Lipschitz Semigroup for an Integro–Differential Equation for Slow Erosion

Graziano Guerra

Work in collaboration with
Debora Amadori, **Rinaldo M. Colombo**, and **Wen Shen**

June, 2012

Outline

Slow erosion limit model

$$\begin{cases} U_t(t,x) - \left(\exp \int_x^{+\infty} f\left(U_x(t,y)\right) dy \right)_x = 0, \\ U(0,x) = U_o(x) \end{cases} \tag{1}$$

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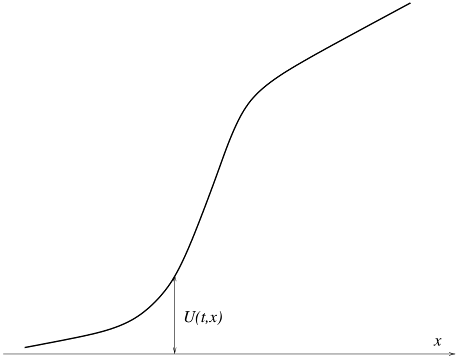

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- Equation (1) was derived by Amadori, Shen (2011) as the slow erosion limit for the 2×2 Hadeler and Kuttler system (1999). In that case $f(w) = \frac{w-1}{w}$.

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- we have a non linear function f of the derivative U_x

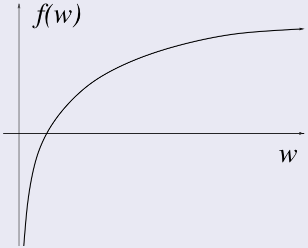
$$\Longrightarrow$$

we have to define in a suitable way $f\left(U_x\right)$ in case of formation of discontinuities;

Continuous solutions

Amadori, Shen, 2011

if $f(1) = 0,\ f' > 0,\ f'' < 0,\ \lim_{w \to 0^+} f(w) = -\infty,\quad \lim_{w \to +\infty} \frac{f(w)}{w} = 0$



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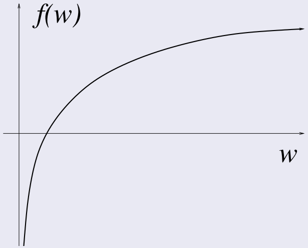

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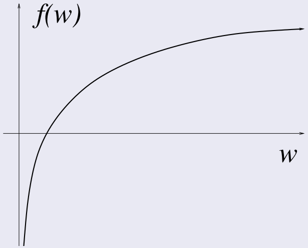

there exists a unique "admissible" solution to (1), $U(t,x)$, Lipschitz continuous in x for all times.

This implies that $U_x(t,x)$ is in L^∞ and $f\left(U_x(t,x)\right)$ is a well defined function.

Continuous solutions

Amadori and Shen observe that the slope $w(t,x) = U_x(t,x)$ solves

$$w_t + \big(f(w)F(w;x)\big)_x = 0 \quad \text{then} \tag{2}$$

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- Uniqueness and continuous dependence.

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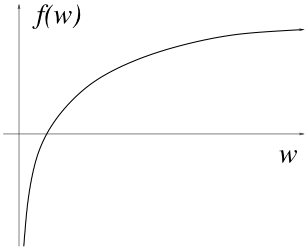
If, instead of $\lim_{w \to +\infty} f(w)/w = 0$



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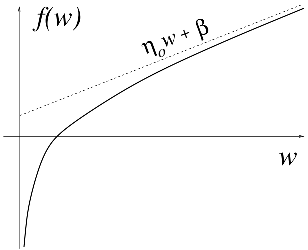
we require $\lim_{w \to +\infty} \left[f(w) - (\eta_o w + \beta) \right] = 0$



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- $U(t,x)$ may develop discontinuities;
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- they give a definition for $f\left(U_x(t,x)\right)$.

Change of variables

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- for smooth U: $\int_x^{+\infty} f\left(U_x(t,y)\right)\ dy = \int_{U(t,x)}^{+\infty} g\left(z(t,v)\right)\ dv$, $g(s) = s \cdot f\left(1/s\right)$, $g(0) = \eta_o$, $g'(0) = \beta$.

Rigorous statement of the problem (*)

Find $z(t,\cdot) \in \boldsymbol{BV}\left(\mathbb{R},\mathbb{R}^+\right)$ with $z(t,\cdot) - 1 \in \boldsymbol{L}^1\left(\mathbb{R},\mathbb{R}\right)$ such that:

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z is also the right unknown to look at if we want to show continuous dependence

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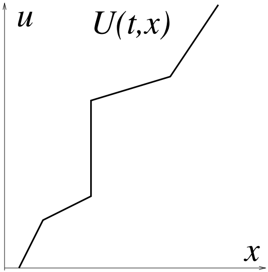
and the L^1 estimate $\|S^g_t z - 1\|_{\boldsymbol{L^1}} \leq \|z - 1\|_{\boldsymbol{L^1}}$.

proof



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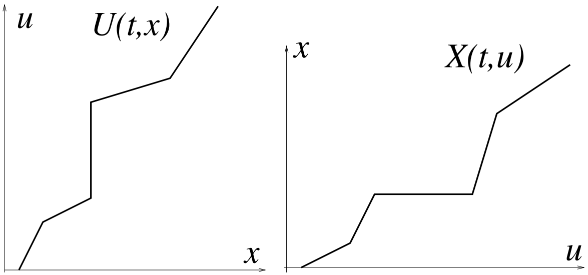

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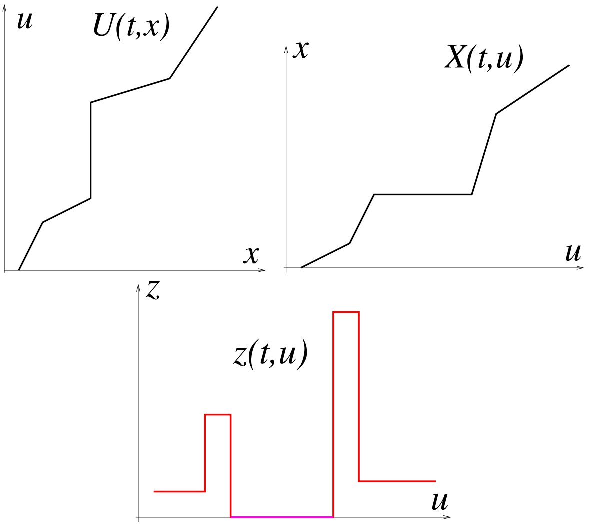

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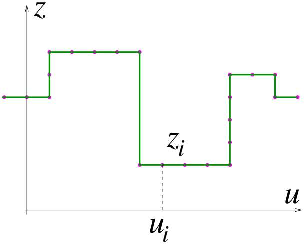
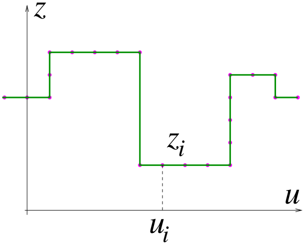
$$\begin{cases} \dfrac{d}{dt}u(t) = -g'(z(t,u(t))G\left(z(t);u(t)\right) \\ \dfrac{d}{dt}z\left(t,u(t)\right) = -g\left(z(t,u(t)\right)^2 G\left(z(t);u(t)\right) \end{cases}$$

we fix approximate initial data



$$z_{i+1} - z_i < \varepsilon$$
$$\left|g\left(z_i\right)\right| \left(u_{i+1} - u_i\right) < \varepsilon$$

we fix approximate initial data



$$z_{i+1} - z_i < \varepsilon$$
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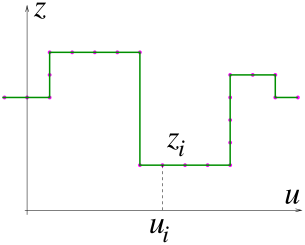
the initial data evolves through a system of ODEs which approximates the characteristic equations

$$\begin{cases} \dot{u}_i = -\frac{g(z_i)-g\left(z_{i-1}\right)}{z_i - z_{i-1}} G\left(z; u_i\right) \\ \\ \end{cases} \qquad \dot{u} = -g'(z) G(z; u)$$

we fix approximate initial data



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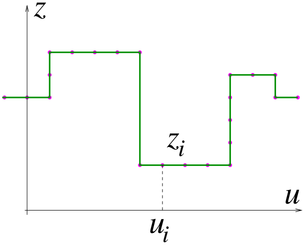
$$\begin{cases} \dot{u}_i = -\frac{g(z_i)-g\left(z_{i-1}\right)}{z_i - z_{i-1}} G\left(z; u_i\right) \\ \dot{z}_i = g\left(z_i\right) \dfrac{G\left(z; u_{i+1}\right) - G\left(z; u_i\right)}{u_{i+1} - u_i} \\ \quad = -g\left(z_i\right)^2 G\left(z; \tilde{u}_i\right) \end{cases}$$

$$\dot{u} = -g'(z) G\left(z; u\right)$$

$$\dot{z} = -g(z)^2 G\left(z; u\right)$$

we fix approximate initial data



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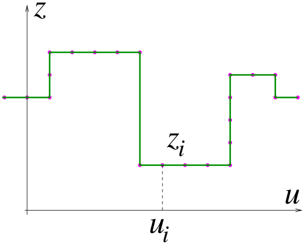
Film

we fix approximate initial data



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Film

Pointwise constraints $z(t,u) \geq 0$

- $z_t(t,u) - \Big(g(z(t,u))G\left(z(t);u\right)\Big)_u = 0$

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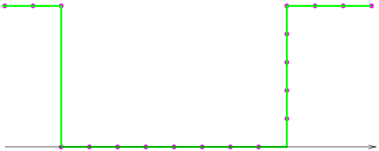

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Film

Pointwise constraints $z(t, u) \geq 0$

- $z_t(t, u) - \Big(g(z(t, u)) G\left(z(t); u\right) \Big)_u = \mu$
- $\dot{z}_i = 0$

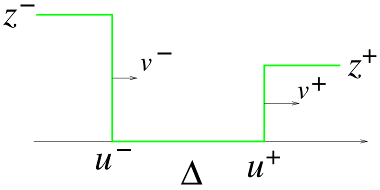
Film

Pointwise constraints $z(t, u) \geq 0$

- $z_t(t, u) - \Big(g(z(t, u))G\left(z(t); u\right)\Big)_u = \mu$
- $\dot{z}_i = 0$

Film

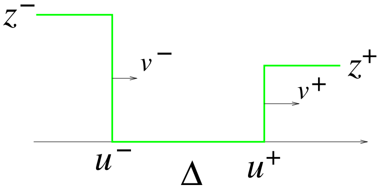
u–shocks velocity



$$v^- = -\frac{g\left(z^-\right) - \frac{1-e^{-\Delta g(0)}}{\Delta}}{z^-} G\left(z; u^-\right) < -\frac{g\left(z^-\right) - g(0)}{z^- - 0} G\left(z; u^-\right)$$

$$v^+ = -\frac{g\left(z^+\right) - \frac{e^{\Delta g(0)}-1}{\Delta}}{z^+} G\left(z; u^+\right) > -\frac{g\left(z^+\right) - g(0)}{z^+ - 0} G\left(z; u^+\right)$$

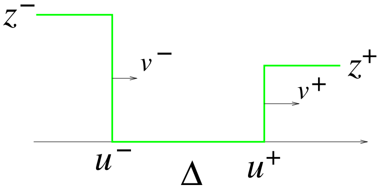
Lax inequalities



$$v^- < -\frac{g\left(z^-\right) - g(0)}{z^- - 0} G\left(z; u^-\right) < -g'\left(z^-\right) G\left(z; u^-\right)$$

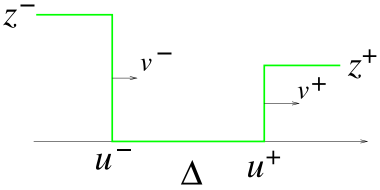
$$v^+ > -\frac{g\left(z^+\right) - g(0)}{z^+ - 0} G\left(z; u^+\right) \not> -g'\left(z^+\right) G\left(z; u^+\right)$$

Lax inequalities



- $v^- < -g'(z^-)\,G(z;u^-)$ always holds;

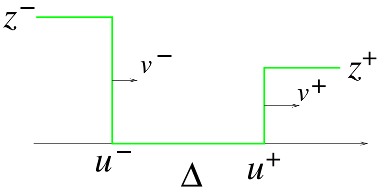
Lax inequalities



- $v^- < -g'(z^-)\, G(z; u^-)$ always holds;
- $v^+ > -g'(z^-)\, G(z; u^+)$ if and only if

$$g(z^+) - z^+ g'(z^+) < \frac{e^{\Delta g(0)} - 1}{\Delta}, \quad \Leftrightarrow z^+ < \zeta(\Delta);$$

Lax inequalities

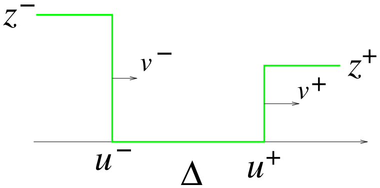


New rarefactions may be generated at time $t > 0$,

$$\Longrightarrow$$

- technical difficulties on wave front tracking algorithm;

Lax inequalities



New rarefactions may be generated at time $t > 0$,

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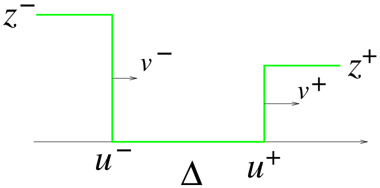
- lack of Oleinik type estimates;

Lax inequalities

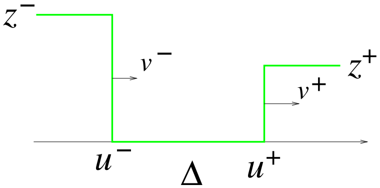


New rarefactions may be generated at time $t > 0$,

$$\Longrightarrow$$

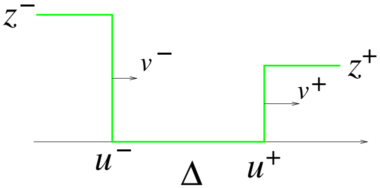
- technical difficulties on wave front tracking algorithm;
- lack of Oleinik type estimates;
- we do not have uniqueness yet $\Rightarrow$ work in progress.

Lax inequalities



Film

Lax inequalities



Using the fact that Lax inequalities are fulfilled by approximate wave front tracking solutions, the main theorem can be proved through a decreasing functional approach.

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Thanks for the attention!!