Incompressible limit of the linearized Navier–Stokes equations.

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Padova, June 25, 2012

Plan of the talk

1. Motivation: Incompressible Limit of Full NSE;
2. Model problem for linearised equations;
3. Discussion of the results.

Incompressible limit of NSE

- Incompressible NSE:

$$\left.\begin{array}{c} \operatorname{div} U = 0, \\ (\varrho U)_t + \operatorname{div}(\varrho U \otimes U) + \nabla P = \mu \Delta U \end{array}\right\} \ \mathtt{IE}(U, P) = 0$$

- Compressible isentropic NSE:

$$\left.\begin{array}{c} \varrho_t + \operatorname{div}(\varrho U) = 0, \qquad \mathscr{F}(\varrho, P) = 0, \\ (\varrho U)_t + \operatorname{div}(\varrho U \otimes U) + \nabla P = \mu \Delta U + \kappa \nabla \operatorname{div} U. \end{array}\right\} \ \mathtt{CE}(\varrho, U, P) = 0$$

- Formally IE can be viewed as CE with $\mathscr{F}(\varrho, P) := \varrho - 1$.

Can we pass to the limit from CE to IE?

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Can we pass to the limit from CE to IE?

Introduction of compressibility

When $\mathscr{F}_\varepsilon(\varrho, P) = \varrho - 1 - \varepsilon P$ the compressible equations take form

$$\left.\begin{array}{c} \varrho_t + \mathrm{div}(\varrho U) = 0, \qquad \varrho = 1 + \varepsilon P, \\ (\varrho U)_t + \mathrm{div}(\varrho U \otimes U) + \nabla P = \mu \Delta U + \lambda \nabla \,\mathrm{div}\, U \end{array}\right\} \mathtt{CE}_\varepsilon(\varrho, U, P) = 0$$

- When $\varepsilon = 0$ we formally obtain the incompressible NSE.
- The speed of sound is $c = \sqrt{dP/d\varrho} = 1/\sqrt{\varepsilon}$.

Definition 1

The parameter $\varepsilon > 0$ in the equations $\mathtt{CE}_\varepsilon(\varrho, U, P) = 0$ is called the *compressibility*.

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Linearization of the original problem

Problem 1 (Original)

Do the solutions $\{\varrho_\varepsilon, U_\varepsilon, P_\varepsilon\}$ *of* $\mathtt{CE}_\varepsilon(\varrho, U, P) = 0$ *converge towards the solution* $\{U, P\}$ *of* $\mathtt{IE}(U, P) = 0$ *as* $\varepsilon \to +0$*?*

$$\frac{d}{d\tau}\mathtt{CE}_\varepsilon(\varrho + \tau\rho, U + \tau u, P + \tau p)\Big|_{\tau=0} = 0 \quad \to \quad \mathtt{LCE}_\varepsilon(\rho, u, p) = 0,$$

$$\frac{d}{d\tau}\mathtt{IE}(U + \tau u, P + \tau p)\Big|_{\tau=0} = 0 \quad \to \quad \mathtt{LIE}(u, p) = 0.$$

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Linearized equations

For simplicity assume that U is smooth and divergence-free and $\varrho = 1$, $\mu = 1$, $\lambda = 0$. Then the linearized equations have the form

$$\left.\begin{aligned} \rho_t + (U,\nabla)\rho + \operatorname{div} u = 0, \qquad \rho = \varepsilon p, \\ u_t + (U,\nabla)u + \nabla p = \Delta u. \end{aligned}\right\} \mathrm{LCE}_\varepsilon(\rho, u, p) = 0$$

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Cauchy problems on torus

On $(0, T) \times \mathbb{T}^d$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $T > 0$ we consider:

1. Compressible problem:

$$\left.\begin{array}{c} \rho_t + (U, \nabla)\rho + \operatorname{div} u = 0, \qquad \rho = \varepsilon p, \\ u_t + (U, \nabla)u + \nabla p = \Delta u, \\ u|_{t=0} = u^\circ, \qquad p|_{t=0} = p^\circ. \end{array}\right\} \quad (1)$$

2. Incompressible problem:

$$\left.\begin{array}{c} \operatorname{div} v = 0, \\ v_t + (U, \nabla)v + \nabla q = \Delta v, \\ v|_{t=0} = v^\circ. \end{array}\right\} \quad (2)$$

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Solvability of the compressible problem

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Theorem 1

$\forall\ \varepsilon > 0$, $u^\circ \in L^2$ and $p^\circ \in L^2$ there exists a unique weak solution

$$\{u, p\} = \{u_\varepsilon, p_\varepsilon\} \in L^2(0, T; H^1) \times L^\infty(0, T; L^2)$$

to the problem (1).

Solvability of the incompressible problem

$$\left.\begin{aligned} \operatorname{div} v &= 0, \\ v_t + (U, \nabla) v + \nabla q &= \Delta v, \\ v\big|_{t=0} &= v^\circ. \end{aligned}\right\} \quad (2)$$

Theorem 2

$\forall\ v^\circ \in H$ there exists a weak solution

$$\{v, q\} \in L^2(0, T; V) \times L^\infty(0, T; L^2)$$

to the problem (2).

Weak convergence of the velocity

$$\left.\begin{aligned} \rho_t + (U,\nabla)\rho + \operatorname{div} u = 0, \qquad \rho = \varepsilon p, \\ u_t + (U,\nabla)u + \nabla p = \Delta u, \\ u|_{t=0} = u^\circ, \qquad p|_{t=0} = p^\circ. \end{aligned}\right\} (1)$$

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Theorem 3

If $P_H u^\circ = v^\circ$ then
$u_\varepsilon \rightharpoonup v$ *in* $L^2(0,T;H^1)$*–weak and* $L^\infty(0,T;L^2)$*–weak**, $\varepsilon \to 0$.

Strong convergence of the velocity

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Theorem 4

If v° is smooth and $u^\circ = v^\circ$ then
$u_\varepsilon \to v$ *in* $L^2(0,T;H^1)$ *and* $L^\infty(0,T;L^2)$ *as* $\varepsilon \to 0$.

Remark: in this case $\|u_\varepsilon - v\| = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

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Convergence of the pressure

What about the convergence of the pressure?

Mass conservation property

$$\left.\begin{aligned} \rho_t + (U,\nabla)\rho + \operatorname{div} u = 0, \qquad \rho = \varepsilon p, \\ u_t + (U,\nabla)u + \nabla p = \Delta u, \\ u|_{t=0} = u^\circ, \qquad p|_{t=0} = p^\circ. \end{aligned}\right\} \quad (1)$$

implies that $\frac{d}{dt}\int_{\mathbb{T}^d} p\,dx = 0$. However in

$$\left.\begin{aligned} \operatorname{div} v = 0, \\ v_t + (U,\nabla)v + \nabla q = \Delta v, \\ v|_{t=0} = v^\circ. \end{aligned}\right\} \quad (2)$$

generally $\frac{d}{dt}\int_{\mathbb{T}^d} q\,dx \neq 0$, since $\forall Q = Q(t)$

$q(t,x) + Q(t)$ is also a pressure for (2).

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Correction of pressure

If $q \in W^{1,2}(0,T;L^2)$, then there exists unique $Q = Q(t)$ such that

$$\widehat{q} = q - Q \in W^{1,2}(0,T;L^2)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{T}^d} \widehat{q}\,dx = 0,$$
$$\left.\int_{\mathbb{T}^d} \widehat{q}\,dx\right|_{t=0} = \int_{\mathbb{T}^d} p^\circ\,dx.$$

Weak convergence of the pressure

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Theorem 5

If v° is smooth and $u^\circ = v^\circ$ then

$$p_\varepsilon \to \widehat{q} \qquad \text{in} \quad L^\infty(0,T;L^2)\text{–weak*} \quad \text{as} \quad \varepsilon \to 0.$$

Strong convergence of the pressure

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Remark: in this case $\|u_\varepsilon - v\| = O(\varepsilon)$ and $\|p_\varepsilon - \widehat{q}\| = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

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Summary

Assumption ↓ Result →	$u_\varepsilon \rightharpoonup v$	$u_\varepsilon \to v$	$p_\varepsilon \overset{*}{\rightharpoonup} \widehat{q}$	$p_\varepsilon \to \widehat{q}$
$P_H u^\circ = v^\circ$	$+$	$-$	$-$	$-$
$u^\circ = v^\circ$	$+$	$\sqrt{\varepsilon}$	$+$	$-$
$u^\circ = v^\circ, \quad p^\circ = q\|_{t=0} + \mathrm{const}$	$+$	ε	$+$	$\sqrt{\varepsilon}$

Some of the results above are similar to well-known results for NSE:

- P.L. Lions, N. Masmoudi ('98); E. Feireisl, A. Novotny ('07)
- Artificial compressibility method (see e.g. R. Temam).

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Summary

Assumption ↓ Result →	$u_\varepsilon \rightharpoonup v$	$u_\varepsilon \to v$	$p_\varepsilon \overset{*}{\rightharpoonup} \widehat{q}$	$p_\varepsilon \to \widehat{q}$
$P_H u^\circ = v^\circ$	$+$	$-$	$-$	$-$
$u^\circ = v^\circ$	$+$	$\sqrt{\varepsilon}$	$+$	$-$
$u^\circ = v^\circ, \quad p^\circ = q\big\|_{t=0} + \mathrm{const}$	$+$	ε	$+$	$\sqrt{\varepsilon}$

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Explicit solution

Suppose that $U = 0$, $d = 2$, $\psi(x) = \sin(2\pi x_1)\sin(2\pi x_2)$. Then $\Delta\psi = -\lambda\psi$ where $\lambda = 8\pi^2$.

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1. $\|u_\varepsilon - v\| \sim \sqrt{\varepsilon}$,
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1. Low Mach Number Limit

Dimensionless form of CE:

$$\Rightarrow \begin{cases} \mathrm{Sr}\,\varrho_t + \operatorname{div}(\varrho U) = 0, \qquad \mathscr{F}(\varrho, P) = 0, \\ \mathrm{Sr}(\varrho U)_t + \operatorname{div}(\varrho U \otimes U) + \dfrac{1}{\mathrm{Ma}^2}\nabla P = \dfrac{1}{\mathrm{Re}_\mu}\Delta U + \dfrac{1}{\mathrm{Re}_\lambda}\nabla \operatorname{div} U, \end{cases}$$

Compressibility $\varepsilon :=$ «Mach Number» $\mathrm{Ma} \to +0$.

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2. Special equation of state

Consider $\begin{cases} \varrho_t + \operatorname{div}(\varrho U) = 0, \qquad \mathscr{F}_\varepsilon(\varrho, P) = 0, \\ (\varrho U)_t + \operatorname{div}(\varrho U \otimes U) + \nabla P = \mu \Delta U + \lambda \nabla \operatorname{div} U, \end{cases}$

with $\mathscr{F}_\varepsilon(\varrho, p) = 0$ equivalent to one of the following:

1. $P = k\varrho^{1/\varepsilon}$, $k = \text{const.}$ (D. Ebin '75)
2. $P = \mathscr{P}(\varrho)/\varepsilon^2$. (A. Majda, S. Klainerman '81)
3. $\varrho = \varrho_0 + \varepsilon \mathscr{R}(P)$. (E. Shifrin '99)

with **compressibility** $\varepsilon \to +0$.

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Comparison with low Mach Number framework

If the compressibility is defined using equation of state, we have

$$\left.\begin{aligned} \rho_t + (U,\nabla)\rho + \operatorname{div} u = 0, \qquad \rho = \varepsilon p, \\ u_t + (U,\nabla)u + \nabla p = \Delta u, \\ u|_{t=0} = u^\circ, \qquad p|_{t=0} = p^\circ. \end{aligned}\right\} \quad (1)$$

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These systems can be identified if $u = u'$, $\rho = \rho'$ and $p = p'/\varepsilon$. Hence $p'_\varepsilon = \varepsilon p_\varepsilon \to 0$ as $\varepsilon \to 0$.

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Thank you for your attention!

Main estimates

Consider the problem with right-hand side:

$$\left.\begin{aligned} \rho_t + (U,\nabla)\rho + \operatorname{div} u = \sigma, \qquad \rho = \varepsilon p, \\ u_t + (U,\nabla)u + \nabla p = \Delta u + s, \\ u|_{t=0} = u^\circ, \qquad p|_{t=0} = p^\circ. \end{aligned}\right\} \quad (1)$$

The energy estimate:

$$\|u\|_{L^2(0,T;H^1)} + \|u\|_{L^\infty(0,T;L^2)} + \sqrt{\varepsilon}\|p\|_{L^\infty(0,T;L^2)} \leqslant$$
$$\leqslant C\left(\|u^\circ\|_{L^2} + \sqrt{\varepsilon}\|p^\circ\|_{L^2} + \|s\|_{L^2(0,T;H^{-1})} + \frac{1}{\sqrt{\varepsilon}}\|\sigma\|_{L^2(0,T;L^2)}\right).$$

A modified estimate:

$$\|u\|_{L^2(0,T;H^1)} + \|u\|_{L^\infty(0,T;L^2)} + \sqrt{\varepsilon}\|p\|_{L^\infty(0,T;L^2)} \leqslant$$
$$\leqslant C\left(\|u^\circ\|_{L^2} + \sqrt{\varepsilon}\|p^\circ\|_{L^2} + \|s\|_{L^2(0,T;H^{-1})} + \|\sigma\|_{W^{1,2}(0,T;L^2/\mathbb{R})}\right).$$

Results by other authors

1. Solvability of (1) when $U \equiv 0$:
   - R. Ikehata , T. Koboyashi , T. Matsuyama '01
   - P.B. Mucha, W.M. Zajaczkowski '02
   - ...
2. Incompressible limit
   - D.G. Ebin '75
   - S. Klainerman, A. Majda '81
   - R. Temam 80
   - P.-L. Lions, N. Masmoudi '98
   - E. Shifrin '99
   - V. Puchnachev '02
   - E. Feireisl, A. Novotný '07
   - ...