Global solutions of the two-component Camassa–Holm system

Katrin Grunert

Norwegian University of Science and Technology (NTNU)

Joint work with H. Holden and X. Raynaud

Two-component Camassa–Holm system

The two-component Camassa–Holm (2CH) system

$$\begin{aligned} u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x &= 0, \\ \rho_t + (u\rho)_x &= 0, \end{aligned}$$

with $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$, has been derived as a model for shallow water [Constantin/Ivanov, 2008].

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In particular, it reduces for $\rho = 0$ to the Camassa–Holm (CH) equation and is one out of many generalizations of the CH equation.

Background

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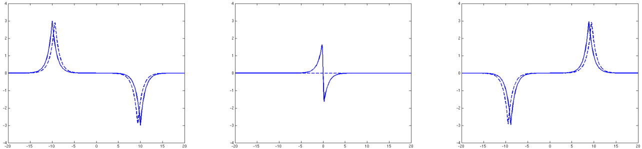
Background

In the case of two symmetric peakons, such that one peakon is moving to the right and one antipeakon to the left wave breaking takes place.



At collision time the energy concentrates at one point. By letting peakon and antipeakon pass through each other we obtain the conservative solution, that means the energy is conserved for all times.

Conservative solutions

Conservative solutions of the CH equation can be described using a reformulation as a semilinear system of ODEs in a Banach space [Bressan / Constantin, 2007; Holden / Raynaud 2007–2008].

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Conservative solutions of the CH equation with nonvanishing asymptotics have been studied by [Grunert / Holden / Raynaud, 2012], following closely the approach of [Holden / Raynaud, 2007].

Let's start

The 2CH system is given by

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \eta\rho\rho_x = 0,$$
$$\rho_t + (u\rho)_x = 0,$$

with $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$. Here we only consider the case $\kappa = 0$ and $\eta = 1$.

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with $\kappa \in \mathbb{R}$ and $\eta \in (0, \infty)$. Here we only consider the case $\kappa = 0$ and $\eta = 1$.

In the weak formulation it reads

$$u_t + uu_x + P_x = 0,$$
$$\rho_t + (u\rho)_x = 0,$$

where P is implicitely defined by

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2.$$

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  $$u_0(x) = \tilde{u}_0(x) + u_{0,\infty}\chi(x),$$

  where $\tilde{u}_0(x) \in H^1(\mathbb{R})$, $u_{0,\infty} \in \mathbb{R}$, and χ denotes a smooth partition function with support in $[0, \infty)$ such that $\chi(x) = 1$ for $x \geq 1$ and $\chi'(x) \geq 0$ for all $x \in \mathbb{R}$.

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- $\rho_0(x) \in L^2_{const}(\mathbb{R})$, i.e. $\rho_0(x)$ can be written as

$$\rho_0(x) = \tilde{\rho}_0(x) + \rho_\infty,$$

where $\tilde{\rho}_0(x) \in L^2(\mathbb{R})$ and $\rho_\infty \in \mathbb{R}$ denotes some constant.

Asymptotic behavior

Assume for the moment being that $u(t,x) \in H_\infty(\mathbb{R})$ and $\rho(t,x) \in L^2_{const}(\mathbb{R})$ for all times, which means in particular that $\lim_{x\to\pm\infty} u(t,x)$ and $\lim_{x\to\pm\infty} \rho(t,x)$ exist for all times. Then

$$
\begin{aligned}
u(t,x) &= \tilde{u}(t,x) + c(t)\chi(x),\\
\rho(t,x) &= \tilde{\rho}(t,x) + k(t).
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$$\begin{aligned} u(t,x) &= \tilde{u}(t,x) + c(t)\chi(x), \\ \rho(t,x) &= \tilde{\rho}(t,x) + k(t). \end{aligned}$$

Thus we can write

$$\begin{aligned} &(P - c^2\chi^2 - \frac{1}{2}k^2) - (P - c^2\chi^2 - \frac{1}{2}k^2)_{xx} \\ &\qquad = 2c\chi\tilde{u} + \tilde{u}^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\tilde{\rho}^2 + k\tilde{\rho} + 2c^2\chi'^2 + 2c^2\chi\chi''. \end{aligned}$$

Hence the function P can be expressed through

$$P(x) = c^2\chi^2(x) + \frac{1}{2}k^2 + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(\frac{1}{2}\tilde{\rho}^2 + k\tilde{\rho})(z)dz$$
$$+ \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(2c\chi\tilde{u} + \tilde{u}^2 + \frac{1}{2}u_x^2 + 2c^2\chi'^2 + 2c^2\chi\chi'')(z)dz,$$

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and $P_x \in L^2(\mathbb{R})$.

Asymptotic behavior

Assuming addionally that $\tilde{u}(t,x) \in H^2(\mathbb{R})$ and $\tilde{\rho}(t,x) \in H^2(\mathbb{R})$, which means that $\lim_{x\to\pm\infty} \tilde{u}_x(t,x) = 0$ and $\lim_{x\to\pm\infty} \tilde{\rho}_x(t,x) = 0$, we conclude from

$$u_t + uu_x + P_x, \quad \text{and} \quad \rho_t + (u\rho)_x = 0,$$

that

$$\lim_{x\to\pm\infty} u_t(t,x) = 0, \quad \text{and} \quad c'(t) = 0,$$

respectively

$$\lim_{x\to\pm\infty} \rho_t(t,x) = 0, \quad \text{and} \quad k'(t) = 0.$$

Global conservative solutions

$$
\begin{aligned}
u_t + uu_x + P_x &= 0,\\
\rho_t + (u\rho)_x &= 0,\\
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The set of Eulerian coordinates $\mathbb{D}$ consists of all triplets (u, ρ, μ) such that

- $u \in H_\infty(\mathbb{R})$,
- $\rho \in L^2_{const}(\mathbb{R})$, and
- μ is a positive, finite Radon measure whose absolute continuous part, μ_{ac}, satisfies

$$\mu_{ac} = (u_x^2 + \tilde{\rho}^2)dx.$$

Global conservative solutions

The set of Lagrangian variables (y, U, h, r) consists of

- the characteristics $y(t, \xi)$ as the solutions of $y_t(t, \xi) = u(t, y(t, \xi))$,
- the Lagrangian velocity $U(t, \xi)$, where $U(t, \xi) = u(t, y(t, \xi))$,
- the energy derivative $h(t, \xi)$ where $h(t, \xi) = u_x^2(t, y(t, \xi))y_\xi(t, \xi) + \tilde{\rho}^2(t, y(t, \xi))y_\xi(t, \xi)$, and
- $r(t, \xi)$ where $r(t, \xi) = \rho(t, y(t, \xi))y_\xi(t, \xi)$.

Global conservative solutions

The following system of ODEs can be obtained, which is formally equivalent to the 2CH system

$$\zeta_t = U,$$
$$U_t = -Q,$$
$$h_t = 2(U^2 + \frac{1}{2}k^2 - P)U_\xi,$$
$$r_t = 0$$

where $\zeta(t,\xi) = y(t,\xi) - \xi$, $U(t,\xi) = \tilde{U}(t,\xi) + c(t)\chi(y(t,\xi))$, and $r(t,\xi) = \tilde{r}(t,\xi) + k(t)y_\xi(t,\xi)$, in the Banach space

$$E = V \times H_\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2_{const}(\mathbb{R})$$

where

$$V = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^2(\mathbb{R})\}.$$

Global conservative solutions

In contrast to the case of vanishing asymptotics $P(t, \xi)$ is not Lipschitz continuous on bounded sets from E to $H^1(\mathbb{R})$, but $P(t, \xi) - \frac{1}{2}k^2 - U^2(t, \xi)$ and $Q(t, \xi)$ are Lipschitz continuous on bounded sets from E to $H^1(\mathbb{R})$.

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Thus the right hand side of

$$\begin{aligned}
\zeta_t &= U, \\
U_t &= -Q, \\
h_t &= 2(U^2+\frac{1}{2}k^2-P)U_\xi, \\
r_t &= 0,
\end{aligned}$$

is Lipschitz continuous on bounded sets of E and therefore one obtains the short time existence of solutions.

Global conservative solutions

To obtain global solutions one needs to introduce the set $\mathcal{F}$ composed of all $(y, U, h, r) \in E$ such that

$$(\zeta, U) \in [W^{1,\infty}(\mathbb{R})]^2, \quad (h, r) \in [L^\infty(\mathbb{R})]^2$$
$$y_\xi \geq 0,\ \ h \geq 0,\ \ y_\xi + h \geq c \text{ a.e., for some constant } c > 0,$$
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The set $\mathcal{F}$ is preserved and the global solutions are unique.

Eulerian vs. Lagrangian coordinates

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Moreover, we have 4 Lagrangian variables in contrast to 3 Eulerian variables and hence equivalence classes have to be identified.

Eulerian vs. Lagrangian coordinates

For any (u, ρ, μ) in $\mathbb{D}$, let

$$\begin{aligned}
y(\xi) &= \sup\{y \mid \mu((-\infty, y)) + y < \xi\}, \\
h(\xi) &= 1 - y_\xi(\xi), \\
U(\xi) &= u \circ y(\xi), \\
r(\xi) &= \rho \circ y(\xi) y_\xi(\xi).
\end{aligned}$$

Then $X = (y, U, h, r) \in \mathcal{F}$.

Eulerian vs. Lagrangian coordinates

Let G be the set of all functions f such that

$$f - \mathbb{I} \text{ and } f^{-1} - \mathbb{I} \text{ both belong to } W^{1,\infty}(\mathbb{R}),$$
$$f_\xi - 1 \text{ belongs to } L^2(\mathbb{R}),$$

where $\mathbb{I}$ denotes the identity function.

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where $\mathbb{I}$ denotes the identity function.

Then to any $(y, U, h, r) \in \mathcal{F}$ can be associated a unique $(\bar{y}, \bar{U}, \bar{h}, \bar{r}) \in \mathcal{F}_0 \subset \mathcal{F}$ such that $\bar{y}_\xi(\xi) + \bar{h}(\xi) = 1$ and

$$(\bar{y}, \bar{U}, \bar{h}, \bar{r}) = (y \circ f, U \circ f, h \circ f f_\xi, r \circ f f_\xi)$$

for a unique $f \in G$.

Eulerian vs. Lagrangian coordinates

For any $X \in \mathcal{F}$, then (u, ρ, μ) given by

$$
\begin{aligned}
u(x) &= U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \\
\mu &= y_{\#}(h d\xi), \\
\tilde{\rho}(x)dx &= y_{\#}(\tilde{r} d\xi), \\
\rho(x) &= \tilde{\rho}(x) + k,
\end{aligned}
$$

belongs to $\mathbb{D}$.

The push-forward of a measure λ by a measureable function f is the measure $f_{\#}\lambda$ defined as $f_{\#}\lambda(B) = \lambda(f^{-1}(B))$ for any Borel set B.

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Note that this mapping is independent of the element of the equivalence class we choose.

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Stability

One possibility to study stability is to use the norm of the underlying Banach space in Lagrangian coordinates, i.e.

$$d_{\mathbb{D}}((u,\rho,\mu),(\bar{u},\bar{\rho},\bar{\mu})) = d_{\mathcal{F}/G}(L(u,\rho,\mu),L(\bar{u},\bar{\rho},\bar{\mu})).$$

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Theorem

The semigroup $(T_t, d_{\mathbb{D}})$ is a continuous semigroup on $\mathbb{D}$ with respect to the metric $d_{\mathbb{D}}$. Moreover, given any intial condition $(u_0,\rho_0,\mu_0) \in \mathbb{D}$, we denote $(u,\rho,\mu)(t) = T_t(u_0,\rho_0,\mu_0)$. Then $(u(t,x),\rho(t,x))$ is a weak global conservative solution of the two-component Camassa–Holm system and μ is a weak solution to

$$(u^2+\mu+\rho^2-\tilde{\rho}^2)_t + (u(u^2+\mu+\rho^2-\tilde{\rho}^2))_x = (u^3-2Pu)_x.$$

For almost every time, μ is absolutely continuous, and in that case $\mu = (u_x^2+\tilde{\rho}^2)\,dx$.

Relation to other topologies

The mapping

$$(u, \rho) \mapsto (u, \rho, (u_x^2 + \tilde{\rho}^2)dx)$$

is continuous from $H_\infty(\mathbb{R}) \times L^2_{const}(\mathbb{R})$ into $\mathbb{D}$.

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Let (u_n, ρ_n, μ_n) be a sequence in $\mathbb{D}$ that converges to (u, ρ, μ) in $\mathbb{D}$. Then

$$u_n \to u \text{ in } L^\infty(\mathbb{R}),\ \tilde{\rho}_n \overset{*}{\rightharpoonup} \tilde{\rho},\ \ , k_n \to k \in \mathbb{R},\ \text{and } \mu_n \overset{*}{\rightharpoonup} \mu.$$

Stability

Given some initial data (u_0, ρ_0, μ_0) such that

- $u_0 \in W^{p,\infty}(I_0)$, $\rho_0 \in W^{p-1,\infty}(I_0)$ and $\mu_{0,ac} = \mu_0$ on I_0
- $\rho_0(x)^2 \geq c > 0$ for $x \in I_0$.

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- $\rho_0(x)^2 \geq c > 0$ for $x \in I_0$.

Then, for any $t \in \mathbb{R}_+$, $(u, \rho, \mu)(t, \cdot)$ satisfies

$$u \in W^{p,\infty}(I_t), \ \rho \in W^{p-1,\infty}(I_t) \ \text{ and } \mu_{ac} = \mu \text{ on } I_t$$

on the interval $I_t = (y(t, \xi_0), y(t, \xi_1))$, where ξ_0 and ξ_1 satisfy $y(0, \xi_0) = x_0$ and $y(0, \xi_1) = x_1$ and are defined as

$$\xi_0 = \sup\{\xi \in \mathbb{R} \mid y(0, \xi) \leq x_0\} \text{ and } \xi_1 = \inf\{\xi \in \mathbb{R} \mid y(0, \xi) \geq x_1\}.$$

Stability

Theorem

Let $u_0 \in H_\infty(\mathbb{R})$. Consider the approximating sequence of initial data $(u_0^n, \rho_0^n, \mu_0^n) \in \mathbb{D}$ given by $u_0^n \in C^\infty(\mathbb{R})$ with $\lim_{n\to\infty} u_0^n = u_0$ in $H_\infty(\mathbb{R})$, $\rho_0^n \in C^\infty(\mathbb{R})$ with $\lim_{n\to\infty} \rho_0^n = 0$ in $L^2_{\mathrm{const}}(\mathbb{R})$, $(\rho_0^n)^2 \geq d_n$ for some constant $d_n > 0$ for all n and $\mu_0^n = ((u_{0,x}^n)^2 + (\tilde{\rho}_0^n)^2)\,dx$. Denote by (u^n, ρ^n) the unique classical solution to the two-component Camassa–Holm system in $C^\infty(\mathbb{R}_+ \times \mathbb{R}) \times C^\infty(\mathbb{R}_+ \times \mathbb{R})$ which corresponds to this initial data. Then for every $t \in \mathbb{R}_+$, the sequence $u^n(t,\cdot)$ converges to $u(t,\cdot)$ in $L^\infty(\mathbb{R})$, where u is the conservative solution of the Camassa–Holm equation with initial data $u_0 \in H_\infty(\mathbb{R})$.

Some references

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