Selective relaxation model for general fluid systems

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joint work

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Outline

- Introduction to (numerical) relaxation, An example of selective relaxation
- Fluid models; from Lagrangian to Eulerian coordinates
- Selective relaxation of fluid models; energy dissipation and approximation results
- Relaxation scheme for a general fluid model

Introduction to relaxation

An example from physics: a simple model for a fluid mixture, composed of 2 immiscible ideal gases: HRM (homogeneous *relaxation* model)

$$\begin{cases} \partial_t m_1 + \partial_x(m_1 u) & = & \frac{1}{\lambda}(m_1^*(\varrho) - m_1) \\ \partial_t \varrho + \partial_x(\varrho u) & = & 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + P) & = & 0 \\ \partial_t(\varrho e) + \partial_x((\varrho e + P)u) & = & 0 \end{cases}$$

(m_1 partial density of fluid 1), and HEM (homogeneous *equilibrium* model), assuming instantaneous thermodynamical equilibrium

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = & 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + p) & = & 0 \\ \partial_t(\varrho e) + \partial_x((\varrho e + p)u) & = & 0. \end{cases}$$

Use the same idea for a class of (physical) equilibrium models (fluid models), introducing a (formal) relaxation model for approximation purposes.

Introduction to relaxation: a simple model

Consider a scalar conservation law

$$\partial_t u + \partial_x f(u) = 0 \tag{1}$$

with $f'' \neq 0$, and an associated *relaxation* $p-$system written

$$\begin{cases} \partial_t u^\lambda + \partial_x v^\lambda = 0 \\ \partial_t v^\lambda + \partial_x p(u^\lambda) = \dfrac{1}{\lambda}(f(u^\lambda) - v^\lambda), \end{cases} \tag{2}$$

λ relaxation time (system (2) is hyperbolic if $p' > 0$). As $\lambda \to 0$, we expect $v^\lambda - f(u^\lambda) \to 0$, and $u^\lambda \to u$ solution of (1). The set of states $(u, v = f(u))$ is called the *equilibrium manifold*.

If (2) 'converges' to (1), a natural stability condition is

$$-\sqrt{p'(u)} < f'(u) < \sqrt{p'(u)}$$

Whitham's condition: speed of *approximate* waves ($\lambda = 0$, solutions of *coarse* (1)) should lie between the fastest and slowest signals of (2) ($\lambda > 0$, speed of waves solution of *finer* description).

Introduction to relaxation: stability, convergence

Question: justify the relaxation process (prove convergence), i.e. prove $u^\lambda \to u$, as $\lambda \to 0$, where (u^λ, v^λ) (resp. u) is solution of the *relaxation* system (2) (resp. of the *equilibrium* equation (1)).
Hint on stability condition: *Chapman-Enskog* expansion.
Assume u^λ, v^λ is solution of (2) and

$$v^\lambda = f(u^\lambda) + \lambda v_1^\lambda + \mathcal{O}(\lambda^2)$$

where the term v_1^λ is a *corrector*, u^λ solution of (2) satisfies at order $\mathcal{O}(\lambda^2)$ the second order equation

$$\partial_t u + \partial_x f(u) = \lambda \partial_x \big((p'(u) - f'(u)^2) \partial_x u \big). \qquad (3)$$

(*diffusive*) under stability condition $p'(u) > f'(u)^2$.
Convergence is proved in the scalar case Chen G.Q., Liu T.P. (1994)

Introduction to numerical relaxation

The other way round: use relaxation as a tool for *numerical approximation* of CL (1), by choosing p such that the system (2) has simple Riemann solutions.
Following Jin-Xin (CPAM 1995) take $p(v) = a^2 v$, the system (2) is linear and writes

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + a^2 \partial_x u = \dfrac{1}{\lambda}(f(u) - v). \end{cases} \tag{4}$$

Idea 'relaxation of the nonlinear flux': $f(u) \sim v$, new variable, larger but simpler system
need an equation for v: is u solution of CL, $f(u)$ satisfies $\partial_t f(u) + f'(u)^2 \partial_x u = 0$.
Linearization: replace $f'(u) \sim a$,
stability requires $-a < f'(u) < a$.

Numerical relaxation scheme

For approximating the scalar CL, given initial data u_0, some grid C_j
$u_j^0 = \frac{1}{\Delta x} \int_{C_j} u_0(x)dx$, define $v_j^0 = f(u_j^0)$ (equilibrium data)
Splitting technique with
1 step: upwind scheme for LHS of (4) which is a linear system with explicit solution
2nd step: relaxation $\lambda \to 0$ (projection on the equilibrium manifold) provides a scheme for u:

$$u_j^{n+1} = u_j^n - \frac{\mu}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\mu}{2} a(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$\mu = \Delta t / \Delta x$; Rusanov scheme optimizes a under the stability constraint:

$$a = \sup_j |f'(u_j^n)|$$

Jin-Xin relaxation scheme for systems

Straightforward extension for a system of n equations: $\mathbf{u} \in \mathbb{R}^n$,

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0} \tag{5}$$

gives a relaxation system of $2n$ equations: $\mathbf{v} \in \mathbb{R}^n$,

$$\left\{ \begin{array}{l} \partial_t \mathbf{u} + \partial_x \mathbf{v} = \mathbf{0}, \\ \partial_t \mathbf{v} + \mathbf{A} \partial_x \mathbf{u} = \frac{1}{\lambda}(\mathbf{f}(\mathbf{u}) - \mathbf{v}) \end{array} \right. \tag{6}$$

$\mathbf{A} = diag(a^2)$ a constant diagonal $n \times n$ matrix with positive entries: (6) is linear. Stability condition writes with λ_j eigenvalues of $\mathbf{f}'(\mathbf{u})$

$$-a < \lambda_j(\mathbf{u}) < a$$

Same splitting technique (upwind + relaxation) gives a sceme for (5). The choice of a in Rusanov's scheme will be

$$a = \sup_{\mathbf{u}} \sup_j |\lambda_j(\mathbf{u})|.$$

Convergence to (a solution of) the equilibrium system is proved by Latanzzio-Serre (2001).

A simple example of selected relaxation

Start from the (equilibrium) $p-$system (gas dynamics with barotropic law in Lagrangian coordinates)

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x p = 0 \end{cases} \quad (7)$$

with non linear $p = p(\tau)$, $p' < 0$, $p" > 0$.
Jin-Xin: relaxation of the whole flux, doubles the size, introduces relaxation even for the 1rst *linear* equation.
With Suliciu (1998), relaxation of the 2nd *nonlinear* pressure equation only: $p \sim \Pi$

$$\begin{cases} \partial_t \tau - \partial_x u = 0 \\ \partial_t u + \partial_x \Pi = 0 \\ \partial_t \mathcal{T} = \frac{1}{\lambda}(\tau - \mathcal{T}) \end{cases} \quad (8)$$

$\Pi = \Pi(\tau, \mathcal{T}) = p(\mathcal{T}) + a^2(\mathcal{T} - \tau)$: *linearization* of the pressure, since $\partial_\tau \Pi = a^2$;$\mathcal{T}$ extended volume fraction; λ relaxation time.
Formally when $\lambda \to 0$, $\tau - \mathcal{T} \to 0$, $\Pi \to p$, we recover (7).

A simple example of selected relaxation

System (8) has only LD fields and the Riemann problem has an explicit solution.
Godunov's scheme leads to an type HLL type scheme for (7).

Stability condition $a^2 > -p'(\tau)$ (Whitham's or subcharacteristic)
Convergence is proved (Chalons-Coulombel, 2008), using Yong's approximation result.

Goal: extend this approach to

- full Euler system in Lagrangian coordinates
- more general fluid systems (Després 01)
- with justification of the relaxation procedure (Yong 99, 04)
- derive a scheme in Eulerian coordinates thanks to an equivalence result: Eulerian $\leftrightarrow$ Lagrangian description

(Wagner 87, Peng 07)

Extension: full Euler system

First point is easy, but there is a trick ! Since

$$\partial_t s = 0 \tag{9}$$

work with the entropy and introduce relaxation only for the pressure: $p(\tau, s) \sim \Pi(\tau, \mathcal{T}, s)$, and just add equation (9) to (8): the relaxation system has now 4 LD fields.
Define as 'entropy' some extended energy Σ which is *conserved* (only LD fields).
For the numerical scheme, use some minimization principle on Σ and a monotonicity argument.
The same for a general fluid model.

from Lagrangian to Eulerian

From

$$\begin{cases} \partial_t \tau - \partial_y u & = 0 \\ \partial_t u + \partial_y \Pi & = 0 \\ \partial_t s & = 0 \\ \partial_t \mathcal{T} & = \frac{1}{\lambda}(\tau - \mathcal{T}) \end{cases}$$

we get the relaxation system in Eulerian coordinates (Wagner, 87)

$$\begin{cases} \partial_t \varrho + \partial_x(\varrho u) & = 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \Pi) & = 0 \\ \partial_t \varrho s + \partial_x(\varrho s u) & = 0 \\ \partial_t(\varrho \mathcal{T}) + \partial_x(\varrho \mathcal{T} u) & = \frac{1}{\lambda}\varrho(\tau - \mathcal{T}) \end{cases} \tag{10}$$

Extends to general systems (Peng, 07): general formalism

$$\partial_t u_1 + \partial_x f_1(U) = 0 \quad \text{gives} \quad \partial_t \frac{1}{\tilde{u}_1} - \partial_y \frac{f_1(\tilde{U})}{\tilde{u}_1} = 0, \text{ with}$$

$dy = u_1 dx - f_1(U)dt$ and assuming $0 < m \leq u_1(x,t) \leq M$ (away from void). Idem for following and for entropy equation.

It leads to a one-to-one correspondence between smooth and weak solutions of Eulerian/Lagrangian formulations.

Extension to fluid systems (Lagrangian)

What works for Euler system works for a general $n \times n$ fluid system $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}$, with $\mathbf{U} = (U_1, \dots U_n)^T$, x Lagrangian coordin.
Assumptions (Després, 01): $n = r + d + 1$, split $\mathbf{U}$ in

- d *velocity* variables $\mathbf{u}$, $r = n - d - 1$ *state* variables $\mathbf{v}$,
$U_n \equiv e = \frac{1}{2}|\mathbf{u}|^2 + \epsilon$
- s and $\epsilon = \epsilon(\mathbf{v}, s)$ are *state* variables and $\partial_t s(\mathbf{U}) = 0$, $s_e(\mathbf{U}) < 0$
- Galilean invariance and reversiblity (smooth solutions)

Then it can also be written (equivalent for smooth solutions)

$$\left\{ \begin{array}{l} \partial_t \mathbf{v} - N \partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \epsilon_{\mathbf{v}}(\mathbf{v}, s) = \mathbf{0} \\ \partial_t s = 0 \end{array} \right. \qquad (11)$$

N is a rectangular $(n - d - 1) \times d$ *constant matrix*.

ex: 1d-Euler system, $\mathbf{U} = (\tau, u, e)^T$, $\mathbf{v} = \tau$, $\epsilon_{\mathbf{v}}(\mathbf{v}, s) = -p$

ex: ideal MHD, $n = 7$, $d = 3$, $\mathbf{U} = (\tau, \tau\mathbf{B}_\perp, \mathbf{u}, e)^T$, B_x const
$\mathbf{v} = (\tau, \tau B_\perp)$, $\epsilon(\mathbf{v}) = \varepsilon(\tau) + \frac{1}{2}\tau|\mathbf{B}|^2 = \varepsilon(\tau) + \frac{1}{2}\tau B_x^2 + \frac{1}{2\tau}|\tau\mathbf{B}_\perp|^2$

Properties of fluid systems

The matrix

$$\begin{pmatrix} 0 & N \\ N^T e_{\mathbf{v},\mathbf{v}} & 0 \end{pmatrix} \tag{12}$$

has eigenvalues in pair (and possibly 0): $\pm\mu$
μ eigenvalue $\Rightarrow \mu^2$ eigenvalue of $N^T e_{\mathbf{v},\mathbf{v}} N$ and $e_{\mathbf{v},\mathbf{v}}$ is symmetric, *positive* (by assumption).
Finally the spectrum of $\mathbf{F}'(\mathbf{U})$ is real valued and *symmetric* and there is at least one 0 which is associated to $\partial_t s = 0$.

ex: Euler system, 0 and 2 eigenvalues $\pm\sqrt{\partial_\tau p(\tau, s)} = \pm c$
ex: ideal MHD, 0 and magnetosonic waves $\pm c_s$ (slow), $\pm c_f$ (fast) and Alfven waves $\pm c_a$ (LD).

Relaxation of (isentropic) fluid systems

To simplify put appart the 'trivial' entropy conservation $\partial_t s = 0$, and $\epsilon_{\mathbf{v}}(\mathbf{v}, s) = \epsilon_{\mathbf{v}}(\mathbf{v}, s_0)$ noted $\epsilon'(\mathbf{v}) = (\partial_{v_1}\epsilon, \partial_{v_2}\epsilon, \dots, \partial_{v_r}\epsilon)$.
Treat the *nonlinear* second block only: $N^T \partial_x \epsilon'(\mathbf{v})$

$$\begin{cases} \partial_t \mathbf{v} - N\partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \mathcal{W} = \mathbf{0} \\ \partial_t \mathcal{V} = \frac{1}{\lambda}(\mathbf{v} - \mathcal{V}) \end{cases} \tag{13}$$

with $\epsilon'(\mathbf{v}) \sim \mathcal{W}$

$$\mathcal{W} = \mathcal{W}(\mathbf{v}, \mathcal{V}) = \epsilon'(\mathcal{V}) + \theta'(\mathbf{v}) - \theta'(\mathcal{V}), \tag{14}$$

Formally, as $\lambda \to 0$, $\mathcal{V} - \mathbf{v} \to 0$, $\mathcal{W} - \epsilon'(\mathbf{v}) \to 0$: equilibrium.
Choice of θ? ensure hyperbolicity by θ convex and $\theta''(\mathcal{V}) > 0$
Ex: for Euler system, $\mathbf{v} = \tau \sim \mathcal{V} = \mathcal{T}$, $\mathcal{W} = -\Pi$
$= -p(\mathcal{T}) + a^2(\tau - \mathcal{T})$, θ quadratic, $\theta'' = a^2 > -p'(\tau) = \epsilon''(\tau) > 0$ is a stability condition.

Relaxation of fluid systems: choice of θ

Prove stability of the relaxation procedure as $\lambda \rightarrow 0$ with a similar stability condition: $\theta''(\mathcal{V}) - \epsilon''(\mathcal{V}) > 0$ (on some subset of $\mathcal{V}$).
Entropy dissipation: define Σ (entropy = energy)
$\Sigma(\mathbf{v}, \mathbf{u}, \mathcal{V}) = \frac{1}{2}|\mathbf{u}|^2 + \epsilon(\mathcal{V}) + \theta(\mathbf{v}) - \theta(\mathcal{V}) + ((\epsilon' - \theta')(\mathcal{V}), \mathbf{v} - \mathcal{V})$.

$$\partial_t \Sigma - \partial_x(\mathbf{u}, N^T \mathcal{W}) = -\frac{1}{\lambda}(\theta''(\mathcal{V}) - \epsilon''(\mathcal{V}))(\mathbf{v} - \mathcal{V}), \mathbf{v} - \mathcal{V})$$

Remark
If θ'' constant matrix, system (13) has only LD fields and the Riemann problem has an explicit solution (ex: $\theta'' = diag(a^2)$, resembles Jin-Xin).
Numerical scheme for (11): Godunov scheme for (13) with projection on the equilibrium manifold

Relaxation of fluid systems: stability

Σ convex, strictly convex on equilibrium manifold, perhaps not on whole set of $\mathcal{V}$ thus entropy extension condition (Chen-Lev.-Liu) is not satisfied. Look for reduced conditions (Bouchut, 2005).

Chapman-Enskog expansion: identify a first order corrector term $\mathcal{V}_\lambda = \mathbf{v}_\lambda + \lambda \mathcal{V}_\lambda^{(1)} + \ldots$ plug it in the equations, retaining the first order term $\mathcal{O}(\lambda)$, we obtain if $(\mathbf{v}_\lambda, \mathbf{u}_\lambda)$ is solution of (13), it satisfies (at least formally) at order $\mathcal{O}(\lambda^2)$

$$\begin{cases} \partial_t \mathbf{v} - N \partial_x \mathbf{u} = \mathbf{0} \\ \partial_t \mathbf{u} - N^T \partial_x \epsilon'(\mathbf{v}) = \lambda \partial_x (N^T (\theta''(\mathbf{v}) - \epsilon''(\mathbf{v})) N \partial_x \mathbf{u}) \end{cases} \tag{15}$$

implies subcharacteristic condition (Bouchut, 2005) and weak entropy inequalities.

Relaxation of fluid systems: approximation

With a little more work, using Yong (04), Yong (99):

• existence of a (global) solution $\mathbf{U}_\lambda = (\mathbf{v}, \mathbf{u}, \mathcal{V})_\lambda$ with initial data $\mathbf{U}_0 \in H^s(\mathbb{R})$ near a constant equilibrium data $\mathbf{U}_{eq}$ and stability results for $||\mathbf{U}_\lambda(., T) - \mathbf{U}_{eq}||_s$ in terms of $||\mathbf{U}_0 - \mathbf{U}_{eq}||_s$

• data $\mathbf{U}_0 = (\mathbf{v}, \mathbf{u}, \mathcal{V})_0 \in H^{s+2}(\mathbb{T})$ near an equilibrium; existence of a (local in time) solution $\mathbf{U} = (\mathbf{v}, \mathbf{u}, \mathcal{V})_\lambda$ and existence of a (local in time) solution $(\bar{\mathbf{v}}, \bar{\mathbf{u}})$ of the equilibrium system with init. data $(\mathbf{v}, \mathbf{u})_0$
and convergence results as $\lambda \to 0$ of $(\mathbf{v}, \mathbf{u})_\lambda$ to $(\bar{\mathbf{v}}, \bar{\mathbf{u}})$ in $C([0, T), \mathbb{H}^s(\mathbb{T}))$ and $\mathcal{V}_\lambda$ converges to $\bar{\mathbf{v}}$ in $\mathbb{L}^1(0, T; \mathbb{H}^s)$

Relaxation scheme

For a numerical scheme: take θ'' constant matrix, system (13) has only LD fields and the Riemann problem has an explicit solution (ex: $\theta'' = diag(a^2)$, or $diag(a_i^2)$).
Use exact Godunov solver: keep good properties (entropy), after relaxation step, it results in a simple Riemann solver (HLL-type) for the fluid system.

The theoretical results explain why the scheme has good properties (robustness, stability, entropy satisfying).

For Euler, very simple scheme.
For ideal MHD, it resembles the scheme derived by Bouchut et al (2007,10)

Some further extensions, prospects.

Thank you for your attention.