The classical Stefan problem: well-posedness theory and asymptotic stability

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Formulation of the problem

Well-posedness framework

Asymptotic stability

Stefan problem

- Stefan problem is one of the best known parabolic two-phase free boundary problems. It is a simple model of phase transitions in liquid-solid systems.
- The unknowns: the temperature $p : \Omega(t) \to \mathbb{R}$ and the boundary $\Gamma(t) = \partial\Omega(t)$.
- The temperature diffuses inside the phase $\Omega(t)$, while the temperature flux at the interface moves the boundary.

Classical Stefan problem

$$p_t - \Delta p = 0, \quad \text{in} \quad \Omega(t) \subset \mathbb{R}^n$$

$$p = 0 \quad \text{on} \quad \Gamma(t).$$

$$\nabla p \cdot n = V_\Gamma \quad \text{on} \quad \Gamma(t).$$

- V_Γ is the normal velocity of $\Gamma(t)$, locally $V_\Gamma = \frac{h_t}{\sqrt{1+|\nabla h|^2}}$.
- It is a *macro-scale model formalizing the intuition that liquid freezes at a constant temperature.*

If the condition $p=0$ on $\Gamma(t)$ is replaced by

$$p=\sigma\kappa_{\Gamma(t)} \quad \text{on} \quad \Gamma(t),$$

we call it the surface tension or the Gibbs-Thomson correction and $\sigma>0$ is the surface tension coefficient. $\kappa_{\Gamma(t)}$ is the mean curvature of $\Gamma(t)$. This model describes a phase transition on a micro-scale.

The natural dissipation law:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} p(t,x)^2\,dx + \int_{\Omega(t)} |\nabla p(t,x)|^2\,dx = 0.$$

What happens if $\sigma > 0$?

If we add surface tension:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} p(t,x)^2\,dx + \int_{\Omega(t)} |\nabla p(t,x)|^2\,dx + \sigma\frac{d}{dt}|\Gamma(t)| = 0.$$

We may thus think of surface tension as a stabilizing effect. However, the limit $\sigma \to 0$ is singular and has to be a-priori justified.

Steady states when $\sigma = 0$

There are infinitely many steady states. Any pair

$$(p, \Gamma) \equiv (0, \bar{\Gamma})$$

for some smooth C^1-hypersurface $\bar{\Gamma}$ is a steady state! This indicates a degeneracy and the asymptotic stability problem has to be formulated with *care*.

Weak Solution Theories

- $\sigma = 0$ - Classical Stefan Problem: Friedman, Kinderlehrer, Caffarelli, Evans, Stampacchia, Athanasopoulos, Salsa,... Kamenomostskaya, Ladyzhenskaya, Uralceva,...
  A common theme: *the maximum principle, viscosity solutions*.
- $\sigma > 0$ - Stefan problem with surface tension: Luckhaus, Almgren & Wang.
  The gradient flow structure of the problem is used. There is NO uniqueness.

$\sigma \to 0$?

- In particular, the results of Luckhaus, Almgren & Wang do not allow enough compactness in σ to pass to a limit. (At least not one with a sharp interface...)
- The limit is SINGULAR as it links two models that are valid on DIFFERENT spatial scales. It is a-priori not clear whether the surface tension actually “stabilizes" the problem.

Classical Solution Theories

- $\sigma = 0$ - Classical Stefan Problem: Meirmanov, Hanzawa: huge loss of derivatives...
  Prüss-Saal-Simonett, Frolova-Solonnikov: L^p-type spaces, $p > n+2$.
- $\sigma > 0$ - Stefan problem with surface tension: Radkevich, Escher-Prüss-Simonett, H.-Guo, H.,Prüss-Simonett-Zacher: all estimates depend on σ and degenerate as $\sigma \to 0$.

Goals and results

- Understand how the regularity of the *free boundary "communicates" to the temperature*.
  To do so, we shall unravel a new structure in the classical Stefan problem that exhibits a formal analogy to the free surface Euler equation.
- Two consequences: well-posedness theory in Sobolev L^2-type spaces for all $\sigma \geq 0$; establish the vanishing surface tension limit (i.e. $\sigma \to 0$).
- Asymptotic stability close to circular steady states: combine the energy method and Harnack-type inequalities.

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Idea 1: Work in the natural energy space.

Recall: the basic energy dissipation law reads:

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Main Obstacle: no control over the moving boundary.

- Note that the boundary information is implicit in the domain of integration.
- It is a-priori not clear how to deal with this obstacle in the "Eulerian" framework.

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Idea 2: Arbitrary Lagrange-Eulerian coordinates.

Let $n=2$ and let us focus on a simple geometric situation:

$$\Omega := \mathbb{T}^1 \times [0,1], \quad \Gamma = \mathbb{T}^1 \times \{x^2 = 0\}.$$

Let $\Gamma(t)$ be parametrized as a graph over $\Gamma(0) = \Gamma$:

$$\Gamma(t) = \{(x', x^2) |\ x^2 = h(t, x'),\ x' \in \mathbb{T}^1\}.$$

ALE-map Ψ

Seek $\Psi : \Omega \to \Omega(t)$ so that

- $$\Psi(\Gamma) = \Gamma(t), \quad \text{i.e. } \Psi(x', 0) = (x', h(t, x')).$$

  Recall $\Gamma = \Gamma(0) = \mathbb{T}^1$.
- most importantly:

$$\|\Psi\|_{H^s(\Omega)} \lesssim \|\Psi\|_{H^{s-0.5}(\Gamma)}, \quad s > 0.5.$$

The second property can be achieved by *constructing* the map (the gauge) Ψ as an elliptic extension of its boundary data.

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Next step: pull-back the Stefan problem onto domain Ω via the ALE-map Ψ. Define

$$q = p \circ \Psi; \quad A := [D\Psi]^{-1}; \quad v = -\nabla p \circ \Psi.$$

Note that v satisfies the Euler-like equation:

$$v^i + A_i^k \partial_k q = 0, \quad i = 1, 2.$$

Also

$$\Delta \to \Delta_\Psi = A_i^k \partial_k (A_i^j \partial_j).$$

Stefan problem in ALE-coordinates

$$v^i + A_i^k \partial_k q = 0 \text{ in } \Omega; \quad \text{(“fluid velocity" equation)}$$

$$q_t - \Delta_\Psi q = -v \cdot \Psi_t \text{ in } \Omega; \quad \text{(heat equation)}$$

$$q = 0 \text{ on } \Gamma; \quad \text{(classical Stefan condition)}$$

$$\Psi_t \cdot n = v \cdot n \text{ on } \Gamma. \quad \text{(evolution of the boundary)}$$

Basic calculation

Let $\partial = \partial_{x^1}$, i.e. the tangential derivative. Apply ∂ to the v-equation and multiply by ∂v^i:

$$\begin{aligned} 0 &= \left(\partial v^i + \partial A_i^k \partial_k q + A_i^k \partial \partial_k q, \partial v^i\right)_{L^2} \\ &= \|\partial v\|_{L^2(\Omega)}^2 + \int_\Omega \partial(A_i^k)\partial_k q \partial v^i \, dx + \left(A_i^k \partial \partial_k q, \partial v^i\right)_{L^2}. \end{aligned}$$

Remember: $A = [D\Psi]^{-1}$. Thus

$$\partial(A_i^k) = -A_r^k \partial \partial_s \Psi^r A_i^s.$$

Thus the second integral reads

$$\int_\Omega \partial(A_i^k)\partial_k q \partial v^i \, dx = -\int_\Omega A_r^k \partial \partial_s \Psi^r A_i^s \partial_k q \partial v^i \, dx$$

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$$0 = \left(\partial v^i + \partial A_i^k \partial_k q + A_i^k \partial\partial_k q, \partial v^i\right)_{L^2}$$
$$= \|\partial v\|^2_{L^2(\Omega)} + \int_\Omega \partial(A_i^k)\partial_k q \partial v^i \, dx + \left(A_i^k \partial\partial_k q, \partial v^i\right)_{L^2}.$$

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Integration by parts with respect to x_s finally gives:

$$\int_\Omega \partial(A_i^k)\partial_k q \partial v^i \, dx = \frac{1}{2}\frac{d}{dt}\int_\Gamma (\partial_2 q)|\partial h|^2 \, dx' + \text{error terms} .$$

- This calculation is inspired by the work of Coutand-Shkoller on the free surface Euler equation without surface tension.

The calculation from the previous slide is “stable" under differentiation. In other words, we systematically differentiate with respect to space and time directions, to obtain high-order Sobolev-type energy spaces.

Energy

$$\mathcal{E}(q,h) \approx \|q\|_{L_t^\infty H_x^4} + \|q\|_{L_t^2 H_x^{4.5}}$$
$$+ \int_\Gamma (\partial_2 q)|\partial^4 h|^2\, dx' + \int_0^t \int_\Gamma (\partial_2 q)|\partial^3 h_t|^2\, dx'.$$

- The “half-a-derivative loss" in the regularity scale above is in fact optimal.

Well-posedness

Theorem (H., Shkoller)

Let (q_0, h_0) *be a set of initial data such that* $\mathcal{E}(q_0, h_0) < \infty$ *and the* Taylor sign condition *holds:*

$$\partial_2 q_0 > 0.$$

Then there exists a $T > 0$ *and a unique solution* $(q(t), h(t))$ *to the classical Stefan problem* ($\sigma = 0$) *on the time interval* $[0, T]$*. Moreover,*

$$\sup_{0 \le t \le T} \mathcal{E}(q(t), h(t)) \lesssim \mathcal{E}(q_0, h_0).$$

Proof

We prove:

$$\mathcal{E}(t) \leq M_0 + CtP(\mathcal{E}(t)).$$

A simple continuity argument shows that there exists a $T > 0$ so that

$$\mathcal{E}(t) \leq 2M_0 \quad 0 \leq t \leq T.$$

Global-in-time result: challenges

- Due to the presence of infinitely many steady states, it is hard to decide where the solution should converge to asymptotically.
- However, we expect the temperature q to decay exponentially fast, since it solves a parabolic problem.
- PROBLEM: the weight $(-\partial_n q)$ in the energy expression

$$\int_\Gamma (-\partial_n q)|\partial h|^2 \, dx'$$

must go to zero as well!
Can we control the derivatives of h?

Lower decay bound on $(-\partial_n q)$??

- Clearly, we need to obtain quantitative lower decay rates for the weight $-\partial_n q$.
- Such a quantitative version of Harnack's inequality was proven by Oddson in 1970's, using the maximum principle, theory of extremal Pucci operators and comparison principle with suitable Bessel functions.

Define

$$\chi(t) := \inf_{x \in \Gamma} \sqrt{(-\partial_n q)(t, x)}$$

Theorem (H.,Shkoller)

[Global stability] Let Γ_0 be close to a circular initial shape and let $q_0(x) > 0$ for all $x \in \Omega$. Then there exists an $\epsilon > 0$ such that if

$$\mathcal{E}(q_0, h_0) < \epsilon$$

then the solution to the classical Stefan problem exists for all times and there is a $\beta > 0$ such that

$$\sup_{0 \le s \le \infty} \mathcal{E}(q(s), h(s)) < 2\epsilon, \qquad \|q(t)\|^2_{H^3} \lesssim e^{-\beta t}, \ t > 0.$$

Bootstrap assumptions

Introduce the decay space for q:

$$E_\beta(t) := e^{\beta t} \|q\|^2_{H^3}.$$

β will be chosen to be $\beta = 3\nu_0/2$, where ν_0 is the smallest eigenvalue of Laplacian on a unit disk.
Assume finally that on a sufficiently large time interval $[0, T]$:

$$\sup_{0 \le s \le T} (\mathcal{E}(s) + E_\beta(s)) \le \epsilon.$$

Proof idea

To run the bootstrap argument, we split the dynamics into three coupled regimes:

- High-order energy. No decay expected:

$$\partial_t \mathcal{E}_1 + \mathcal{E}_2 \leq p(t)\epsilon \frac{e^{-\beta t}}{\chi(t)} \mathcal{E}_1 + \epsilon \mathcal{E}_2.$$

- “Heat regime"

$$\frac{1}{2}\partial_t \|q\|_{H^3}^2 + \|q\|_{H^4}^2 \lesssim \left(\epsilon + \frac{e^{-\beta t}}{\chi(t)}\right)\|q\|_{H^4}^2$$

- Oddson bound

$$\chi(t) = \inf_{x\in\Gamma} \sqrt{-\partial_n q(t,x)} \gtrsim e^{-\nu_0 t + O(\epsilon)t}$$

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Note however, under the bootstrap assumption

$$\mathcal{E} + E_\beta \lesssim \epsilon$$

the heat regime implies the *near optimal* decay for $\|q\|_{H^3}$:

$$\|q\|_{H^3}^2 \lesssim e^{-2\nu_0 t + O(\epsilon) t}$$

Thus, if $\beta = \frac{3\nu_0}{2}$

$$\frac{e^{-\beta t}}{\chi(t)} = \frac{e^{-\frac{3}{2}\nu_0 t + O(\epsilon) t}}{e^{-\nu_0 t - O(\epsilon t)}} \lesssim e^{-\gamma t}$$

where $\gamma = \nu_0/2 - O(\epsilon) > 0$!

Vanishing surface tension limit

Theorem (H., Shkoller)

Let $(p_0^\sigma, \Gamma_0^\sigma)$ *be a sequence of initial data satisfying*

1. $$(p_0^\sigma, \Gamma_0^\sigma) \to (p_0, \Gamma_0) \quad \textit{as } \sigma \to 0 \textit{ in energy norm},$$
2. $$\nabla p_0^\sigma \cdot n < -\delta < 0, \quad \sigma \geq 0.$$

Then, on a σ*-independent time interval* $[0, \mathcal{T}]$

$$(p^\sigma(t), \Gamma^\sigma(t)) \to (p(t), \Gamma(t)) \quad \textit{in } C^{1,2}.$$

Proof

In the presence of surface tension

$$\mathcal{E}^{\sigma}(t) = \mathcal{E}(t) + \sigma \|h\|_{H^5}^2.$$

We prove

$$\mathcal{E}^{\sigma} \le M_0^{\sigma} + CtP(\mathcal{E})\mathcal{E}^{\sigma}.$$

The key is: the highest order derivatives of h enter only *linearly* into the energy estimate. The uniform estimate follows:

$$\mathcal{E}^{\sigma} \le 2M_0^{\sigma} \le 2M_0 + 1$$

over a σ-independent time interval $[0, \mathcal{T}]$.

Construction of the solution...

- In many free boundary problems, the step from a given a-priori estimate to the construction of a solution is not straightforward.
- We rely on a method introduced by Coutand & Shkoller in the well-posedness treatment of free-surface Euler equations: *Horizontal convolution by layers*.
- In a suitable coordinate localization of the free-boundary, one convolves the unknown h with a standard mollifier, but *only in tangential directions*.
- This kind of construction scheme respects the non-linear energy structure of the problem.

Further comments

- The results apply to the two-phase Stefan problem (work in preparation).
- Similar techniques can be applied to two-phase Muskat and Hele-Shaw problem (note the difference to the two-phase Stefan problem!)

Thank you for your attention!