Stability of Small Shock Waves Associated with Métivier-Convex Modes

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Overview

- Spectral stability of extreme shocks
- The case of non-extreme shocks

I. Spectral stability of extreme shocks

Hyperbolic systems of conservation laws

$$\frac{\partial}{\partial t}u + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(f_j(u)) = 0$$

system: $u \in \mathbb{R}^n$, multidimensional: $x \in R^d$

$$\sum_{j=1}^{d} \zeta_j D f_j(u) \quad \text{symmetric,} \qquad u \in U \subset \mathbb{R}^n, \quad \zeta \in \mathbb{R}^d.$$

eigenvalues $\kappa_l(u,\zeta)$, eigenvectors $r_l(u,\zeta)$, $l = 1, \ldots, n$

Parabolic extension

$$\frac{\partial}{\partial t}u + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(f_j(u)) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}u$$

Inviscid shock wave

planar shock wave

$$u(x,t) = \begin{cases} u^-, & x \cdot n - st < 0 \\ u^+, & x \cdot n - st > 0 \end{cases}$$

direction $n \in \mathbb{R}^d$, speed $s \in \mathbb{R}$

left state u^-, right state u^+

jump condition

$$\left(-su + \sum_{j=1}^{d} n_j f_j(u) \right) \Bigg|_-^+ = 0$$

Viscous shock wave

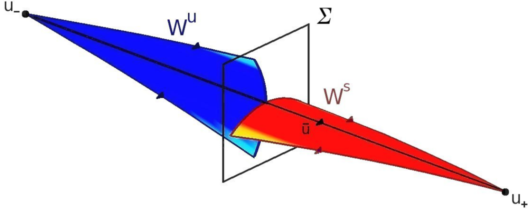
Travelling wave solution

$$u(x,t) = \phi(\xi), \qquad \xi := x \cdot n - st, \qquad \lim_{\xi \to \pm\infty} \phi(\xi) = u^{\pm}$$

w.l.o.g.: $n = (1, 0, \dots, 0)$

$$\phi' = f_1(\phi) - s\phi - c$$

Profile ϕ is a heteroclinic orbit connecting equilibria u^- and u^+:



Stability of multidimensional viscous shock waves

Theorem (Zumbrun 2001): If a multi-dimensional planar viscous shock wave satisfies the spectral stability conditions

(E) $D(\lambda, \omega) \neq 0$, $Re\lambda \geq 0$, $\omega \in \mathbb{R}^{d-1}$, $(\lambda, \omega) \neq (0, 0)$

(L) $\Delta(\bar{\lambda}, \bar{\omega}) \neq 0$, $Re\bar{\lambda} \geq 0$, $\bar{\omega} \in \mathbb{R}^{d-1}$, $|\bar{\lambda}|^2 + |\bar{\omega}|^2 = 1$,

then the shock wave is nonlinearly stable.
Here, D is the Evans function and Δ is the Majda determinant (cf. below), with λ spectral parameter, ω transverse wave number.

Briefly:
spectral stability $\Rightarrow$ **linear stability** $\Rightarrow$ **nonlinear stability**

Remark: According to Gues, Métivier, Williams, Zumbrun (2002, ..., 2009, ...), (E) and (L) imply also the validity of the vanishing viscosity limit, even for curved shock fronts.

Spectral stability of small-amplitude shocks

Theorem 1 (F. and Szmolyan 2010):
For a symmetric system of viscous conservation laws

$$\frac{\partial}{\partial t}u + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(f_j(u)) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}u$$

assume near some state u_* and ζ near n :

i) The smallest (or largest) eigenvalue $\kappa(u,\zeta)$ of $\sum_{j=1}^{d} \zeta_j Df_j(u)$ is simple with eigenvector $r(u,\zeta)$

ii) κ is genuinely nonlinear: $r(u_*,n)D_u\kappa(u_*,n) > 0$

iii) κ is Métivier convex with resp. to ζ: $D_\zeta^2\kappa(u_*,n)|_{n^\perp} > 0$

$\Rightarrow$ family of small-amplitude viscous κ-shocks $\phi_\varepsilon(x \cdot n - st)$ with

$$u_\varepsilon^\pm = \phi_\varepsilon(\pm\infty) = u_* - \varepsilon r(u_*,n) + O(\varepsilon^2)$$

satisfy spectral stability conditions (E) and (L).

The Majda determinant

The (Kreiss-)Majda (Lopatinski) determinant $\Delta(\bar\lambda,\bar\omega) \equiv$

$$\det(R_1^-(\bar\lambda,\bar\omega),..,R_{p-1}^-(\bar\lambda,\bar\omega),\bar\lambda[u]+i[f^{\bar\omega}(u)],R_{p+1}^+(\bar\lambda,\bar\omega),..,R_n^+(\bar\lambda,\bar\omega))$$

of the given shock is defined on

$$H \equiv \{(\bar\lambda,\bar\omega)\in\mathbb{C}\times\mathbb{R}^{d-1}\ :\ \mathrm{Re}\,\bar\lambda\geq 0, |\bar\lambda|^2+\bar\omega^2=1\},$$

with $f^{\bar\omega}\equiv\sum_{j=2}^d \bar\omega_j f_j$ and

$$\{R_1^-(\bar\lambda,\bar\omega),\dots,R_{p-1}^-(\bar\lambda,\bar\omega)\},\quad \{R_{p+1}^+(\bar\lambda,\bar\omega),\dots,R_n^+(\bar\lambda,\bar\omega)\}$$

continuous bases for the (extensions to H of) the stable/unstable spaces $E^-(\bar\lambda,\bar\omega)$, $E^+(\bar\lambda,\bar\omega)$ of

$$A^{\mp}(\bar\lambda,\bar\omega)\equiv(\bar\lambda I+iDf^{\bar\omega}(u^{\mp}))(Df_1(u^{\mp}))^{-1}.$$

Eigenvalue problem

linearization along profile in co-moving frame

$$\frac{\partial p}{\partial t} = Lp := \Delta p - \frac{\partial}{\partial \xi}[(df_1(\phi) - sI)p] - \sum_{j=2}^{d} df_j(\phi)\frac{\partial p}{\partial x_j}$$

eigenvalue problem: $Lp = \lambda p$, $\lim_{\xi \to \pm\infty} p = 0$

translation invariance: eigenvalue $\lambda = 0$, $p = \phi'$

spectral stability condition (E): no spectrum in $\{\Re\lambda \geq 0,\ \lambda \neq 0\}$

Eigenvalue problem as a first order ODE system

Fourier-transform in transversal directions

$$(x_2, \dots, x_d) \to (\omega_2, \dots, \omega_d) =: \omega \in \mathbb{R}^{d-1}$$

and introduce $q := p' - (Df_1(\phi) - sI)p$ to obtain

$$\begin{aligned} p' &= (Df_1(\phi) - sI)p + q \\ q' &= (\lambda I + iB^\omega(\phi) + |\omega|^2 I)p \end{aligned}$$

$\lambda \in \mathbb{C}, \quad (p, q) \in \mathbb{C}^{2n}, \quad B^\omega(\phi) := \sum_{j=2}^{d} \omega_j Df_j(\phi)$

briefly:

$$z' = \mathbb{A}(\lambda, \omega, \xi)z, \qquad z := (p, q)$$

- asymptotically constant coefficients: $\lim_{\xi \to \pm\infty} \mathbb{A}(\lambda, \omega, \xi) = \mathbb{A}^\pm(\lambda, \omega)$, exponential rate!
- $\omega = 0 \;\Leftrightarrow\;$ stability problem in one space dimension $(d = 1)$

Unstable and stable bundles, Evans function

Theorem: for $\Re\lambda \geq 0$, $(\lambda, \omega) \neq (0, 0)$:

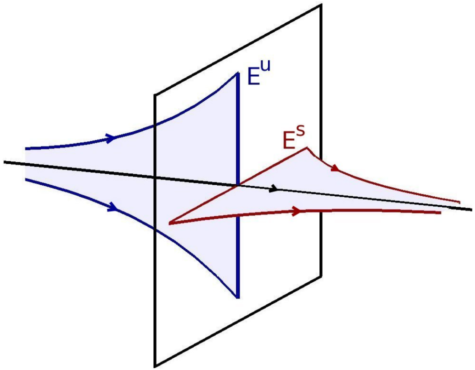
- $\exists$ n-dimensional unstable and stable spaces $U(\lambda, \omega)$, $S(\lambda, \omega)$ analytic in λ, i.e. spaces of initial values (at $\xi = 0$) of solutions to $z' = \mathbb{A}(\lambda, \omega, \xi)z$ decaying for $\xi \to -\infty$, $\xi \to \infty$, respectively.

$$\lambda \text{ eigenvalue of } L \quad \Leftrightarrow \quad \exists \omega : \quad U(\lambda, \omega) \cap S(\lambda, \omega) \neq 0$$

- Evans function $D(\lambda, \omega) := det[U(\lambda, \omega), S(\lambda, \omega)]$
  i) analytic in λ,
  ii) λ eigenvalue of $L \quad \Leftrightarrow \quad D(\lambda, \omega) = 0$,
  iii) algebraic multiplicity of eigenvalue equals order of zero

Evans, Jones, Gardner, Alexander, Sandstede, Kapitula, Zumbrun,. . .

Evans function, unstable and stable bundles



intersection only along $(p, q) = (0, 0)$ $\Rightarrow$ λ no eigenvalue

First difficulty: singular perturbation w. r. t. $\varepsilon \to 0$!

Assume $f(0) = 0$ and $Df(0) = \mathrm{diag}(\kappa_1^0, \dots, \kappa_n^0)$ with $\kappa_k^0 = 0$.

Via

$$\text{scaling} \quad \phi = \varepsilon u \quad \text{etc.},$$

spectral problem (incl. profile equation) reads

$$\begin{aligned} u' &= \varepsilon^{-1} f_1(\varepsilon u) - \varepsilon s u - \varepsilon c \\ p' &= (D f_1(\varepsilon u)) - \varepsilon s I) p + \varepsilon q \\ q' &= \varepsilon(\lambda I + i B^\omega(u) + |\omega|^2 I) p \end{aligned}$$

Different scales for $\varepsilon \to 0$:

Fast scale

$$
\begin{aligned}
u' &= \varepsilon^{-1} f_1(\varepsilon u) - \varepsilon s u - \varepsilon c \\
p' &= (D f_1(\varepsilon u)) - \varepsilon s I) p + \varepsilon q \\
q' &= \varepsilon(\lambda I + i B^{\omega}(u) + |\omega|^2 I) p
\end{aligned}
$$

Slow scale

$$
\begin{aligned}
\varepsilon \dot{u} &= \varepsilon^{-1} f_1(\varepsilon u) - \varepsilon s u - \varepsilon c \\
\varepsilon \dot{p} &= (D f_1(\varepsilon u)) - \varepsilon s I) p + \varepsilon q \\
\dot{q} &= (\lambda I + i B^{\omega}(u) + |\omega|^2 I) p
\end{aligned}
$$

Tool: Geometric Singular Perturbation Theory

slow problem

$$\begin{aligned}\dot{x} &= f(x,y)\\ \varepsilon\dot{y} &= g(x,y)\end{aligned}$$

fast problem

$$\begin{aligned}x' &= \varepsilon f(x,y)\\ y' &= g(x,y)\end{aligned}$$

reduced problem

$$\begin{aligned}\dot{x} &= f(x,y)\\ 0 &= g(x,y)\end{aligned}$$

layer problem

$$\begin{aligned}x' &= 0\\ y' &= g(x,y)\end{aligned}$$

Assume on $M_0 := g^{-1}(0)$: det$\frac{\partial g}{\partial y} \neq 0$. Then

$g(x,y) = 0 \Leftrightarrow y = h_0(x)$ and reduced flow on $M_0 = \text{graph}(h_0)$:

$$\dot{x} = \hat{f}_0(x) := f(x, h_0(x))$$

Question: Letting $0 < \varepsilon << 1$, what happens to the reduced flow?

Answer:

Theorem (Fenichel, 1971):
If

$$\frac{\partial g}{\partial y} \text{ normally hyperbolic, i. e., } \sigma(\frac{\partial g}{\partial y}) \cap i\mathbb{R} = \emptyset,$$

then M_0 perturbs regularly to an invariant manifold

$$M_\varepsilon = \mathsf{graph}(h_\varepsilon)$$

with slow flow

$$\dot{x} = \hat{f}_\varepsilon(x) = f(x, h_\varepsilon(x)).$$

Second difficulty: $D(\lambda, \omega)$ close to $(0,0)$

polar coordinates

$$\lambda = \rho\bar{\lambda}, \quad \omega = \rho\bar{\omega}, \qquad |\bar{\lambda}|^2 + |\bar{\omega}|^2 = 1, \quad \Re\bar{\lambda} \geq 0, \quad \rho \geq 0$$

spectral problem coupled to profile equation

$$\begin{aligned} u' &= f_1(u) - su - c \\ p' &= (Df_1(u)) - sI)p + q \\ q' &= \rho(\bar{\lambda}I + iB^{\bar{\omega}}(u) + \rho|\bar{\omega}|^2 I)p \end{aligned}$$

- $\rho \to 0$ long wave limit
- ρ small : another singular perturbation!

Summary of the proof of Theorem 1

- $\epsilon \to 0$ for ρ fixed: singular perturbation problem, rescaled profile governed by Burgers equation

- $(\epsilon, \rho) \to (0, 0)$, multi-scale singular perturbation problem

  inner regime: $0 < \rho \leq \varepsilon^2 r_0$

  middle regime: $\varepsilon^2 r_0 \leq \rho \leq r_1$

  outer regime: $r_1 \leq \rho$

- slow-fast decomposition of unstable and stable spaces, viewed as points or manifolds in suitable Grassmann manifolds

- various rescalings to regain hyperbolicity and transversality at certain points in $(\bar{\lambda}, \bar{\omega})$ space

- Melnikov type arguments to show transversal breaking of $\rho = 0$ intersections of unstable and stable spaces.

II. The case of non-extreme shocks

Theorem 2 (F. and Szmolyan 2011): The assumption that the shock be extreme (i. e., correspond to smallest or largest characteristic speed), cannot be removed without losing something. There are systems for which arbitrarily small shocks $(U_\epsilon^-, U_\epsilon^+)$ associated with a non-extreme Métivier convex mode have

$$\Delta(\lambda_\epsilon, \omega_\epsilon) = 0$$

for certain $(\bar\lambda_\epsilon, \bar\omega_\epsilon)$ with Re $\bar\lambda_\epsilon = 0$, Im $\bar\lambda_\epsilon \neq 0$.

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Remark: One might have thought that a 1989 result of G. Métivier would imply that this cannot happen ... .

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Proof: By example:

The counterexample

Consider a system

$$\partial_t u + \partial_{x_1} f_1(u) + \partial_{x_2} f_2(u) = 0, \quad (1)$$

$$\partial_t v + \partial_{x_1} g_1(v) + \partial_{x_2} g_2(v) = 0, \quad (2)$$

where (1) by itself is a symmetrizable hyperbolic system of conservation laws with modes of constant multiplicity, among which at least one, λ_p, is Métivier convex. Assume that for some point u_* in the state space of (1) and for propagation direction $N_* = (1, 0)^\top$, this mode has zero speed,

$$\lambda_p(u_*, N_*, 0) = 0.$$

E. g., (1) could be the Euler equations for compressible fluid flow and λ_p the acoustic mode.

Tune (2) by choosing $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$g_1(v_1, v_2) = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad g_2(v_1, v_2) = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

with some

$$s > \max\{|\lambda| : \lambda \text{ eigenvalue of } Df_1(u_*)\}. \tag{3}$$

Fix a family of small-amplitude p-shock waves $(u_\epsilon^-, u_\epsilon^+)$ of (1) with speed 0 and propagation direction N_*, perturbing from $u_0^\pm = u_*$. Augmenting u to $U = (u, 0)$, this family trivially induces a family of small-amplitude shock waves $(U_\epsilon^-, U_\epsilon^+)$ of (1). Since the characteristic speeds of (2) are $-s$ and $+s$ and by virtue of (3), the augmentation increases both the number of outgoing and that of incoming modes by 1, on either side of each shock wave.

As (1) and (2) are completely independent from each other ...

..., the Lopatinski determinant Δ_ϵ of the shock wave $(U_\epsilon^-, U_\epsilon^+)$ contains a factor

$$\delta = \det(r^-(\tau,\xi), r^+(\tau,\xi))$$

with $r^-(\tau,\xi), r^+(\tau,\xi)$ spanning the stable/unstable spaces of

$$b(\tau,\xi) = \left(\tau I + i\xi \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix}\right) \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}^{-1}.$$

Writing

$$\tau = i\sigma, \text{ and } \mu = i\beta$$

for the possible eigenvalue μ of $b(\tau,\xi)$, we see that as

$$\det\left(\sigma I + \xi \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} + \beta \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}\right) = -s^2\beta^2 + (\sigma^2 - s^2\xi^2),$$

these spaces coincide at the branchpoints given by

$$\sigma \in \{\pm s(1+s^2)^{-1/2}\} \text{ and } \xi \in \{\pm(1+s^2)^{-1/2}\}.$$

At these (four) points, δ and thus Δ_ϵ vanish!

Work in progress ...

Trying to show that

- there are Métivier convex modes with arbitrarily small shocks that violate even the weak Kreiss-Lopatinski condition, or
- all small shocks for Métivier convex modes are weakly stable at the inviscid level, and perhaps even
- all small shocks for Métivier convex modes are stable at the viscous level

THANK YOU

PS to I. The Zumbrun-Serre Lemma

Evans function and Majda determinant

Lemma (Zumbrun & Serre): Consider a fixed viscous profile. Then:

$$D(\rho\bar{\lambda}, \rho\bar{\omega}) = \rho\,\Gamma\,\Delta(\bar{\lambda}, \bar{\omega}) + O(\rho^2) \quad \text{as } \rho \to 0.$$

$\Delta(\bar{\lambda}, \bar{\omega})$ is the Lopatinski-Kreiss-Majda determinant governing inviscid stability of the planar shock wave (u^-, u^+) as a solution of the hyperbolic conservation law

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(f_j(u)) = 0.$$

The constant Γ is nonzero iff the profil ϕ is realized as transverse intersection of the unstable and stable manifolds of the end-states u^- resp. u^+ of the profile equation $u' = f_1(u) - su - c$.

Geometric proof of the Zumbrun-Serre lemma

$$\begin{aligned} u' &= f_1(u) - su - c \\ p' &= (Df_1(u)) - sI)p + q \\ q' &= \rho(\bar{\lambda} I + iB^{\bar{\omega}}(u) + \rho|\bar{\omega}|^2 I)p \end{aligned}$$

is for $\rho \to 0$ a singularly perturbed system on fast time scale ξ

layer problem (fast dynamics):

$$\begin{aligned} u' &= f_1(u) - su - c \\ p' &= (Df_1(u) - sI)p + q \\ q' &= 0 \end{aligned}$$

normally hyperbolic manifold M of equilibria

$$u = u^{\pm}, \quad p = -(Df_1(u^{\pm}) - sI)^{-1} q, \quad q \in \mathbb{C}^n$$

reduced dynamics on M:

$$\dot{q} = -\left(\bar{\lambda} I + iB^{\bar{\omega}}(u^{\pm})\right)\left(Df_1(u^{\pm}) - sI\right)^{-1} q$$

reduced problem: slow dynamics on M

$$u = u^{\pm}, \quad p = -(Df_1(u^{\pm}) - sI)^{-1}q, \quad q \in \mathbb{C}^n$$

$$\dot{q} = -\left(\bar{\lambda} I + i B^{\bar{\omega}}(u^{\pm})\right)\left(Df_1(u^{\pm}) - sI\right)^{-1} q$$

define slow unstable space U_0^s at u^-, slow stable space S_0^s at u^+.

slow - fast decomposition of stable and unstable spaces:

$$D(\rho\bar{\lambda}, \rho\bar{\omega}) = \det(U_\rho, S_\rho)$$

$$= \rho \det(U_0^f, S_0^f) \det(U_0^s, \bar{\lambda}[u] + i[f^{\bar{\omega}}], S_0^s) + O(\rho^2)$$

$$= \rho \, \Gamma \, \Delta(\bar{\lambda}, \bar{\omega}) + O(\rho^2)$$