Domain continuity
for the
Euler and Navier-Stokes equations

(Based on joint works with C. Lacave and A.L. Dalibard)

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1. Domain continuity

Starting point: Fluid in a domain Ω^n, parametrized by $n \gg 1$.

Assumption:

$$\boxed{\Omega^n \to \Omega \quad \text{in a suitable topology}}$$

Question: Does the fluid have an asymptotic behavior ? Namely:

- Does the fluid velocity field $u^n \to u$ in a suitable sense ?
- Does u satisfy the same equations as the u^n's ?
- Does u satisfy the same conditions at $\partial\Omega$ as the u^n's at $\partial\Omega^n$?

Plan: To adress this kind of questions, for Euler and Navier-Stokes.

2. The 2D Euler equation in non-smooth sets

Let $\Omega \subset \mathbb{R}^2$ an open set.

$$\boxed{\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \quad u \cdot \nu|_{\partial\Omega} = 0 \end{cases}} \tag{E}$$

<u>Aim</u>: To solve (E) with minimal requirements on Ω.

We focus on the construction of weak solutions, inspired by the whole space case.

<u>Here</u>: weak solutions with vorticity in $L^p(\Omega)$, $p > 1$.

<u>Problem:</u> Global weak solutions for general Ω ?

Until recently, limited results.

- *Most works deal with $C^{1,1}$ boundaries.*

  [Wolibner,33], [Yudovitch,63], [Kato,67], [Bardos,72] ...

  Reason: Use of the Biot and Savart law: $\boxed{u = \nabla^\perp \Delta^{-1} \omega.}$

  Regularizing effect of $\nabla^\perp \Delta^{-1}$ weakens in non-smooth sets.

- Convex domains [Taylor,00]

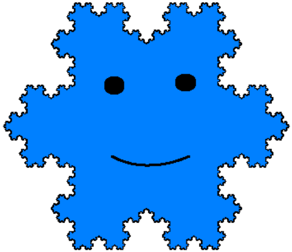
  $\longrightarrow\ \Delta^{-1} : L^2(\Omega) \mapsto H^2(\Omega).$

  $\longrightarrow$ weak solutions with vorticity in $L^p(\Omega),\ \ p > 2.$

- Exterior of a smooth Jordan curve [Lacave, 09]

  Relies on an explicit Biot and Savart law, through conformal mapping. The smoothness of the curve is needed.

Objective: To go beyond such specific cases.



$$\boxed{\Omega := \tilde{\Omega} \setminus \cup_{i=1}^{k} \mathcal{C}^i, \quad k \in \mathbb{N},}$$ with

(A1) (Connectedness):

- $\tilde{\Omega}$ bdd simply connected domain.
- $\mathcal{C}^i$'s connected compact subsets.

(A2) (Positive H^1 capacity):

for all $i = 1, \ldots, k$, $\mathrm{cap}(\mathcal{C}^i) > 0$.

Definition: $E \subset \mathbb{R}^N$:

$$\boxed{\mathrm{cap}(E) := \inf\{ \|v\|_{H^1(\mathbb{R}^N))}, \; v \geq 1 \textit{ a.e. in a neighborhood of } E\}}$$

Very roughly:

$\mathrm{cap}(E) \approx \mathrm{Leb}(E) + (n-1)$-dimensional "measure" of ∂E.

Remarks:

1. The $\mathcal{C}^i$'s can be of positive measure, smooth curves ...
2. $k = 0$: any bounded simply connected domain.

Theorem: *Let Ω satisfying (A1)-(A2), $p > 1$, and*

$$u_0 \in L^2(\Omega), \quad \omega_0 \in L^p(\Omega), \quad \textit{with} \int_\Omega u_0 \cdot \nabla \psi_0 = 0, \quad \forall \psi_0 \in C^1_c(\mathbb{R}^2).$$

Then, there exists $u = u(t,x)$ such that

$$u \in L^\infty(\mathbb{R}_+; L^2(\Omega)), \quad \omega = curl\, u \in L^\infty(\mathbb{R}_+; L^p(\Omega))$$

$$\textit{with} \boxed{\int_{\mathbb{R}_+} \int_\Omega u \cdot \nabla \psi = 0, \quad \forall \psi \in \mathcal{D}([0, +\infty[; C^1_c(\mathbb{R}^2))},$$

and s.t. for all $\varphi \in \mathcal{D}([0, +\infty[\times \Omega)$ with $div\, \varphi = 0$,

$$\boxed{\int_{\mathbb{R}_+} \int_\Omega (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi) = - \int_\Omega u_0 \cdot \varphi(0, \cdot).}$$

Ideas of proof

<u>Basic idea:</u> Smoothing procedure

- Approximate Ω by smooth Ω^n, u_0 by smooth u_0^n.
- u^n, solution of Euler in Ω^n "$\rightarrow$" u, solution of Euler in Ω.

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a) Approximation of Ω:

Lemma: *There exists sequences $\left(\tilde{\Omega}^n\right)$ and $\left(O^{i,n}\right)$ of smooth Jordan domains such that*

$$\boxed{\Omega \;=\; \lim_{H} \Omega^n, \quad \Omega^n = \tilde{\Omega}^n \setminus \cup_{i=1}^{k} \overline{O}^{i,n}}$$

Definition: *Let (Ω^n) be a sequence of confined open sets in $\mathbb{R}^N$, B a compact set with $\Omega^n \subset B$ for all n.*

$$\boxed{\Omega^n \xrightarrow{H} \Omega \quad \text{if} \quad d_H(B \setminus \Omega^n, B \setminus \Omega) \rightarrow 0.}$$

Sketch of Proof.

- Approximation of $\tilde{\Omega}$ by $\tilde{\Omega}^n$: use conformal mapping.

- Approximation of $\mathcal{C}^i$ by $O^{i,n}$:

  - One approximates $\mathcal{C}^i$ by a finite union of disks.

  - One makes slits in this union of disks to get it simply connected.

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b) Weak compactness. Continuity of the tangency condition

First ingredient: Explicit Hodge decomposition.

The field u^n reads

$$\boxed{u^n(t,x) = \nabla^\perp \left(\psi^{0,n}(t,x) + \sum_{i=1}^{k} \alpha^{i,n}(t) \psi^{i,n}(x) \right)}$$

with

- ψ_n^0 (rotational part) satisfying

$$\boxed{\Delta\psi^{0,n} = \omega^n \quad \text{in } \Omega^n, \quad \psi^{0,n}|_{\partial\Omega^n} = 0.}$$

- ψ_n^i, $i \geq 1$ (harmonic part) satisfying

$$\boxed{\Delta\psi^{i,n} = 0 \quad \text{in } \Omega^n, \quad \psi^{i,n}|_{\partial\tilde{\Omega}^n} = 0, \quad \psi^{i,n}|_{\partial O^{j,n}} = \delta_{ij}.}$$

<u>Questions</u>:

- Bounds on stream functions $\psi^{i,n}$, $i \geq 0$, and coeffts $\alpha^{i,n}$?
- $\partial_\tau \psi^{i,n}|_{\partial\Omega^n} = 0 \Rightarrow \partial_\tau \psi^i|_{\partial\Omega} = 0$?

Second ingredient: γ-convergence of open sets.

Definition: $\Omega^n \Subset D$. *We note* $\Omega^n \xrightarrow{\gamma} \Omega$ *if the solution* v^n *of*

$$\Delta v^n = 1 \quad in\ \Omega^n, \quad v^n|_{\partial\Omega^n} = 0$$

converges in $H_0^1(D)$*(after extension by* 0*) to the solution* v *of*

$$\Delta v = 1 \quad in\ \Omega, \quad v|_{\partial\Omega} = 0.$$

Remark: Equivalent to the Γ-convergence of the associated Dirichlet functionals.

Remark: γ-convergence means domain continuity of elliptic equations and Dirichlet conditions.

Proposition (Sverak): *If the number of connected components of* $D \setminus \Omega^n$ *is uniformly bounded in* n*, then*

$$\Omega^n \xrightarrow{H} \Omega \quad \Rightarrow \quad \Omega^n \xrightarrow{\gamma} \Omega.$$

Allows to handle the asymptotic boundary behavior of $\psi^{i,n}$.

Question: Coefficients $\alpha^{i,n}$ in the decomposition ?

Broadly: they satisfy a linear system with

- a source term involving the circulations of u^n around the $\mathcal{O}_n^i$.
- a matrix close to $(\int_{\Omega^n} \nabla\psi^{i,n} \cdot \nabla\psi^{j,n})_{i,j}$

Broadly:

- The source is controlled thanks to Bernoulli's theorem.
- The matrix is controlled by (A2).

$$\mathrm{cap}(\mathcal{C}^i) > 0 \;\Rightarrow\; (\int_\Omega \nabla\psi^i \cdot \nabla\psi^j)_{i,j} \text{ non-singular} \;\Rightarrow\; \alpha^{i,n} \to \alpha^i$$

c) Asymptotics of the momentum equation in Euler

Main point: No Aubin-Lions lemma in Ω^n. What about $\Omega' \Subset \Omega$?

Not really ! Roughly, one has

$$P_{\Omega'} u^n \quad \text{is compact in } \; L^q(0, T \times \Omega'), \quad \text{for some } q > 2.$$

$\longrightarrow$ harmonic functions p_h^n such that

$$\boxed{\tilde{u}^n = u^n + \nabla p_h^n \quad \text{is compact in } \; L^q(0, T \times \Omega'), \quad \text{for some } q > 2.}$$

Then, one uses an algebraic identity well-known in the theory of Navier-Stokes [Lions et al, 1998], [Woelf, 2004]:

$$\begin{aligned} \operatorname{div}(u^n \otimes u^n) = \operatorname{div}(\tilde{u}^n \otimes \tilde{u}^n) \; + \; \text{weak-strong terms} \\ + \frac{1}{2} \nabla |\nabla p_h^n|^2 \; + \; \Delta p_h^n \nabla p_h^n. \end{aligned}$$

4. Navier-Stokes equation in rough domains

Physical motivation: Microfluidics.

Goal: To make fluids flow through very small devices.

Minimizing drag at the walls is crucial.

Many theoretical and experimental works.
[Tabeling, 2004], [Bocquet, 2007 and 2012], [Vinogradova, 2012].

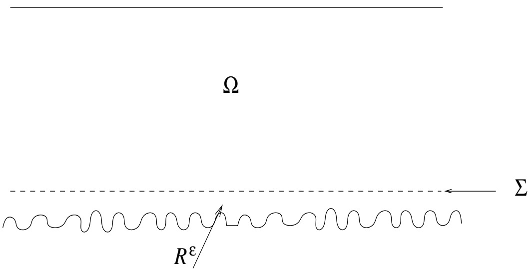
Some of these works claim that the usual no-slip condition is not always satisfied at the micrometer scale:

Some rough surfaces may generate a substantial slip.

However, these results have raised controversies . . .

... Maths may help, notably through a homogenization approach.

Simple model: *2D rough channel* : $\Omega^\varepsilon = \Omega \cup \Sigma \cup R^\varepsilon$



- Ω : smooth part: $\mathbb{R} \times (0,1)$.
- R^ε : rough part, *typical size* $\varepsilon \ll 1$.

$$\boxed{R^\varepsilon = \varepsilon R, \quad R \; = \; \{y = (y_1, y_2), \quad 0 > y_2 > \omega(y_1)\}}$$

ω with values in $(-1, 0)$, and K-Lipschitz.

- Σ : interface: $\mathbb{R} \times \{0\}$.

Stationary Navier-Stokes, with given flux:

$$\boxed{\begin{cases} u \cdot \nabla u - \Delta u + \nabla p \ = \ 0, \quad x \in \Omega^\varepsilon, \\ \operatorname{div} u = 0, \quad x \in \Omega^\varepsilon, \\ \displaystyle\int_\sigma u_1 = \phi, \end{cases}} \qquad (\mathrm{NS}^\varepsilon)$$

with $\phi > 0$, σ vertical cross-section.

Question: Can we get, for some boundary condition at $\partial\Omega^\varepsilon$, an effective (meaning *asymptotic*) slip condition at $\partial\Omega$?

Intuition: yes, at least if we consider some *pure slip* at $\partial\Omega^\varepsilon$:

$$\boxed{u \cdot \nu^\varepsilon|_{\partial\Omega^\varepsilon} = 0, \quad D(u)\nu^\varepsilon \times \nu^\varepsilon|_{\partial\Omega^\varepsilon} = 0.} \qquad (\mathrm{S})$$

Answer: No !

As soon as the roughness is "non-degenerate", any weak limit u of a sequence of solutions (u^ε) in $H^1_{loc}(\Omega)$ will satisfy $u|_{\partial\Omega} = 0$!

[Casado-Diaz et al, 03], [Bucur et al, 08].

Here: Refined result, under the non-degeneracy assumption

(A) There is $C > 0$, such that for all $u \in C_c^\infty(\overline{R})$,

$$\boxed{u \cdot \nu|_{\{y_2=\omega(y_1)\}} = 0 \quad \Rightarrow \quad \|u\|_{L^2(R)} \leq C \, \|\nabla u\|_{L^2(R)}}$$

Remarks:

- Not satisfied for flat boundaries.
- Satisfied if there is $A > 0$ such that

$$\inf_{y_1 \in \mathbb{R}} \int_0^A |\omega'(y_1 + t)|^2 dt > 0.$$

- Satisfied by non-cst periodic and quasiperiodic ω.

Theorem: *There exists $\phi_0 > 0$ such that for all $\phi < \phi^0$, $\varepsilon \leq 1$, system $(\mathrm{NS}^\varepsilon)$-(S) has a unique solution $u^\varepsilon \in H^1_{uloc}(\Omega^\varepsilon)$.*

Moreover, if (A) holds,

$$\boxed{\|u^\varepsilon - u\|_{H^1_{uloc}(\Omega)} \leq C\phi\sqrt{\varepsilon}, \quad \|u^\varepsilon - u\|_{L^2_{uloc}(\Omega)} \leq C\phi\varepsilon,}$$

where u is the Poiseuille flow in Ω (that satisfies $u|_{\partial\Omega} = 0$).

Remark: The theorem shows that the effective slip can not be more than $O(\varepsilon)$. Does not support some physics papers ...

Boundary layer analysis: under ergodicity properties of ω, one shows that the effective slip is indeed $O(\varepsilon)$.

Formal idea:

Non-vanishing of the tangential component $+$ high frequency oscillations of the boundary $\Rightarrow$ blow up of ∇u^ε as ε goes to zero.

Incompatible with the control of ∇u^ε in Navier-Stokes.