On Cavitation in Elastodynamics

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Outline

- Introduction
- 1D: Concept for discontinuous solutions to second order equations
- 1D: Energy of solutions
- 3D: Solution concept and energy
- Summary & Prospects

Introduction

- Study fracture and cavitation in elastical solids.
- Once a bar breaks, or a hole forms, coherence is lost and the hypothesis of continuum implicit in the elastodynamics equations is no longer valid.
- Transition region from a range of loading where the model is valid to a range where it loses validity.
- Explore this transition at the onset of fracture.
- Irregular solutions to the compressible, non-linear elastodynamics equations, where a hole forms in the interior.
- i.e. a discontinuity of the displacement field at one point.
- Study the energy of the irregular solution compared to trivial solutions (which also exist).

Introduction

Equations of elastodynamics: Search displacements

$$\mathbf{y} : B_1(0) \times [0, T) \longrightarrow \mathbb{R}^d \quad \text{such that } \det(\nabla \mathbf{y}) > 0.$$

satisfying the wave equation

$$\mathbf{y}_{tt} - \operatorname{div}(\tau(\nabla \mathbf{y})) = 0 \qquad \text{(WAVE)}$$

with stress response

$$W : \mathbb{R}^{d\times d}_{+} \longrightarrow \mathbb{R}; \quad \tau = \frac{\partial W}{\partial F} : \mathbb{R}^{d\times d}_{+} \longrightarrow \mathbb{R}^{d\times d}_{+}.$$

The wave equation is equivalent to the following system of first order conservation laws with $\mathbf{u} = \nabla \mathbf{y}$, $\mathbf{v} = \mathbf{y}_t$:

$$\begin{aligned} \mathbf{u}_t - \nabla \mathbf{v} &= 0 \\ \mathbf{v}_t - \operatorname{div}(\tau(\mathbf{u})) &= 0. \end{aligned} \qquad \text{(CONS)}$$

Energy and admissibility

Smooth solutions of (WAVE) satisfy the energy equality

$$\frac{d}{dt}\left(\frac{1}{2}(\mathbf{y}_t)^2 + W(\nabla \mathbf{y})\right) - \operatorname{div}\left(\tau(\nabla \mathbf{y})\mathbf{y}_t\right) = 0.$$

For weak solutions (having discontinuities in $\nabla \mathbf{y}$, $\mathbf{y}_t$) the classical admissibility criterion is the energy inequality

$$\frac{d}{dt}\left(\frac{1}{2}(\mathbf{y}_t)^2 + W(\nabla \mathbf{y})\right) - \operatorname{div}\left(\tau(\nabla \mathbf{y})\mathbf{y}_t\right) \leqslant 0,$$

which has to be satisfied in a weak sense.

Solutions with cavitation

We study solutions of the form

$$\mathbf{y}(\mathbf{x},t) = t\varphi\left(\frac{|\mathbf{x}|}{t}\right)\frac{\mathbf{x}}{|\mathbf{x}|} \qquad \text{(ANSATZ)}$$

for some $\varphi : [0,\infty) \to [0,\infty)$, with $\varphi(0) > 0$, $\varphi'(s) > 0 \,\forall\, s > 0$.

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In 1D: these solutions describe discontinuous shear or fracture,
in 3D: such solutions describe opening holes/cavities.

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Displacement initial boundary value problem

$$\mathbf{y}(\mathbf{x},0) = \lambda\mathbf{x},\ \mathbf{y}_t(\mathbf{x},0) = 0 \quad \forall \mathbf{x} \in B_1(0),$$
$$\mathbf{y}(\mathbf{x},t) = \lambda\mathbf{x} \quad \forall \mathbf{x} \in \partial B_1(0) \text{ and } 0 \leqslant t < T$$

for some $\lambda > 0$.

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for some $\lambda > 0$.

Note: There is the trivial solution $\mathbf{y}(\mathbf{x},t) = \lambda\mathbf{x}$ to this IBVP.

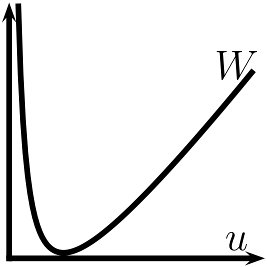
Situation in 1D

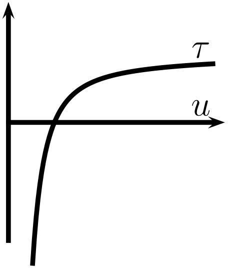
Ansatz

$$y(x,t) = tY\left(\frac{x}{t}\right) \text{ with } Y(-\xi) = -Y(\xi),\ Y'(\xi) > 0\,\forall\,\xi > 0,\ \lim_{\xi\to 0, \xi>0} Y(\xi) > 0.$$

Then (WAVE) amounts to

$$\xi^2\, Y'' = (\tau(Y'))'$$





We impose the conditions $W'' > 0$ and $W''' < 0$.

Rankine Hugoniot conditions and admissibility

$$\begin{aligned} y_t(x,t) &= Y(\xi) - \xi\, Y'(\xi) =: V(\xi), \\ y_x(x,t) &= Y'(\xi) =: U(\xi) \end{aligned} \qquad \text{where } \xi = \frac{x}{t}$$

We can write down the equations for U, V

$$\begin{aligned} \xi\, U' + V' &= 0 \\ \xi\, V' + (\tau(U))' &= 0 \end{aligned} \qquad (*)$$

For a shock with speed σ the Rankine–Hugoniot conditions read

$$-\sigma[U] = [V] \quad \text{and} \quad -\sigma[V] = [\tau(U)] \quad \Longrightarrow \sigma = \sqrt{\frac{[\tau(U)]}{[U]}}.$$

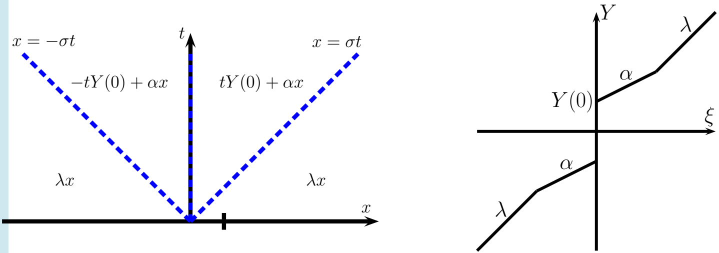
The eigenvalues of $(*)$ are $\pm\sqrt{\tau'(U)}$ and the Lax condition becomes

$$U_l > U_r \quad \text{for } 1-\text{shocks} \quad \text{and} \quad U_l < U_r \quad \text{for } 2-\text{shocks}.$$

One class of possible solutions

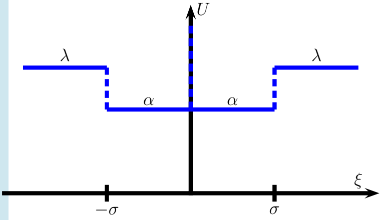
Thus, we investigate

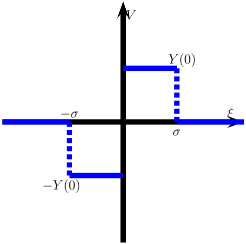
$$Y(\xi) := \begin{cases} Y(0) + \alpha\xi & : \quad 0 < \xi < \sigma \\ -Y(0) + \alpha\xi & : \quad -\sigma < \xi < 0 \\ \lambda\xi & : \quad |\xi| > \sigma \end{cases} \qquad \text{(1D-ANSATZ)}$$



One class of possible solutions

$$U = 2Y(0)\delta_{\xi=0} + \alpha\chi_{\{|\xi|<\sigma\}} + \lambda\chi_{\{|\xi|>\sigma\}}$$
$$V = Y(0)\chi_{\{0<\xi<\sigma\}} - Y(0)\chi_{\{-\sigma<\xi<0\}}$$





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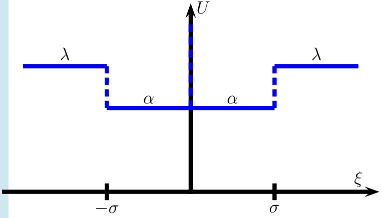

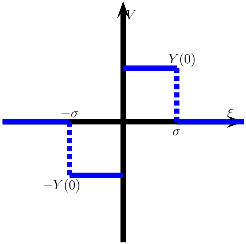

What is the meaning of $\tau(U)$ in this case?

slic–solutions

We propose the following notion of solution

Definition

We call $y \in C([0,T), L^p(B_1(0)))$ a **s**ingular **l**imiting **i**nduced from **c**ontinuum (slic)–solution provided for all $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp}(\psi) \subset [-1,1], \quad \int_{-1}^{1} \psi(x)dx = 1, \quad \text{and} \quad \psi(x) = \psi(-x)$$

the following holds

$$\lim_{n\to\infty} [y^n_{tt} - (\tau(y^n_x))_x] = 0 \quad \text{in } \mathcal{D}'$$

where

$$y^n(x,t) = y *_x n\psi(n\cdot).$$

δ-shocks

- This problem is conected to the notion of δ-shocks for hyperbolic conservation laws. There is a vast literature on this field, see the work of Danilov, Shelkovic.
- In contrast, our notion of solution is based on the underlying structure of the second order problem.
- For the p-system in fluid mechanics we have $\lim_{U\to\infty} p(U) = 0$ such that $(p(\delta_{x=0}))(0) = 0$ seems reasonable. See chapter 9.7 in Dafermos' book.

Are there slic–solutions?

Lemma

Let $y \in H^1([0,T], L^2(B_1(0))) \cap L^2([0,T], H^1(B_1(0)))$ *with* $\operatorname{essinf} y_x > 0$ *satisfy*

$$\int_0^T \int_{B_1(0))} y_t \varphi_t - \tau(y_x)\varphi_x \, dxdt = 0 \quad \forall \varphi \in C_0^1((0,T) \times B_1(0)),$$

then y *is a slic–solution. Thus, slic–solutions generalize standard weak solutions.*

Lemma

A function y *given by (1D-ANSATZ) with* $Y(0) + \alpha\sigma = \lambda\sigma$ *is a slic–solution if and only if*

$$Y(0) = \frac{\tau(\lambda) - \tau(\alpha)}{\sigma} \qquad \text{and} \qquad \lim_{u \to \infty} \frac{\tau(u)}{u} = 0.$$

Energy of slic–solutions

We define the energy at time t of a slic–solution y to be

$$E(y,t) := \lim_{n\to\infty} \int_{B_1(0)} \frac{1}{2}(y_t^n(x,t))^2 + W(y_x^n(x,t))\,dx.$$

Because of l'Hospitals Theorem it holds

$$\lim_{u\to\infty} \frac{W(u)}{u} = \lim_{u\to\infty} \tau(u).$$

Lemma

In case $\lim_{u\to\infty} \tau(u) = \infty$ *a slic–solution* y *given by (1D-ANSATZ) satisfies* $E(y,t) = \infty$ *for all* $t > 0$ *and* $E(y,0) < \infty$.

Solutions with finite energy

Lemma

In case $\tau_\infty = \lim_{u\to\infty} \tau(u) = \lim_{u\to\infty} \tau(u) < \infty$ *slic–solutions given by (1D-ANSATZ) satisfy*

$$E(y,t) = \underbrace{2W(\lambda)}_{\textit{initial energy}} \underbrace{-2t\sigma W(\lambda) + t\sigma Y(0)^2 + 2t\sigma W(\alpha) + 2tY(0)\tau(\alpha)}_{\textit{energy dissipation at the shock}} + \underbrace{2Y(0)t(\tau_\infty - \tau(\alpha))}_{\textit{energy of the cavity}}.$$

Sketch of the proof:

- Decompose the integral over right half of the wave fan into 4 parts

- For $\frac{1}{n} < |x| < \sigma t - \frac{1}{n}$

$$|y_t^n| = Y(0), \qquad W(y_x^n) = W(\alpha).$$

Solutions with finite energy

Sketch of the proof, continued:

- For $\sigma t - \frac{1}{n} < |x| < \sigma t + \frac{1}{n}$

$$|y_t^n| \leqslant Y(0), \qquad |W(y_x^n)| \leqslant \max_{u \in [\alpha, \lambda]} |W(u)|.$$

- For $\sigma t + \frac{1}{n} < |x|$

$$|y_t^n| = 0, \qquad W(y_x^n) = W(\lambda).$$

- For $|x| < \frac{1}{n}$

$$|y_t^n| \leqslant Y(0), \qquad W(y_x^n(x)) = W\left(\alpha + 2tY(0)n\psi\left(\frac{x}{n}\right)\right)$$

$$\lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{1}{2} Y(0)^2 + W\left(\alpha + 2tY(0)n\psi\left(\frac{x}{n}\right)\right)\, dx$$
$$= \lim_{n\to\infty} \int_{-1}^{1} \frac{1}{n} W\left(\alpha + 2tY(0)n\psi\left(\tilde{x}\right)\right)\, d\tilde{x} = 2tY(0)\tau_\infty.$$

For (1D-ANSATZ) solutions the energy increases

Lemma

In case $\tau_\infty = \lim_{u\to\infty} \frac{W(u)}{u} = \lim_{u\to\infty} \tau(u) < \infty$ *all slic–solutions given by (1D-ANSATZ) satisfy*

$$\frac{d}{dt} E(y,t) > 0$$

and

$$\int_{B_1(0)} (\tau(y_x^n) y_t^n)_x \, dx = 0.$$

for n *sufficiently large.*

Sketch of the proof:

- As y^n is smooth

$$\int_{B_1(0)} (\tau(y_x^n) y_t^n)_x \, dx = \tau(y_x^n(1,t)) y_t^n(1,t) - \tau(y_x^n(-1,t)) y_t^n(-1,t)$$

and

$$y_t^n(1,t) = y_t^n(-1,t) = 0.$$

For (1D-ANSATZ) the energy increases

- Recall:
  $E(y,t) = (2-2t\sigma)\,W(\lambda) + t\sigma\,Y(0)^2 + 2t\sigma\,W(\alpha) + 2\,Y(0)\,t\tau_\infty.$
-

$$\frac{d}{dt}E(y,t) = -2\sigma\,W(\lambda) + \sigma\,Y(0)^2 + 2\sigma\,W(\alpha) + 2\,Y(0)\tau_\infty$$

$$\Longrightarrow \frac{d}{dt}E(y,t) = \underbrace{\sigma\,Y(0)^2}_{>0} - 2\underbrace{\sigma(\lambda-\alpha)}_{=Y(0)}\underbrace{\frac{W(\lambda)-W(\alpha)}{\lambda-\alpha}}_{=W'(\bar{u})=\tau(\bar{u})} + 2\,Y(0)\tau_\infty$$

$$\Longrightarrow \frac{d}{dt}E(y,t) > -2\,Y(0)\tau(\bar{u}) + 2\,Y(0)\tau_\infty > 0$$

as $\tau' = W'' > 0.$

Energies in $d = 3$ dimensions

We consider

$$\mathbf{y}_{tt} - \operatorname{div}(\tau(\nabla \mathbf{y})) = 0; \qquad \tau = \frac{\partial W}{\partial F}.$$

We assume that $W : \mathbb{R}^{3\times 3}_{+} \longrightarrow \mathbb{R}$ is frame indifferent, isotropic and polyconvex, i.e.

$$W(F) = \Phi(\lambda_1, \lambda_2, \lambda_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\sqrt{FF^T}$ and

$$\Phi : \{x \in \mathbb{R}^3 : x_i > 0 \ \forall \ i\} \longrightarrow \mathbb{R}$$

is symmetric. We will consider energy densities of the form

$$W(F) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) + h(\lambda_1 \lambda_2 \lambda_3)$$

where $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies $h'' > 0, h''' < 0$.

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where

$$\mathbf{y}^n(x,t) = t\varphi_n\left(\frac{|\mathbf{x}|}{t}\right)\frac{\mathbf{x}}{|\mathbf{x}|}$$

and

$$\varphi_n = \int_0^T \int_0^\infty n\psi\left(\frac{s-\tilde{s}}{n}\right)\varphi(\tilde{s})\,d\tilde{s} - \int_0^T \int_{-\infty}^0 n\psi\left(\frac{s-\tilde{s}}{n}\right)\varphi(-\tilde{s})\,d\tilde{s}.$$

Observe that with our regularization $\nabla \mathbf{y}^n$ and $\mathbf{y}_t^n$ only depend on s.

Are the existing weak solutions slic–solutions?

- Pericak-Spector and Spector '88 constructed weak solutions of (WAVE) with cavity.
- In their work the cavity itself does not contribute to the energy.
- It just makes shocks possible, which in turn dissipate energy.

Lemma

The solutions constructed by Pericak-Spector and Spector are

- *slic–solutions provided*

$$\lim_{v\to\infty} \frac{h'(v)}{\sqrt[3]{v}} = 0.$$

- *not slic–solutions in case*

$$\liminf_{v\to\infty} \frac{h'(v)}{\sqrt[3]{v}} > 0.$$

There are intermediate cases here, where the answer is not clear.

Energy of the existing solutions

Lemma With our definition of energy the solutions constructed by Pericak-Spector and Spector satisfy

$$E(\mathbf{y},t) = \begin{cases} \infty & \text{for } \lim_{v\to\infty} \frac{h(v)}{v} = \infty \\ \underbrace{\frac{3}{2}\lambda^2 + h(\lambda^3)}_{\text{initial energy}} + \underbrace{\frac{t^3\sigma^d 4\pi}{3} J}_{\text{energy dissipated at the shock}} + \underbrace{\frac{t^3 4\pi}{3}\varphi(0)^3 L}_{\text{energy of the cavity}} \\ \text{for } L := \lim_{v\to\infty} \frac{h(v)}{v} < \infty \end{cases}$$

$$\text{with } J = \frac{1}{2}(\varphi'^-)^2 + h(\varphi'^-\lambda^2) - \frac{1}{2}\lambda^2 - h(\lambda^3) + \frac{1}{2}\Big(\varphi'^- + h'(\varphi'^-\lambda^2)\lambda^2 + \lambda + h'(\lambda^3)\lambda^2\Big)(\lambda - \varphi'^-)$$

where $\varphi'^- = \lim_{s\to\sigma, s<\sigma} \varphi'(s)$.
In particular: For the known weak solutions the energy is increasing.

Summary and Prospects

Summary:

- New solution concept for solutions including δ-shocks.
- Notion of energy for these solutions.
- We constructed solutions containing δ-shocks.
- For these solutions the energy increases in time.
- Similar situation in 3d.

Prospects:

- Can we use slic–solutions to describe vacuum in gas-dynamics?
- Are there slic–solutions with cavity, which dissipate energy?

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Thank you for your attention!