Nonlinear hyperbolic balance laws coupled with ordinary differential equations

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Joint works with R. Borsche and R. M. Colombo

PDE-ODE Model

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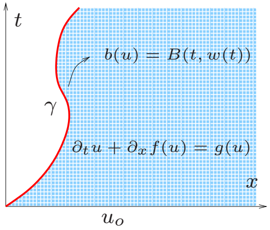

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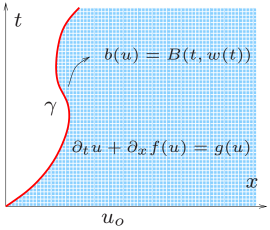

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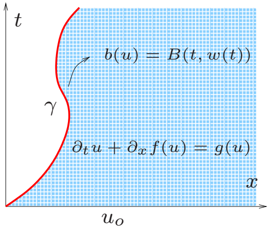

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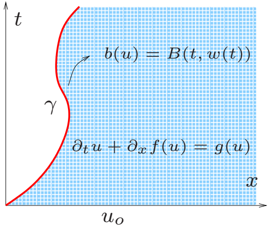

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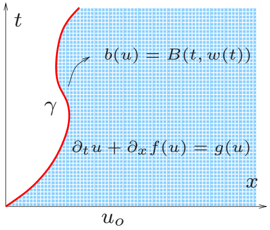

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- b "prescribes" the boundary data for waves with speed greater than $\sup \Pi + c$.

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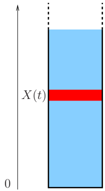
2. Lagoutière-Seguin-Takahashi and Andreianov-Lagoutière-Seguin-Takahashi: interactions between fluids and solids.

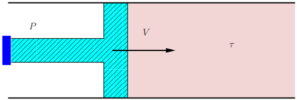
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = (h'(t) - u(t, h(t)))\delta_{h(t)}(x) \\ h''(t) = -h'(t) + u(t, h(t)) \\ u(0, x) = u_o(x) \\ (h(0), h'(0)) = (0, v_o) \end{cases}$$

Applications. 1 - Gas-particle interaction



ρ	$\rightarrow$ density
q	$\rightarrow$ linear momentum density
p	$\rightarrow$ pressure
X and V	$\rightarrow$ particle position and speed
m	$\rightarrow$ particle mass
g	$\rightarrow$ gravity

$$\begin{cases} \partial_t \rho - \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = -g\rho \\ V = v(t, 0+) \\ \dot{V} = -g - \frac{p(\rho^+) - p(\rho^-)}{m} \end{cases}$$



τ $\rightarrow$ specific volume
v $\rightarrow$ Lagrangian speed
p $\rightarrow$ pressure
V $\rightarrow$ speed of the piston
P $\rightarrow$ pressure to the left of the piston

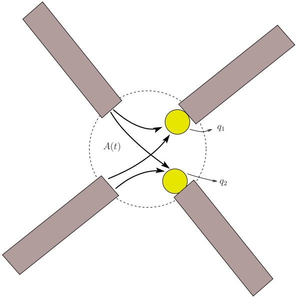
$$\begin{cases} \partial_t \tau - \partial_x v = 0 \\ \partial_t v + \partial_x p(\tau) = 0 \\ V = v(t, 0+) \\ \dot{V} = (P(t) - p\,(\tau(t, 0+))) \end{cases}$$

Applications. 3 - Moving bottleneck

ρ $\rightarrow$ traffic density
v $\rightarrow$ average speed
p $\rightarrow$ “pressure”
X $\rightarrow$ truck position

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t \left(\rho \left(v + p(\rho)\right)\right) + \partial_x \left(\rho v \left(v + p(\rho)\right)\right) = 0 \\ \ddot{X} = -\frac{1}{T^*} \left(\dot{X} - V_* \left(1 - \frac{\rho^+(t, X(t)+)}{R}\right)\right) \end{cases}$$

Applications. 4 - Supply chains



ρ_i → material density
v_i → production speed
q_i → queue load
$A(t)$ → distribution matrix
μ_k → maximal production capacity

$$\begin{cases} \partial_t \rho_i(t,x) + \partial_x \left(v_i \rho_i(t,x)\right) = 0 & i = 1,\ldots,n \\ v_k\, \rho_k(t,0+) = \min\left(\frac{q_k(t)}{\epsilon}, \mu_k\right) & k = \ell+1,\ldots,n \\ \dot q_k(t) = \sum_{j=1}^{\ell} a_{jk}(t)\, v_k\, \rho_k(t,0) - \min\left(\frac{q_k(t)}{\epsilon}, \mu_k\right) & k = \ell+1,\ldots,n\,. \end{cases}$$

Definition of solution

The triple (u, w, γ) such that $\begin{cases} u \in \mathbf{C}^0\left(\mathbb{R}^+; L^1(\mathbb{R}^+;\Omega)\right) \\ u(t) \in \mathbf{BV}(\mathbb{R}^+;\Omega) \text{ for a.e. } t > 0 \\ w \in \mathbf{W}^{1,1}(\mathbb{R}^+;\mathbb{R}^m) \\ \gamma \in \mathbf{W}^{1,\infty}(\mathbb{R}^+;\mathbb{R}) \end{cases}$

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(g) $\|g(u) - g(u')\|_{L^1} \leq L_1 \|u - u'\|_{L^1}$ and $TV(g(u)) \leq L_2$.

Theorem: Well posedness of the system

there exist $T > 0$, $L > 0$, domains $\mathcal{D}_t$ and maps $P(t, t_0) : \mathcal{D}_{t_0} \to \mathcal{D}_{t_0+t}$ such that

1. for $(u_o, w_o, x_o) \in \mathcal{D}_0$,

$$t \mapsto P(t, 0)\,(u_o, w_o, x_o)$$

   is a solution

2. for $t_0 \in [0, T[$, $t_1 \in [0, T - t_0[$, $t_2 \in [0, T - t_0 - t_1[$,

$$P(t_2, t_0 + t_1) \circ P(t_1, t_0) = P(t_1 + t_2, t_0); \qquad P(0, t_0) = Id$$

3. for $t_0 \in [0, T[$, $t \in [0, T - t_0]$, $(u, w, x), (\bar{u}, \bar{w}, \bar{x}) \in \mathcal{D}_{t_0}$

$$\|P(t, t_0)(u, w, x) - P(t, t_0)(\bar{u}, \bar{w}, \bar{x})\| \leq L\,(\|u - \bar{u}\| + \|w - \bar{w}\| + |x - \bar{x}|)$$

Proof — 1

• (Colombo-Guerra) Theorem about systems of balance laws with boundary

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$$\begin{cases} u_t + f(u)_x = g(u) \\ b(u(t, \gamma_*(t))) = B_*(t) \\ u(0, x) = u_o(x) \end{cases}$$

Under suitable assumptions, there exists a process $P_{B_*}(t, t_o) \colon \mathcal{D}_{t_o} \to \mathcal{D}_{t_o+t}$ such that

1. $P_{B_*}(0, t_o)\, u = u$ and $P_{B_*}(t+s, t_o)\, u = P_{B_*}(t, t_o + s) \circ P_{B_*}(s, t_o) u$

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2. Lipschitz estimates

$$\begin{aligned} \left\| P_{B_*}(t, t_o)u - P_{\bar{B}_*}(t, t_o)v \right\| &\leq L\left[\|u - v\| + \|B_* - \bar{B}_*\| + \left\| \gamma_*^1 - \gamma_*^2 \right\| \right] \\ \int_{t_0}^{t_0+t} \| Tr\,(P_1(\tau, t_0)u_1) - Tr\,(P_2(\tau, t_0)u_2)\| d\tau &\leq L\left[\|u - v\| + \|B_* - \bar{B}_*\| + \left\| \gamma_*^1 - \gamma_*^2 \right\| \right] \end{aligned}$$

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3. $u(t,x) = \left(P_{B_*}(t,0)u_o\right)(x)$ is the solution

Proof — 1

• (Colombo-Guerra) Theorem about systems of balance laws with boundary
• Well posedness results for ODEs

$$\begin{cases} \dot{w}(t) = F(t, w(t)) \\ w(0) = w_o \end{cases}$$

IP1: F Carathéodory
IP2: F Lipschitz continuous w.r.t. w
IP2: Grow conditions for F
TH: Existence + continuous dependence + stability

Proof — 1

- (Colombo-Guerra) Theorem about systems of balance laws with boundary
- Well posedness results for ODEs
- Completeness

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• (Colombo-Guerra) Theorem about systems of balance laws with boundary
• Well posedness results for ODEs
• Completeness
• Stability inequalities

Proof — 2

- Fix the constant functions: $u_0(t, x) = u_o(x)$, $w_0(t) = w_o$ and $\gamma_0(t) = x_o$.

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- Fix the constant functions: $u_0(t,x) = u_o(x)$, $w_0(t) = w_o$ and $\gamma_0(t) = x_o$.
- Define the sequences γ_k, u_k and w_k such that

γ_k as $\gamma_k = x_o + \int_0^t \Pi(w_{k-1}(\tau))d\tau$

u_k solution of
$$\begin{cases} \partial_t u + \partial_x f(u) = g(u) \\ b\left(u(t,\gamma_k(t))\right) = B\left(w_{k-1}(t)\right) \\ u(0,x) = u_o(x) \end{cases}$$

w_k solution of
$$\begin{cases} \dot{w} = F(t, u_{k-1}(t,\gamma_{k-1}(t)), w) \\ w(0) = w_o\,. \end{cases}$$

Proof — 3

- w_k is a Cauchy sequence

$$\begin{aligned}\|w_k(t) - w_{k-1}(t)\| &\leq C_1 \int_0^t \|w_k(\tau) - w_{k-1}(\tau)\| d\tau \\ &\quad + C_2 \int_0^t \|w_{k-2}(\tau) - w_{k-3}(\tau)\| d\tau\end{aligned}$$

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Gronwall inequality

$$\|w_k(t) - w_{k-1}(t)\| \leq C_3 e^{C_3 t} \int_0^t \|w_{k-2}(\tau) - w_{k-3}(\tau)\| d\tau$$

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By recursion

$$\|w_k(t) - w_{k-1}(t)\| \leq \frac{C_4^k}{k!} [\|w_1 - w_0\| + \|w_2 - w_1\|]$$

Proof — 3

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- u_k satisfies

$$\begin{aligned}\|u_k(t) - u_h(t)\|_{L^1} &\leq C \sup_{\tau\in[0,T]} \|w_{k-1}(\tau) - w_{h-1}(\tau)\| \\ &+ C \sup_{\tau\in[0,T]} \|\gamma_k(\tau) - \gamma_h(\tau)\|\end{aligned}$$

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- Lebesgue Theorem: $\gamma_k \to \gamma_* = x_o + \int_0^t \Pi(w_*(\tau))d\tau$
- $u_k \to u_*$
- (u_*, w_*, γ_*) solves the problem

Proof — 4

- $(u_{o,1}, w_{o,1}, x_{o,1})$ and $(u_{o,2}, w_{o,2}, x_{o,2})$ initial conditions

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- the term $\|u_{k,1}(t) - u_{k,2}(t)\|_1$ can be estimated by

$$L\left[\left\|u_{0,1} - u_{0,2}\right\|_1 + C\int_0^t \left\|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right\| d\tau + \left|\gamma_{k,1}(\cdot) - \gamma_{k,2}(\cdot)\right|\right]$$

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$$L\left[\|u_{0,1} - u_{0,2}\|_1 + C\int_0^t \|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\| d\tau + |\gamma_{k,1}(\cdot) - \gamma_{k,2}(\cdot)|\right]$$

- $|\gamma_{k,1}(t) - \gamma_{k,2}(t)| \leq |x_{o,1} - x_{o,2}| + L\int_0^t \|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\| d\tau$

Proof — 4

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- $w_{k,1}$, $u_{k,1}$, $\gamma_{k,1}$ sequences for $(u_{o,1}, w_{o,1}, x_{o,1})$
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- the term $\left\| u_{k,1}(t) - u_{k,2}(t) \right\|_1$ can be estimated by

$$L \left[\left\| u_{0,1} - u_{0,2} \right\|_1 + C \int_0^t \left\| w_{k-1,1}(\tau) - w_{k-1,2}(\tau) \right\| d\tau + \left| \gamma_{k,1}(\cdot) - \gamma_{k,2}(\cdot) \right| \right]$$

- $\left| \gamma_{k,1}(t) - \gamma_{k,2}(t) \right| \leq \left| x_{o,1} - x_{o,2} \right| + L \int_0^t \left\| w_{k-1,1}(\tau) - w_{k-1,2}(\tau) \right\| d\tau$
- the term $\left\| w_{k,1}(t) - w_{k,2}(t) \right\|$ can be estimated by

$$M \left[\left\| w_{0,1} - w_{0,2} \right\| + \left\| u_{0,1} - u_{0,2} \right\|_1 + \left| x_{o,1} - x_{o,2} \right| + \int_0^t \left\| w_{k-2,1}(s) - w_{k-2,2}(s) \right\| ds \right]$$

Proof — 4

- $(u_{o,1}, w_{o,1}, x_{o,1})$ and $(u_{o,2}, w_{o,2}, x_{o,2})$ initial conditions
- $w_{k,1}$, $u_{k,1}$, $\gamma_{k,1}$ sequences for $(u_{o,1}, w_{o,1}, x_{o,1})$
- $w_{k,2}$, $u_{k,2}$, $\gamma_{k,2}$ sequences for $(u_{o,2}, w_{o,2}, x_{o,2})$
- the term $\left\|u_{k,1}(t) - u_{k,2}(t)\right\|_1$ can be estimated by

$$L\left[\left\|u_{0,1} - u_{0,2}\right\|_1 + C\int_0^t \left\|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right\| d\tau + \left|\gamma_{k,1}(\cdot) - \gamma_{k,2}(\cdot)\right|\right]$$

- $\left|\gamma_{k,1}(t) - \gamma_{k,2}(t)\right| \leq \left|x_{o,1} - x_{o,2}\right| + L\int_0^t \left\|w_{k-1,1}(\tau) - w_{k-1,2}(\tau)\right\| d\tau$
- the term $\left\|w_{k,1}(t) - w_{k,2}(t)\right\|$ can be estimated by

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- Inductively

$$\left\|w_{k,1}(t) - w_{k,2}(t)\right\| \leq \tilde{M}\left[\left\|w_{0,1} - w_{0,2}\right\| + \left\|u_{0,1} - u_{0,2}\right\|_1 + \left|x_{o,1} - x_{o,2}\right|\right]$$

$$\left\|u_{k,1} - u_{k,2}\right\|_1 \leq \tilde{M}\left[\left\|w_{0,1} - w_{0,2}\right\| + \left\|u_{0,1} - u_{0,2}\right\|_1 + \left|x_{o,1} - x_{o,2}\right|\right]$$

$$\sup_{t\in[0,T_\delta]} \left|\gamma_{k,1}(t) - \gamma_{k,2}(t)\right| \leq \tilde{M}\left[\left\|w_{0,1} - w_{0,2}\right\| + \left\|u_{0,1} - u_{0,2}\right\|_1 + \left|x_{o,1} - x_{o,2}\right|\right]$$

Piston movie

(Piston)

Large vehicle movie

(Piston)