Complete synchronization of particle and kinetic Kuramoto models on networks

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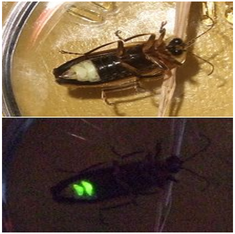
Outline

Short tour of Kuramoto's mean-field theory

Synchronization

- Fireflies(lightening bugs)



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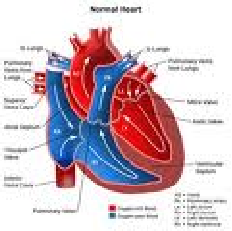

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Synchronization

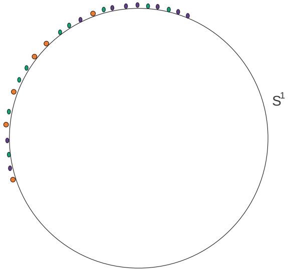
"Synchronization (=syn (same, common) + chronous (time))" is an adjustment of rhythms of oscillating objects due to their weak interaction.

◇ Examples:

- Flashing of fireflies in South-East Asia
- Firing of coupled cardiac pacemaker cells (heart's contraction)
- Synchronous firing of many neurons (Parkinson's disease)
- Hands clapping in a concert

How to model the synchronized dynamics ?

Consider an ensemble of rotors moving along S^1 with natural frequency Ω_i which is randomly drawn from some probability distribution with a density $g(\Omega)$.



- Dynamics of x_k: (Phase dynamics)

$$x_k \in S^1 (\subset C): \text{ position of } k\text{-th oscillator} = e^{i\theta_k}, \quad \theta_k \in R.$$

State of system is determined by the dynamics of θ_k.

In the absence of interactions(couplings), we have

$$\dot{\theta}_k = \Omega_k, \quad \text{i.e.} \quad \theta_k(t) = \theta_{k0} + \Omega_k t.$$

Then a natural question is

How to model "interactions" ?

The Kuramoto model (1975)

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

The Kuramoto model is a prototype for the **synchronization**

Kuramoto's mean-field analysis

$$\text{KM}: \quad \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

Define order parameters r and ϕ:

$$re^{i\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \quad r \in [0, 1].$$

This yields

$$re^{i(\phi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}, \quad r\sin(\phi - \theta_i) = \frac{1}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i).$$

$$\text{KM} \quad \Leftrightarrow \quad \dot{\theta}_i = \Omega_i + Kr\sin(\phi - \theta_i).$$

WLOG, we can assume $\phi = 0$, i.e.,

$$\dot{\theta}_i = \Omega_i - Kr\sin(\theta_i).$$

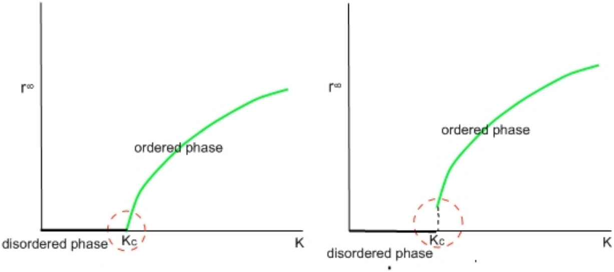
If $|\Omega_i| > Kr$, then i-th oscillator will drift over the circle.

If $|\Omega_i| \leq Kr$, then i-th oscillator may approach to some equilibrium state.

- Mean-field limit:

$$r^\infty(K) := \lim_{N\to\infty} \lim_{t\to\infty} r^N(K,t).$$

- Phase transition



cf. 1 Acebron et al. The Kuramoto model: a simple paradigm for synchronization phenomena, Rev. Modern Phys. 77 (2005) 137-185.
2. Jian-Guo Liu's talk in this meeting

$$
\begin{aligned}
&\partial_t f + \partial_\theta(\omega[f]f) = 0,\\
&\omega[f](x,\Omega,t) := \Omega - K\int_0^{2\pi}\int_R \sin(\theta_* - \theta) f(\theta_*,\Omega_*,t) g(\Omega_*) d\Omega_* d\theta.
\end{aligned}
$$

What I want to address today:

Complete synchronization

In Steve Strogatz's survey paper (2000)

"From Kuramoto to Crawford exploring the onset of synchronization in populations of coupled oscillators, Phys. D 143, 1-20 (2000)."

"In the last of her three Bowen lectures at Berkeley in 1986, Kopell pointed out that Kuramoto's argument contained a few intuitive leaps that were far from obvious. In fact, they began to seem paradoxical the more one thought about them, and she wondered whether one could prove some theorems that would put the analysis on firmer footing. In particular, she wanted to redo the analysis rigorously for large but finite N, and then prove a convergence result as $N \to \infty$. But it would not be easy. Whereas Kuramoto's approach had relied on the assumption that r was strictly constant, Kopell emphasized that nothing like that could be strictly true for any finite N".

- Questions: In this talk, I would like to discuss:

1. Complete synchronization of the particle KM with $N < \infty$ for large coupling regime $K \gg 1$.

2. Long-time dynamics of the kinetic KM.

Complete synchronization of Particle KM

The complete synchronization problem

Consider the Kuramoto model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N,$$

subject to initial data

$$\theta_i(0) = \theta_{i0}.$$

where $\Omega_i, K > 0, N$ are given constants satisfying

$$\sum_{i=1}^{N}\Omega_i = 0.$$

- Static questions:

$$\Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i) = 0, \quad \sum_{i=1}^{N}\Omega_i = 0, \quad i = 1, \cdots, N.$$

1. Are there solutions for the above system ? **(Existence)**
2. If yes, how do they look like ? **(Structure)**

- Dynamic questions:

1. Are phase-locked states stable ? **(Stability)**
2. If phase-locked states can emerge from initial configurations, how does the relaxation process look like ? **(Relaxation)**

cf. Mirollo-Strogatz('05, '07), Aeyels-Rogge '04, De Smet-Aeyels '07, $\cdots$

- **Simple observation**:

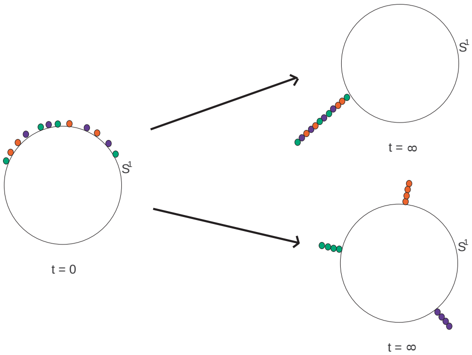
1. If $|\Omega_i| > K$, then there are no steady solutions at all.
2. One solution generates a one-parameter family of solutions:

$$\theta = (\theta_1, \cdots, \theta_N) : \text{ solution } \implies$$
$$\theta + 2\pi\alpha = (\theta_1 + 2\pi\alpha_1, \cdots, \theta_N + 2\pi\alpha_N), \ \alpha \in Z^N : \text{ solution.}$$

Definition: The Kuramoto system $\mathcal{P}$ has asymptotic complete synchronization if and only if the following condition holds.

$$\lim_{t\to\infty} |\dot\theta_i(t) - \dot\theta_j(t)| = 0, \qquad \forall \ i \neq j.$$

- Formation of asymptotic synchronization



Problem and Strategy

- **Problem**:

Find conditions for initial configurations and parameters leading to complete synchronization.

- **Strategy**:

1. Consider the functionals: diameters of the phase and frequency configurations:

$$D(\theta(t)) := \max_{1\le i,j\le N} |\theta_i(t)-\theta_j(t)|,$$
$$D(\omega(t)) := \max_{1\le i,j\le N} |\omega_i(t)-\omega_j(t)|.$$

2. Derive Gronwall's inequalities for $D(\theta)$ and $D(\omega)$.

3. Finally we show

$$\lim_{t\to\infty} D(\theta(t)) = 0, \qquad \lim_{t\to\infty} D(\omega(t)) = 0.$$

• **Theorem** (Formation of phase-locked states): Choi-H-Jung-Kim '11

Suppose initial data, natural frequencies and coupling strength satisfy

$$0 < D(\theta_0) < \pi, \quad D(\Omega) > 0, \quad K > K_e := \frac{D(\Omega)}{\sin D_0}.$$

Then there exists t_0 such that

$$D(\omega(t_0))e^{-K(t-t_0)} \leq D(\omega(t)) \leq D(\omega(t_0))e^{-K(\cos D_0^*)(t-t_0)}, \quad t \geq t_0.$$

cf. 1. Chopra-Spong '09, Ha-Lattanzio-Rubino-Slemrod '10, H-Ha-Kim '10

2. Dong-Xue ('12): $0 < D(\theta_0) < 2\pi$ for identical oscillators

Orbital stability in ℓ_1

• **Theorem** Let θ and $\tilde{\theta}$ be the global smooth solution to the Kuramoto model with initial data θ_0 and $\tilde{\theta}_0$, respectively satisfying

$$0 < \mathcal{D}(\tilde{\theta}_0) \leq \mathcal{D}(\theta_0) < \pi \quad \text{and} \quad K > \frac{\mathcal{D}(\Omega)}{\sin \mathcal{D}_0}.$$

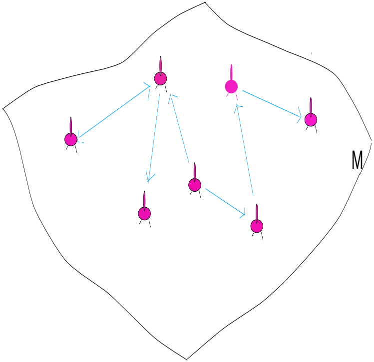
Then if $\theta_c(0) = \tilde{\theta}_c(0)$, then we have

$$||(\theta - \tilde{\theta})(t_0)||_1 e^{-K(t-t_0)} \leq ||(\theta - \tilde{\theta})(t)||_1 \leq ||(\theta - \tilde{\theta})(t_0)||_1 e^{-\frac{K \sin 2\mathcal{D}^\infty}{2\mathcal{D}^\infty}(t-t_0)}.$$

Original Kuramoto model with all-to-all couplings can be understood as a synchronization process on the complete graph.

Network interpretation for complex system

A complex system can be interpreted as a network with suitable topology (network structure)



Node(vertex) = agent, particle,   edge = capacity of interaction.

The complete synchronization problem on two types of networks

- Symmetric network
- Network with hierarchical leadership structure

Joint work with Zhuchun Li (SNU) and Xiaoming Xue (Harbin Institute of Technology)

A network with symmetric couplings

$$\dot{\theta}_i = \Omega_i + K \sum_{i=1}^{N} \psi_{ji} \sin(\theta_j - \theta_i), \quad \psi_{ji} = \psi_{ij}.$$

WLOG, we may assume

$$\sum_{i=1}^{N} \Omega_i = 0, \quad \sum_{i=1}^{N} \theta_i = 0.$$

cf. KM : $\psi_{ij} = \frac{1}{N}$

• Difficulty: some kind of "*completeness property*" lacks in the symmetric networks: For each $i = 1, \cdots, N$,

$$\sum_{j\in\Lambda_i} \psi_{ji}\theta_j = 0, \quad ?? \qquad \Lambda_i := \{j : 1 \le j \le N, \ \psi_{ji} > 0\}.$$

Thus the previous approach involving with $D(\theta)$ and $D(\omega)$ does not seem to work.

• Remedy: We instead use "ℓ_2-energy method" and gradient structure of KM flow.

$$\mathcal{E}(\theta) := \sum_{i=1}^{N} |\theta_i|^2.$$

KM as a gradient flow

ψ : symmetric coupling $\Longleftrightarrow$

$$\partial_t\theta = -\nabla f(\theta), \quad \theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N,$$

where the analytic potential f is given

$$f(\theta) = -\sum_{k=1}^{N} \Omega_k\theta_k + \frac{K}{2}\sum_{k,l=1}^{N} \psi_{kl}(1 - \cos(\theta_k - \theta_l)) \ : \ \text{analytic.}$$

- **Theorem**: Dong-Xue '12, H-Li-Xue'12

Bounded fluctuations $\Longleftrightarrow$ Asymptotic complete synchronization.

Proof. Direct application of Lojasiewicz's gradient inequality for a gradient system with analytical potential

- **Lemma**:
Suppose that the network $(V(G), E(G))$ is connected and $\{z_i\}$ has a mean zero:

$$\sum_{i=1}^{N} z_i = 0.$$

Then we have

$$2L_* N\mathcal{E}(z) \le \sum_{i,j \in E(G)} |z_i - z_j|^2 \le 2N\mathcal{E}(z), \quad t \ge 0,$$

where the constant L_* is given by

$$\mathcal{E}(z) := \sum_{i=1}^{N} |z_i|^2, \qquad L_* := \frac{1}{1 + \mathrm{diam}(G)|E^c|}.$$

- **Lemma**:

For $T \in [0, \infty]$, let $\theta = \theta(t)$ be the smooth solutions in the time-interval $[0, T)$ to KM satisfying a priori boundedness condition:

$$\max_{1 \le i,j \le N} \max_{t \in [0,T]} |\theta_i(t) - \theta_j(t)| \le D_0 < \pi.$$

Then the energy $\mathcal{E}(\theta)$ satisfies

$$\frac{d\mathcal{E}(\theta)}{dt} \le 2\sigma(\Omega)\sqrt{\mathcal{E}(\theta)} - L_* K N \psi_m \Big(\frac{\sin D_0}{D_0}\Big)\mathcal{E}(\theta), \quad \text{on } [0, T].$$

- **Theorem**: H-Li-Xue'12

Let $D_0 \in (0, \pi)$, and suppose that the coupling strength and initial data satisfy

$$K > \frac{\sqrt{2}\sigma(\Omega)}{L_* N \psi_m \sin D_0} \quad \text{and} \quad \mathcal{E}(\theta_0) < \frac{{D_0}^2}{2}.$$

Then the complete synchronization occurs asymptotically.

$$D(\theta(t)) < D_0, \quad \lim_{t \to \infty} |\dot{\theta}_i(t)| = 0, \quad 1 \le i, j \le N.$$

- Proof. We set $y(t) := \sqrt{\mathcal{E}(\theta(t))}$ and

$$\dot{y} \le \sigma(\Omega) - L_* K N \psi_m \Big(\frac{\sin D_0}{D_0} \Big) y.$$

- **Corollary**:
Let $D_0 \in (0, \frac{\pi}{2})$, and suppose that the coupling strength and initial data satisfy

$$K > \frac{\sqrt{2}\sigma(\Omega)}{L_* N \psi_m \sin D_0} \quad \text{and} \quad \mathcal{E}(\theta_0) < \frac{{D_0}^2}{2}.$$

Then the exponential complete synchronization occurs asymptotically and

$$|\dot{\theta}_i(t)| \le \sqrt{\mathcal{E}(\omega_0)} \exp\Big[-(2K\psi_m L_* \cos D_0) t \Big].$$

Sketch of proof

Note that

$$D(\theta(t)) < D_0 < \frac{\pi}{2},$$

and the frequency set $\{\omega_i\}$ has a mean zero:

$$\sum_{i=1}^{N} \omega_i = \sum_{i=1}^{N} \Omega_i + K \sum_{i,j} \psi_{ji} \sin(\theta_j - \theta_i) = \sum_{i=1}^{N} \Omega_i = 0.$$

From KM,

$$\dot{\omega}_i = K \sum_{j=1}^{N} \psi_{ji} \cos(\theta_j - \theta_i)(\omega_j - \omega_i).$$

Energy method implies

$$
\begin{aligned}
\frac{d\mathcal{E}(\omega)}{dt} &= -\frac{K}{N}\sum_{i,j=1}^{N} \psi_{ji}\cos(\theta_j-\theta_i)|\omega_j-\omega_i|^2 \\
&\le -\frac{K\psi_m\cos D_0}{N}\sum_{(i,j)\in E(G)} |\omega_j-\omega_i|^2 \\
&= -\frac{2K\psi_m\cos D_0}{1+\mathrm{diam}(G)|E^c|}\mathcal{E}(\omega).
\end{aligned}
$$

A network with hierarchical leadership

$$\dot{\theta}_i = \Omega_i + K \sum_{i=1}^{N} \psi_{ji} \sin(\theta_j - \theta_i).$$

The KM has an "*HL*" structure iff the matrix ψ satisfies:

1. Followers are influenced only by leaders, i.e.,

$$\psi_{ji} > 0 \quad \Longrightarrow \quad j < i.$$

2. The leader set of the i-th oscillator $\mathcal{L}(i) := \{j \;:\; 1 \le j \le i,\ \psi_{ji} > 0\}$ is not empty for all $i > 1$.

- Note: the leader rotates with a constant speed:

$$\dot{\theta}_1 = \Omega_1, \quad \theta_1(t) = \theta_{10} + \Omega_1 t.$$

- Difficulty: No conservation law

$$\sum_{i=1}^{N} \theta_i \text{ is not conserved along KM flow}$$

- Remedy: Use Induction due to HL

A large system with $N \geq 4$

Recall the Kuramoto model with an HL topology:

$$\begin{aligned} \dot\theta_1 &= \Omega_1, \\ \dot\theta_i &= \Omega_i + K \sum_{j=1}^{i-1} \psi_{ji} \sin(\theta_j - \theta_i), \quad i = 2, \cdots, N, \end{aligned} \tag{1}$$

We set $\varphi_i := \theta_i - \theta_{i+1}, \quad \Delta_i := \Omega_i - \Omega_{i+1}, \quad \omega_i := \dot\theta_i - \dot\theta_{i+1}.$
Then φ_i satisfy

$$\begin{aligned} \dot\varphi_1 &= \Delta_1 - K\psi_{12} \sin\varphi_1, \\ \dot\varphi_i &= \Delta_i + K \sum_{j=1}^{i-1} \psi_{ji} \sin\Big(\sum_{l=j}^{i-1} \varphi_l\Big) - K \sum_{j=1}^{i} \psi_{j(i+1)} \sin\Big(\sum_{l=j}^{i} \varphi_l\Big). \end{aligned}$$

We set Γ^* and D_n:

$$
\begin{aligned}
D_\infty &< \frac{\pi}{2}, \\
\Gamma^* &:= \max_{1\le l\le N-1}\left[\frac{\sum_{j=1}^{l-1}\psi_{jl}+\sum_{j=1}^{l}\psi_{j(l+1)}}{\sqrt{1-(\sin D_\infty)^2}\sum_{j=1}^{l-1}\psi_{j(l+1)}+\psi_{l(l+1)}}\right], \\
S_n(t) &:= \sum_{i=1}^{n}|\varphi_i(t)|, \quad t\ge 0, \\
D_n &:= \sup_{t>0} S_n(t), \quad n=1,\cdots,N-1.
\end{aligned}
$$

Lemma 3: For a given $N \geq 4$ and $D_\infty \in (0, \frac{\pi}{2})$, let $\kappa \in (0, D_\infty)$ and ε be positive constants such that

$$(i) \left[\frac{(\Gamma^* + 1)^{N-2} - 1}{\Gamma^*}\right] \sin\kappa < \frac{1}{2}\sin D_\infty.$$

$$(ii)\ (\Gamma^* + 1)^{N-2} \sin\varepsilon < \frac{1}{2}\sin D_\infty, \quad \varepsilon < \frac{D_\infty}{N-1},$$

$$\psi_{13}\sqrt{1-\varepsilon^2} + \psi_{23} - 2\sqrt{2}(\psi_{12} + \psi_{13})\sin\varepsilon > 0.$$

Suppose that the coupling strength K and initial data θ_0 satisfy

$$K > \max\Bigg\{\frac{|\Delta_1|}{\psi_{12}\sin\varepsilon},$$

$$\max_{2\leq l\leq N-1}\Bigg\{\frac{|\Delta_l|}{\sin\kappa\Big(\sqrt{1-\sin^2 D^\infty}\sum_{j=1}^{l-1}\psi_{j(l+1)} + \psi_{l(l+1)}\Big)}\Bigg\}\Bigg\},$$

$$\theta_{i0} \in (0, \varepsilon), \quad i = 1, \cdots, N.$$

Then we have

$$\sum_{i=1}^{N-1} |\varphi_i(t)| \leq D_\infty, \quad t > 0.$$

Exponential complete frequency synchronization

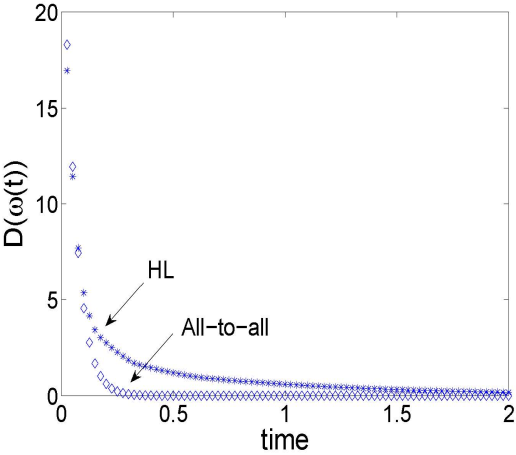
Theorem: Based on the same assumptions in the above lemma, there exist $\{\mu_i\}_{i=1}^{N-1}$ such that

$$|\dot{\theta}_i(t) - \dot{\theta}_j(t)| \le Ce^{-\mu t}, \quad t > 0,$$

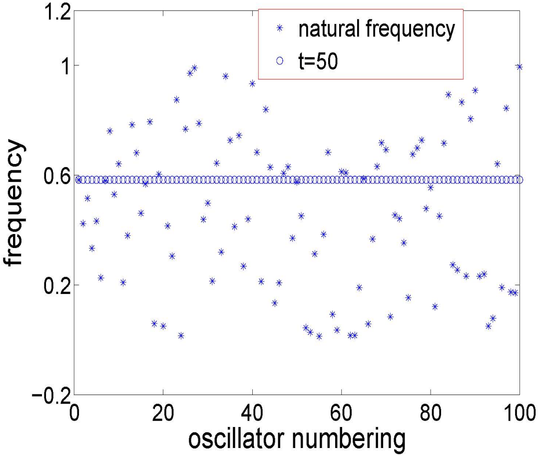
i.e., exponential complete synchronization will occur asymptotically.

- The proof is given by *induction* again.

Simulations for non-identical oscillators



Simulations for non-identical oscillators



Long-time dynamics of kinetic KM

Contraction of the kinetic KM

- The kinetic Kuramoto model: Chiba '10, Lancellotti '05

$$\partial_t f + \partial_\theta(\omega[f]f) = 0, \quad (\theta, \Omega) \in T \times R, \ t > 0,$$
$$\omega[f](\theta, \Omega, t) = \Omega - K \int_0^{2\pi} \sin(\theta - \theta_*)\rho(\theta_*, t) d\theta_*,$$
$$\rho(\theta_*, t) := \int_R f d\Omega_*,$$

subject to prepared initial data:

$$f(\theta, \Omega, 0) = f_0(\theta, \Omega),$$

where initial datum f_0 is assumed to satisfy constraints:

$$f_0(\theta, \Omega) = f_0(\theta + 2\pi, \Omega), \ \int_0^{2\pi} f_0(\theta, \Omega) d\theta = g(\Omega), \ \int_0^{2\pi} \int_R f_0 d\Omega d\theta = 1.$$

• **Theorem**: Carrillo-H-Kang-Kim '11
Suppose that the oscillators are identical, and let $\mu_0 \in \mathcal{M}(T \times R)$ be a given initial Radon measure satisfying

$$||\mu_0|| = 1, \quad \langle \mu_0, \theta \rangle = 0,$$
$$\mathrm{Proj}_\theta(\mathrm{Spt} f_0) \subset (0, D_\theta(0)), \quad D_\theta(0) < \pi.$$

Then the measure valued solution μ_t to KKE with initial datum μ_0 satisfies

$$\lim_{t \to \infty} d(\mu_t, \mu_\infty) = 0,$$

where $d = d(\cdot, \cdot)$ is the bounded Lipschitz distance and $\mu_\infty(d\theta, d\Omega)$ is defined by

$$\mu_\infty(d\theta, d\Omega) := M\delta_{\theta_c(0)}(\theta) \otimes \delta(\Omega).$$

Define the cumulative distribution function of f and its pseudo-inverse:

$$\begin{aligned} F(\theta, \Omega, t) &:= \int_0^\theta f(\theta_*, \Omega, t) d\theta_*, \quad (\theta, \Omega, t) \in T \times R \times R_+, \\ \phi(\eta, \Omega, t) &:= \inf\{\theta \ : \ F(\theta, \Omega, t) > \eta\}, \quad \eta \in [0, 1]. \end{aligned}$$

ϕ satisfies the following integro-differential equation:

$$\partial_t \phi = \Omega + K \int_0^1 \int_R \sin(\phi_* - \phi) d\Omega_* d\eta_*,$$

where we used abbreviated handy notations:

$$\phi_* := \phi(\eta_*, \Omega_*, t), \qquad \phi := \phi(\eta, \Omega, t).$$

- **Lemma** Let $\Phi = \Phi(\eta)$ be a measurable function defined on $[0, 1]$ satisfying

$$|\Phi(\eta_*)| < \frac{\pi}{2}, \quad |\Phi(\eta)| < \frac{\pi}{2}, \quad |\Phi(\eta_*) - \Phi(\eta)| < \pi, \quad \eta_*, \ \eta \in [0, 1],$$

$$\text{and} \quad \int_0^1 \Phi(\eta) d\eta = 0.$$

Then for $1 \leq p < \infty$, we have

$$\int_0^1 \int_0^1 \Big[|\Phi(\eta)|^{p-1} \mathrm{sgn}(\Phi(\eta)) - |\Phi(\eta_*)|^{p-1} \mathrm{sgn}(\Phi(\eta_*)) \Big] \times \sin\left(\frac{\Phi(\eta_*) - \Phi(\eta)}{2} \right) d\eta_* d\eta \leq -\frac{2}{\pi} \int_0^1 |\Phi(\eta)|^p d\eta.$$

Recall that

$$W_p(f_1, f_2)(\Omega, t) := ||\phi_1(\cdot, \Omega, t) - \phi_2(\cdot, \Omega, t)||_{L^p((0,1))}, \quad 1 \le p \le \infty.$$

Since $W_p(f_1, f_2)$ depends on Ω, we need to introduce a modified metric on the phase-space $\theta - \Omega$:

$$\tilde{W}_p(f_1, f_2)(t) := ||W_p(f_1, f_2)(\cdot, t)||_{L^p(R)}, \quad 1 \le p \le \infty.$$

• **Theorem**: Carrillo-H-Kang-Kim '11
Suppose that two Radon measures μ_0, ν_0 and K satisfy

$$
\begin{aligned}
&(i)\ \mu_0 = f_{10}\, d\theta d\Omega, \quad \nu_0 = f_{20}\, d\theta d\Omega, \quad f_{i0} \in L^1(R), \quad i = 1, 2.\\
&(ii)\ 0 < D_\theta(\nu_0) \le D_\theta(\mu_0) < \pi, \qquad 0 < D_\Omega(\mu_0) = D_\Omega(\nu_0) < \infty.\\
&(iii)\ \int_{T\times R} \theta \mu_0(d\theta, d\Omega) = \int_{T\times R} \theta \nu_0(d\theta, d\Omega).\\
&(iv)\ K > D_\Omega(\mu_0) \max\left\{ \frac{1}{\sin D_\theta(\mu_0)}, \frac{1}{\sin D_\theta(\nu_0)} \right\}.
\end{aligned}
$$

Let μ_t and ν_t be two measure valued solutions to KKE corresponding to initial data μ_0 and ν_0 respectively. Then there exists $t_0 > 0$ such that

$$
\widetilde{W}_p(f_1(t), f_2(t)) \le \widetilde{W}_p(f_1(t), f_2(t)) \exp\left[-\frac{2K \cos D^\infty}{\pi}(t - t_0) \right], \quad t \ge t_0,
$$

where $p \in [1, \infty]$ and f_1 and f_2 are corresponding density functions of μ_t and ν_t respectively.

Conclusion

- We have provided complete synchronization estimates of Kuramoto oscillators on some networks, but the results are still far from the completeness
- For networks with non-symmetric couplings, the complete synchronization problem for Kuramoto oscillators needs to be studied.

Thank you for your attention

• **Proposition**: H-Li-Xue'12
Let $\theta = \theta(t)$ be the uniformly bounded solution to the gradient system (**??**) with non-equilibrium initial data. Then there exists an $f_\infty \in$ such that

$$f(\theta(t)) > f_\infty, \quad t > 0 \qquad \text{and} \qquad \lim_{t\to\infty} f(\theta(t)) = f_\infty.$$

• **Theorem**: Łojasiewicz Let $U \subset \mathbb{R}^N$ be open and $f : U \to \mathbb{R}$ be real-analytic. Then for any $z_0 \in U$, there exist constants $\gamma \in [\frac{1}{2}, 1)$, $C_L, r > 0$ such that

$$|f(z) - f(z_0)|^\gamma \le C_L \|\nabla f(z)\|, \qquad \forall\, z \in B_r(z_0) \subset U.$$