactions

$$\mathbf{W}_{1fan} = \begin{cases} v = a + \frac{x}{t} \\ \rho = \exp\left(\frac{v' - v}{a} + \ln \rho'\right) \end{cases}$$

$$\mathbf{W}_{2fan} = \begin{cases} v = -a + \frac{x}{t} \\ \rho = \exp\left(\frac{v - v''}{a} + \ln \rho''\right) \end{cases} .$$

Shocks

$$S_1 = v' - \frac{\sqrt{a^2 \rho' \rho''}}{\rho'} \qquad v'' = v' - \frac{a^2(\rho'' - \rho')}{\sqrt{a^2 \rho' \rho''}}$$

$$S_2 = v' + \frac{\sqrt{a^2 \rho' \rho''}}{\rho'} \qquad v'' = v' + \frac{a^2(\rho'' - \rho')}{\sqrt{a^2 \rho' \rho''}}$$

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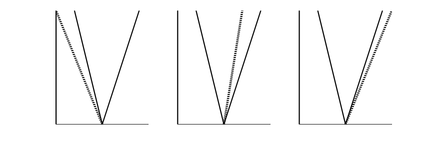
$$S_2 = v' + \frac{\sqrt{a^2\rho'\rho''}}{\rho'} \qquad v'' = v' + \frac{a^2(\rho''-\rho')}{\sqrt{a^2\rho'\rho''}}$$

Wave configuration

Riemann problem

$$\rho(x,0) = \begin{cases} \rho_V & \text{for} \quad x < 0 \\ \rho_L & \text{for} \quad x > 0 \end{cases} \quad \text{and} \quad v(x,0) = \begin{cases} v_V & \text{for} \quad x < 0 \\ v_L & \text{for} \quad x > 0 \,. \end{cases}$$



solid: classical waves, dashed: vapor-liquid phase boundary

Lemma 4

There exists no solution of wave pattern types a) und c), which include the cases of coincidence of the classical waves with the phase boundary.

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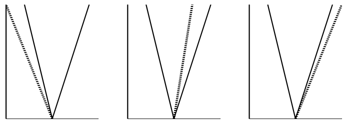

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Solution of isothermal Euler equations, Case 1

THEOREM 1: Let f be given as
$f(p, \mathbf{W}_V, \mathbf{W}_L) = f_V(p, \mathbf{W}_V) + f_L(p, \mathbf{W}_L) + \Delta v$, where the functions f_V and f_L are given by

$$f_V(p, \mathbf{W}_V) = \begin{cases} \frac{p-p_V}{\sqrt{\rho_V p}} & \text{if } p > p_V \text{ (shock)} \\ -a_V \ln p_V + a_V \ln p & \text{if } p \leq p_V \text{ (rarefaction)} \end{cases}$$

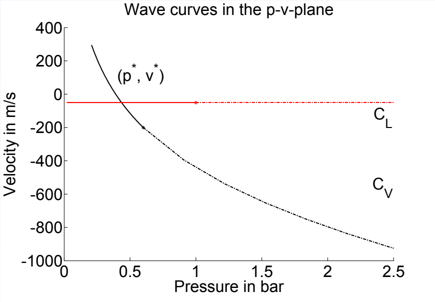
$$f_L(p, \mathbf{W}_L) = \begin{cases} \frac{p-p_L}{\sqrt{K_0 \rho_L \left(\frac{p-p_0}{K_0}+1\right)}} & \text{if } p > p_L \text{ (sh.)} \\ -a_L \ln \frac{\rho_L}{\rho_0} + a_L \ln \left(\frac{p-p_0}{K_0} + 1\right) & \text{if } p \leq p_L \text{ (rf.).} \end{cases}$$

If the function $f(p, \mathbf{W}_V, \mathbf{W}_L)$ has a root p^* with $0 < p^* \leq \tilde{p}$, this root is unique and is the unique solution for pressure p_V^* of the above Riemann problem. The velocity v_V^* can be calculated as follows

$$v_V^* = \frac{1}{2}(v_V + v_L) + \frac{1}{2}\left(f_L(p_*, \mathbf{W}_L) - f_V(p_*, \mathbf{W}_V)\right) .$$

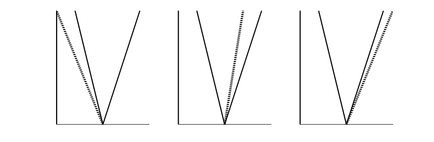
Wave curves

$v_V = -200\frac{m}{s}$	$v_L = -50\frac{m}{s}$	$T_0 = 473.15K$	$K_0 = \frac{10^9}{0.88383}Pa$
$p_V = 60000Pa$	$p_L = 100000Pa$	$\rho_0 = \frac{1000}{1.15651}\frac{kg}{m^3}$	$p_0 = 1554670Pa$



Wagner, Kretzschmar. *International steam tables*, Springer-Verlag, Berlin - Heidelberg, 2008.

Wave configuration



solid: classical waves, dashed: vapor-liquid phase boundary

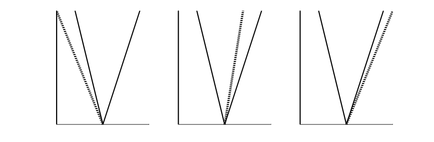
Lemma 5

There exists no solution of wave pattern type a), which include the cases of coincidence of the classical waves with the phase boundary.

Lemma 6

For $p_L \geq p_0$, there exists no solution of wave pattern type c).
For $p_L < p_0$, the condition $p_V^*(p_L) \geq p_0 \exp(-\sqrt{2\pi})$ is sufficient, that there is no solution of type c).

Wave configuration



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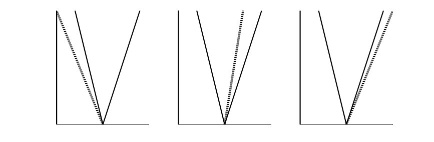

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Solution of isothermal Euler equations, Case 2

THEOREM 2: Let f_z be given as
$f_z(p, \mathbf{W}_V, \mathbf{W}_L) = f_V(p, \mathbf{W}_V) + f_L(p_L^*(p), \mathbf{W}_L) + z[\![\frac{1}{\rho}]\!] + \Delta v$ with

$$f_V(p, \mathbf{W}_V) = \begin{cases} \frac{p - p_V}{\sqrt{\rho_V p}} & \text{if } p > p_V \text{ (shock)} \\ -a_V \ln p_V + a_V \ln p & \text{if } p \le p_V \text{ (rarefaction)} \end{cases}$$

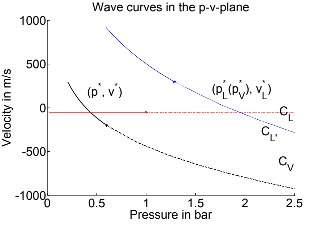
$$f_L(p, \mathbf{W}_L) = \begin{cases} \frac{p_L^*(p) - p_L}{\sqrt{K_0 \rho_L \left(\frac{p_L^*(p) - p_0}{K_0} + 1\right)}} & \text{if } p_L^*(p) > p_L \text{ (sh.)} \\ -a_L \ln \frac{\rho_L}{\rho_0} + a_L \ln\left(\frac{p_L^*(p) - p_0}{K_0} + 1\right) & \text{if } p_L^*(p) \le p_L \text{ (rf.)} . \end{cases}$$

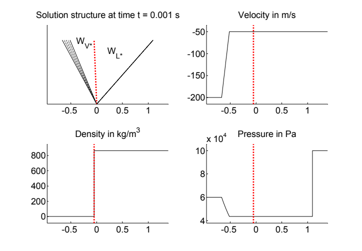
If the function f_z has a root p^* with $0 < p^* \le \tilde{p}$, this root is unique. If further

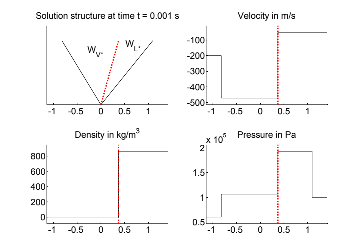
$$p^* > p_V \qquad \text{we must have} \qquad z > -a_V \sqrt{\rho_V \rho_V^*} \,. \qquad (1)$$

In this case the root $p^*$is the unique solution for the pressure p_V^* for a b)-type solution of the above Riemann problem with phase transition and the complete solution is uniquely determined.

Wave curves, complete solution







Solution properties

THEOREM 3: (**Sufficient condition for solvability**) Let us consider the above Riemann problem. If the Riemann problem considered for Case 1 is solvable, then the same Riemann problem is also solvable taking into account phase transition due to the kinetic relation.

THEOREM 4: Let p^* be the solution of the pressure in the star region of the above Riemann problem for Case 1. Then for the solutions p_V^* and $p_L^*(p_V^*)$ of the same Riemann problem for Case 2 we have

1. $p^* = p_0$ implies that $p_V^* = p_L^*(p_V^*) = p_0$.
2. $p^* < p_0$ implies that $p^* < p_L^*(p_V^*) < p_0$.
3. $p^* > p_0$ implies that $p_0 < p_V^* < p^*$.

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Single phase flow, two gases

Riemann problem

$$\rho(x,0) = \begin{cases} \rho_{V-} & \text{for} \quad x < 0 \\ \rho_{V+} & \text{for} \quad x > 0 \end{cases} \quad \text{and} \quad v(x,0) = \begin{cases} v_{V-} & \text{for} \quad x < 0 \\ v_{V+} & \text{for} \quad x > 0\,. \end{cases}$$

THEOREM 5: Let the function f_{VV} be given as

$$f_{VV}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+}) = f_{V-}(p, \mathbf{W}_{V-}) + f_{V+}(p, \mathbf{W}_{V+}) + \Delta v \ .$$

If the function $f_{VV}(p, \mathbf{W}_{V-}, \mathbf{W}_{V+})$ has a root p^* with $0 < p^* \leq \tilde{p}$, this root is unique and is the unique solution for pressure p_V^* of the above Riemann problem. The velocity v_V^* is given by

$$v_V^* = \frac{1}{2}(v_{V-} + v_{V+}) + \frac{1}{2}\left(f_{V+}(p_*, \mathbf{W}_{V+}) - f_{V-}(p_*, \mathbf{W}_{V-})\right) \ .$$

Nucleation criterion, wave configuration

Definition 1

If there is no solution to the above Riemann problem according to Theorem 5, then nucleation occurs.

Lemma 7

If there is a solution of the Riemann problem consisting of two classical waves and two phase boundaries, then no wave is propagating through the liquid. Waves may only occur in the gas.

Nucleation criterion, wave configuration

Euler equations with phase transition

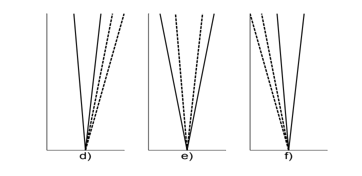
M. Hantke

Definition 1

If there is no solution to the above Riemann problem according to Theorem 5, then nucleation occurs.

Lemma 7

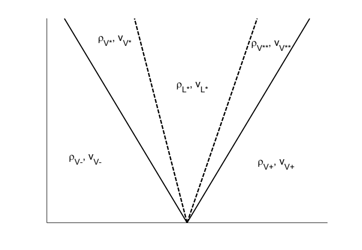
If there is a solution of the Riemann problem consisting of two classical waves and two phase boundaries, then no wave is propagating through the liquid. Waves may only occur in the gas.



Wave configuration

Lemma 8

There are no solutions of wave pattern types d) and f).



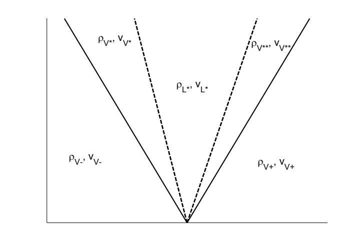
Lemma 9

Assume, there is a solution of wave pattern type e). Then $p_{V*} = p_{V**}$.

Wave configuration

Lemma 8

There are no solutions of wave pattern types d) and f).



Lemma 9

Assume, there is a solution of wave pattern type e). Then $p_{V*} = p_{V**}$.

Nucleation

THEOREM 6: Consider the above Riemann problem and assume the nucleation criterion is satisfied. Let f_{VVz} be given as

$$f_{VVz}(p,\mathbf{W}_{V-},\mathbf{W}_{V+}) = f_{V-}(p,\mathbf{W}_{V-}) + f_{V+}(p,\mathbf{W}_{V+}) + 2z[\![\frac{1}{\rho}]\!] + \Delta v = 0 \ .$$

Here z is given by the kinetic relation and $[\![\frac{1}{\rho}]\!] = \frac{1}{\rho_{L*}} - \frac{1}{\rho_{V*}}$. The function $p_L^*(p)$ is implicitely defined.
If the function f_{VVz} has a root with $p_0 < p \le \tilde{p}$, then this root is the only one. Furthermore, this root is the unique solution for pressure $p_{V*} = p_{V**}$ of the Riemann problem for the vapor pressure in the star regions. The liquid velocity v_{L*} can be calculated by

$$v_{L*} = \frac{1}{2}(v_{V-} + v_{V+}) + \frac{1}{2}\left(f_{V+}(p_*) - f_{V-}(p_*)\right) \ .$$

Single phase flow, two liquids

Riemann problem

$$\rho(x,0) = \begin{cases} \rho_{L-} & \text{for} \quad x<0 \\ \rho_{L+} & \text{for} \quad x>0 \end{cases} \quad \text{and} \quad v(x,0) = \begin{cases} v_{L-} & \text{for} \quad x<0 \\ v_{L+} & \text{for} \quad x>0\,, \end{cases}$$

THEOREM 7: Let f_{LL} be given as

$$f_{LL}(p,\mathbf{W}_{L-},\mathbf{W}_{L+}) = f_{L-}(p,\mathbf{W}_{L-}) + f_{L+}(p,\mathbf{W}_{L+}) + \Delta v\ .$$

If the function $f_{LL}(p,\mathbf{W}_{L-},\mathbf{W}_{L+})$ has a root p^* with $p_{\min} \leq p^*$, this root is unique and is the unique solution for pressure p_L^* of the above Riemann problem. The velocity v_L^* is calculated from

$$v_L^* = \frac{1}{2}(v_{L-} + v_{L+}) + \frac{1}{2}\left(f_{L+}(p_*) - f_{L-}(p_*)\right)\ .$$

Cavitation criterion, cavitation

Definition 2

If there is no solution to the above Riemann problem according to Theorem 7, then cavitation occurs.

THEOREM 8: Consider the above Riemann problem and assume the cavitation criterion is satisfied. Let f_{LLz} be given as

$$f_{LLz}(p, \mathbf{W}_{L-}, \mathbf{W}_{L+}) = f_{L-}(p_L(p), \mathbf{W}_{L-}) + f_{L+}(p_L(p), \mathbf{W}_{L+}) + 2z[\![\frac{1}{\rho}]\!] + \Delta v = 0 \ .$$

Here z is calculated from the kinetic relation and $[\![\frac{1}{\rho}]\!] = \frac{1}{\rho_{L*}} - \frac{1}{\rho_{V**}}$. The function $p_L^*(p)$ is implicitely defined.
If the function f_{LLz} has a root with $p_{\min} \leq p$, then this root is unique. Further, this root uniquely determines the pressure p_V^* of the Riemann problem for the vapor pressure in the star region. Further, the vapor velocity v_{V*} is given by

$$v_{V*} = \frac{1}{2}(v_{L-} + v_{L+}) + \frac{1}{2}\left(f_{L+}(p_L^*(p_*)) - f_{L-}(p_L^*(p_*))\right) \ .$$

Cavitation criterion, cavitation

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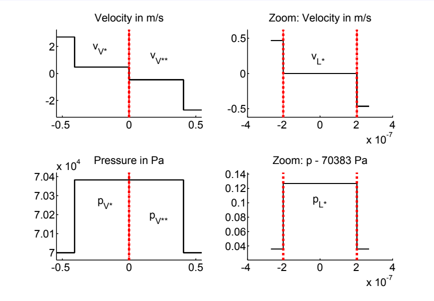
Here z is calculated from the kinetic relation and $[\![\frac{1}{\rho}]\!] = \frac{1}{\rho_{L*}} - \frac{1}{\rho_{V**}}$. The function $p_L^*(p)$ is implicitely defined.
If the function f_{LLz} has a root with $p_{\min} \le p$, then this root is unique. Further, this root uniquely determines the pressure p_V^* of the Riemann problem for the vapor pressure in the star region. Further, the vapor velocity v_{V*} is given by

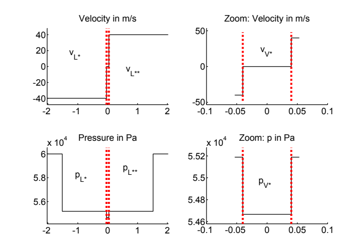
$$v_{V*} = \frac{1}{2}(v_{L-} + v_{L+}) + \frac{1}{2}\left(f_{L+}(p_L^*(p_*)) - f_{L-}(p_L^*(p_*))\right) \; .$$

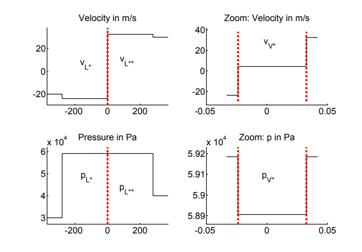
Solvability, example

THEOREM 9: Consider the above Riemann problem and assume the cavitation criterion is satisfied. If we admit phase transition, this problem is always solvable.



Further examples

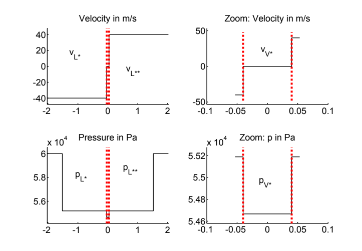


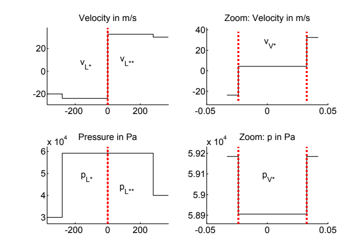


Future work

- take into account temperature
- numerical solver

Further examples





Future work

- take into account temperature
- numerical solver