An efficient splitting technique for two-layer shallow-water model

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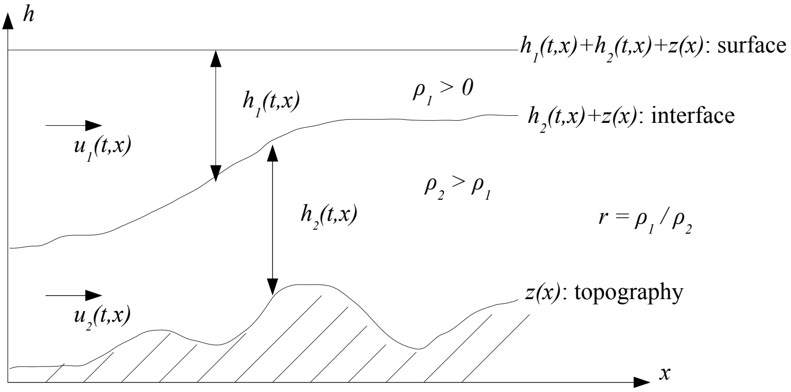
Outline

Two superposed layers on a non flat bottom



Equations

We consider the following system of equations modelling the flow of two shallow water layers :

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0 \\ \partial_t(h_1 u_1) + \partial_x(h_1 u_1^2 + \frac{g}{2} h_1^2) = -g h_1 \partial_x(h_2 + z) \\ \partial_t h_2 + \partial_x(h_2 u_2) = 0 \\ \partial_t(h_2 u_2) + \partial_x(h_2 u_2^2 + \frac{g}{2} h_2^2) = -g h_2 \partial_x(r\, h_1 + z) \end{cases}$$

This problem has been already adressed, we can cite C. Parés, M. Castro, J. Macías, F. Bouchut, T. Morales, T. Chacón, E. Fernández, J. García, E. Audusse, J. Sainte-Marie, ...

Motivations

We are looking for a scheme which is expected to :

- preserve $h_1 \geq 0$, $h_2 \geq 0$,
- preserve the steady states at rest (well-balancing property) :

$$\begin{cases} u_1 = u_2 = 0 \\ h_1 + h_2 + z = \mathrm{cst} \\ r\,h_1 + h_2 + z = \mathrm{cst} \end{cases}$$

- be in agreement with real results especially if r approaches 1.

Idea : apply what we did for the one-layer problem :

C. Berthon, F. Foucher, Efficient well-balanced hydrostatic upwind schemes for shallow-water equations, *JCP*, 2012.

One-layer system

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x(hu^2 + \frac{g}{2}h^2) = -hg\partial_x z \end{cases}$$

We see that the steady states at rest are given by :

$$\begin{cases} u = 0 \\ h + z = \text{cst} \end{cases}$$

In order to derive a well-balanced scheme, the idea is to introduce the free surface :

$$H = h + z$$

Then, using the associated fraction of water $X = \frac{h}{H}$ and writing :

$$h^2 = h(H - z) = hH - hz = XH^2 - hz,$$

New one-layer system

we transform for weak solutions the initial system into :

$$\begin{cases} \partial_t h + \partial_x (XHu) = 0 \\ \partial_t (hu) + \partial_x \left(X(Hu^2 + \frac{g}{2}H^2) - \frac{g}{2}hz \right) = -gh\partial_x z \end{cases}$$

which can be written :

$$\partial_t w + \partial_x \left(Xf(W) \right) = S(H,h)$$

where

$$\begin{cases} w = \begin{pmatrix} h \\ hu \end{pmatrix} \text{ and } W = \begin{pmatrix} H \\ Hu \end{pmatrix} \text{ are state vectors, } h \geq 0,\ H > 0, \\ f(W) = \begin{pmatrix} Hu \\ Hu^2 + \frac{g}{2}H^2 \end{pmatrix} \text{ and } S(H,h) = \begin{pmatrix} 0 \\ \frac{g}{2}\partial_x \left(h(H-h) \right) - gh\partial_x (H-h) \end{pmatrix} \end{cases}$$

New two-layer system

We introduce :

$$\begin{cases} H_1 = h_1 + h_2 + z \text{ and } X_1 = \frac{h_1}{H_1} \\ H_2 = r\,h_1 + h_2 + z \text{ and } X_2 = \frac{h_2}{H_2} \end{cases}$$

and state vectors :

$$w_j = \begin{pmatrix} h_j \\ h_j u_j \end{pmatrix} \text{ and } W_j = \begin{pmatrix} H_j \\ H_j u_j \end{pmatrix},\ h_j \geq 0,\ H_j > 0,\ j = 1, 2$$

In order to transform the two-layers system, we write that

$$\begin{cases} h_1^2 = X_1 H_1 (H_1 - h_2 - z) \\ h_2^2 = X_2 H_2 (H_2 - r\,h_1 - z) \end{cases}$$

Two-layer splitting system

Then we derive the new system :

$$\begin{cases} \partial_t h_1 + \partial_x (X_1 H_1 u_1) = 0 \\ \partial_t (h_1 u_1) + \partial_x \left(X_1 (H_1 u_1^2 + \frac{g}{2} H_1^2) - \frac{g}{2} h_1 (h_2 + z) \right) = -g h_1 \partial_x (h_2 + z) \\ \partial_t h_2 + \partial_x (X_2 H_2 u_2) = 0 \\ \partial_t (h_2 u_2) + \partial_x \left(X_2 (H_2 u_2^2 + \frac{g}{2} H_2^2) - \frac{g}{2} h_2 (r\, h_1 + z) \right) = -g h_2 \partial_x (r\, h_1 + z) \end{cases}$$

Finally using that

$$h_2 + z = H_1 - h_1 \text{ and } r\, h_1 + z = H_2 - h_2$$

we turn the system into two similar systems with source terms

$$\partial_t w_j + \partial_x (X_j f(W_j)) = S(H_j, h_j), \; j = 1, 2$$

Discretization of $f(W)$

- Uniform mesh in space $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$
- Time step $\Delta t = t^{n+1} - t^n$
- Values in the cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time t^n :

$$h_i^n,\ u_i^n,\ H_i^n,\ X_i^n,\ w_i^n,\ W_i^n,\ z_i$$

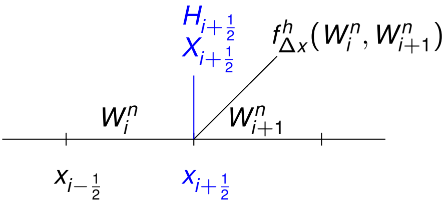
- $f_{\Delta x}(W_i^n, W_{i+1}^n) = \begin{pmatrix} f_{\Delta x}^h(W_i^n, W_{i+1}^n) \\ f_{\Delta x}^{hu}(W_i^n, W_{i+1}^n) \end{pmatrix}$ computed by a numerical scheme (HLLC, VFRoe, relaxation, ...) well-known for the homogeneous system $\partial_t w + \partial_x f(w) = 0$:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{\Delta x}(w_i^n, w_{i+1}^n) - f_{\Delta x}(w_{i-1}^n, w_i^n)\right)$$

Discretization of $S(H, h)$

- We introduce upwind values on interfaces $x_{i+\frac{1}{2}}$:

$$H_{i+\frac{1}{2}},\ X_{i+\frac{1}{2}} = \begin{cases} H_i^n,\ X_i^n \text{ si } f^h(W_i^n, W_{i+1}^n) > 0 \\ H_{i+1}^n,\ X_{i+1}^n \text{ else} \end{cases}$$



- Then we define values $h_{i+\frac{1}{2}}$ by :

$$h_{i+\frac{1}{2}} = H_{i+\frac{1}{2}} X_{i+\frac{1}{2}}$$

One-layer scheme

- We write the approximation :

$$
\begin{aligned}
&\frac{g}{2}\partial_x h(H-h) - gh\partial_x(H-h) \\
&\simeq \frac{g}{2}\Big(h_{i+\frac{1}{2}}(H_{i+\frac{1}{2}} - h_{i+\frac{1}{2}}) - h_{i-\frac{1}{2}}(H_{i-\frac{1}{2}} - h_{i-\frac{1}{2}})\Big) \\
&- \frac{g}{2}(h_{i+\frac{1}{2}} + h_{i-\frac{1}{2}})\Big((H_{i+\frac{1}{2}} - h_{i+\frac{1}{2}}) - (H_{i-\frac{1}{2}} - h_{i-\frac{1}{2}})\Big) \\
&= \frac{g}{2}(h_{i+\frac{1}{2}}H_{i-\frac{1}{2}} - h_{i-\frac{1}{2}}H_{i+\frac{1}{2}}) \\
&= \frac{g}{2}H_{i+\frac{1}{2}}H_{i-\frac{1}{2}}(X_{i+\frac{1}{2}} - X_{i-\frac{1}{2}})
\end{aligned}
$$

- We deduce the one-layer scheme :

$$
\begin{aligned}
w_i^{n+1} = w_i^n &- \frac{\Delta t}{\Delta x}\big(X_{i+\frac{1}{2}} f_{\Delta x}(W_i^n, W_{i+1}^n) - X_{i-\frac{1}{2}} f_{\Delta x}(W_{i-1}^n, W_i^n)\big) \\
&+ \frac{g}{2}\frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ H_{i-\frac{1}{2}}H_{i+\frac{1}{2}}(X_{i+\frac{1}{2}} - X_{i-\frac{1}{2}}) \end{pmatrix}
\end{aligned}
$$

Two-layers scheme

We recall the splitting two-layers system :

$$\partial_t w_j + \partial_x (X_j f(W_j)) = S(H_j, h_j),\ j = 1,2$$

Now, we derive the following scheme to approximate this system, writing the previous one-layer scheme for each layer :

$$w_{j,i}^{n+1} = w_{j,i}^{n} - \frac{\Delta t}{\Delta x}\left(X_{j,i+\frac{1}{2}} f_{\Delta x}(W_{j,i}^{n}, W_{j,i+1}^{n}) - X_{j,i-\frac{1}{2}} f_{\Delta x}(W_{j,i-1}^{n}, W_{j,i}^{n})\right) + \frac{g}{2}\frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ H_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}}) \end{pmatrix}$$

The scheme is well-balanced

Let's suppose $u^n_{j,i} = 0$, $H^n_{j,i} = H_j$, where H_j constant, $j = 1, 2$.

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- Since $f_{\Delta x}$ is consistent, we've got $f_{\Delta x}(w, w) = f(w)$, so that

$$f_{\Delta x}(W^n_{j,i}, W^n_{j,i+1}) = \begin{pmatrix} 0 \\ \frac{g}{2} H_j^2 \end{pmatrix}$$

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- Putting it in the scheme, we get : (i) $h_{j,i}^{n+1} = h_{j,i}^n$ and

$$
\begin{aligned}
(ii)\ (hu)_{j,i}^{n+1} = (hu)_{j,i}^n - \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})\frac{g}{2}H_j^2 \\
+ \frac{g}{2}\frac{\Delta t}{\Delta x} H_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})
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$$(ii)\ (hu)_{j,i}^{n+1} = (hu)_{j,i}^n - \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})\frac{g}{2}H_j^2 + \frac{g}{2}\frac{\Delta t}{\Delta x}H_{j,i-\frac{1}{2}}H_{j,i+\frac{1}{2}}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})$$
- Then from (i), we deduce :
$$\begin{cases} H_{1,i}^{n+1} = h_{1,i}^{n+1} + h_{2,i}^{n+1} + z_i = h_{1,i}^n + h_{2,i}^n + z_i = H_{1,i}^n = H_1 \\ H_{2,i}^{n+1} = r\, h_{1,i}^{n+1} + h_{2,i}^{n+1} + z_i = r\, h_{1,i}^n + h_{2,i}^n + z_i = H_{2,i}^n = H_2 \end{cases}.$$

The scheme is well-balanced

Let's suppose $u_{j,i}^n = 0$, $H_{j,i}^n = H_j$, where H_j constant, $j = 1, 2$.

- Since $f_{\Delta x}$ is consistent, we've got $f_{\Delta x}(w, w) = f(w)$, so that $f_{\Delta x}(W_{j,i}^n, W_{j,i+1}^n) = \begin{pmatrix} 0 \\ \frac{g}{2} H_j^2 \end{pmatrix}$
- Putting it in the scheme, we get : (i) $h_{j,i}^{n+1} = h_{j,i}^n$ and

$$(ii)\ (hu)_{j,i}^{n+1} = (hu)_{j,i}^n - \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})\frac{g}{2}H_j^2 \\ + \frac{g}{2}\frac{\Delta t}{\Delta x} H_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})$$

- Then from (i), we deduce :

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- This implies $H_{j,i-\frac{1}{2}} = H_{j,i+\frac{1}{2}} = H_j$ so that (ii) leads to $(hu)_{j,i}^{n+1} = (hu)_{j,i}^n$.

The scheme is well-balanced

Let's suppose $u^n_{j,i} = 0$, $H^n_{j,i} = H_j$, where H_j constant, $j = 1, 2$.

- Since $f_{\Delta x}$ is consistent, we've got $f_{\Delta x}(w, w) = f(w)$, so that
  $f_{\Delta x}(W^n_{j,i}, W^n_{j,i+1}) = \begin{pmatrix} 0 \\ \frac{g}{2} H_j^2 \end{pmatrix}$
- Putting it in the scheme, we get : (i) $h^{n+1}_{j,i} = h^n_{j,i}$ and

$$(ii)\ (hu)^{n+1}_{j,i} = (hu)^n_{j,i} - \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})\frac{g}{2}H_j^2 + \frac{g}{2}\frac{\Delta t}{\Delta x}H_{j,i-\frac{1}{2}}H_{j,i+\frac{1}{2}}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})$$

- Then from (i), we deduce :

$$\begin{cases} H^{n+1}_{1,i} = h^{n+1}_{1,i} + h^{n+1}_{2,i} + z_i = h^n_{1,i} + h^n_{2,i} + z_i = H^n_{1,i} = H_1 \\ H^{n+1}_{2,i} = r\,h^{n+1}_{1,i} + h^{n+1}_{2,i} + z_i = r\,h^n_{1,i} + h^n_{2,i} + z_i = H^n_{2,i} = H_2 \end{cases}.$$

- This implies $H_{j,i-\frac{1}{2}} = H_{j,i+\frac{1}{2}} = H_j$ so that (ii) leads to $(hu)^{n+1}_{j,i} = (hu)^n_{j,i}$.
- Finally $u^{n+1}_{j,i} = \frac{(hu)^{n+1}_{j,i}}{h^{n+1}_{j,i}} = \frac{(hu)^n_{j,i}}{h^n_{j,i}} = u^n_{j,i} = 0$

The scheme preserves h_1 and h_2 non negative ?

We use for each layer exactly the same proof as for one layer. Let's look to the first equation of the scheme :

$$h_{j,i}^{n+1} = h_{j,i}^{n} - \frac{\Delta t}{\Delta x}\left(X_{j,i+\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) - X_{j,i-\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})\right)$$

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Reminding the definition of $X_{j,i+\frac{1}{2}}$, we write :

$$\begin{aligned} X_{j,i+\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) = & \frac{1}{2}(X_{j,i}^{n} + X_{j,i+1}^{n}) f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) \\ & - \frac{1}{2}(X_{j,i+1}^{n} - X_{j,i}^{n}) |f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n})| \end{aligned}$$

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Reminding the definition of $X_{j,i+\frac{1}{2}}$, we write :

$$\begin{aligned} X_{j,i+\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) = & \frac{1}{2}(X_{j,i}^{n} + X_{j,i+1}^{n}) f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) \\ & - \frac{1}{2}(X_{j,i+1}^{n} - X_{j,i}^{n}) |f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n})| \end{aligned}$$

We obtain $h_{j,i}^{n+1} = \alpha_{j,i} X_{j,i-1}^{n} + (\tilde{H}_{j,i}^{n+1} - \alpha_{j,i} - \beta_{j,i}) X_{j,i}^{n} + \beta_{j,i} X_{j,i+1}^{n}$ with :

$$\begin{cases} \alpha_{j,i} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n}) + |f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})|\right) \geq 0 \\ \beta_{j,i} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(-f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) + |f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n})|\right) \geq 0 \\ \tilde{H}_{j,i}^{n+1} = H_{j,i}^{n} - \frac{\Delta t}{\Delta x}\left(f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) - f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})\right) \end{cases}$$

The scheme preserves h_1 *and* h_2 *non negative*

Let's suppose $h^n_{j,i} \geq 0, \ j = 1, 2$.

The scheme preserves h_1 and h_2 non negative

Let's suppose $h^n_{j,i} \geq 0,\ j = 1, 2$.

- Assume that the numerical flux we use satisfies :

$$h^n_i > 0 \Rightarrow h^n_i - \frac{\Delta t}{\Delta x}\left(f^h_{\Delta x}(w^n_i, w^n_{i+1}) - f^h_{\Delta x}(w^n_{i-1}, w^n_i)\right) > 0$$

Then : $h^n_{j,i} \geq 0 \Rightarrow H^n_{j,i} > 0 \Rightarrow \tilde{H}^{n+1}_{j,i} > 0$.

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Let's suppose $h_{j,i}^n \geq 0, \; j = 1, 2$.

- Assume that the numerical flux we use satisfies :

$$h_i^n > 0 \Rightarrow h_i^n - \frac{\Delta t}{\Delta x}\left(f_{\Delta x}^h(w_i^n, w_{i+1}^n) - f_{\Delta x}^h(w_{i-1}^n, w_i^n)\right) > 0$$

  Then : $h_{j,i}^n \geq 0 \Rightarrow H_{j,i}^n > 0 \Rightarrow \tilde{H}_{j,i}^{n+1} > 0$.

- The sign of $\gamma_{j,i} = \tilde{H}_{j,i}^{n+1} - \alpha_{j,i} - \beta_{j,i}$ depends on the sign of $f_{\Delta x}^h(W_{j,i-1}^n, W_{j,i}^n)$ and $f_{\Delta x}^h(W_{j,i}^n, W_{j,i+1}^n)$. We find $\gamma_{j,i} > 0$ in all cases, under the additional CFL like condition :

$$\frac{\Delta t}{\Delta x}\left(\max(0, f_{\Delta x}^h(W_{j,i}^n, W_{j,i+1}^n)) - \min(0, f_{\Delta x}^h(W_{j,i-1}^n, W_{j,i}^n))\right) \leq H_{j,i}^n$$

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Let's suppose $h^n_{j,i} \geq 0,\ j = 1, 2$.

- Assume that the numerical flux we use satisfies :

$$h^n_i > 0 \Rightarrow h^n_i - \frac{\Delta t}{\Delta x}\left(f^h_{\Delta x}(w^n_i, w^n_{i+1}) - f^h_{\Delta x}(w^n_{i-1}, w^n_i)\right) > 0$$

Then : $h^n_{j,i} \geq 0 \Rightarrow H^n_{j,i} > 0 \Rightarrow \tilde{H}^{n+1}_{j,i} > 0$.

- The sign of $\gamma_{j,i} = \tilde{H}^{n+1}_{j,i} - \alpha_{j,i} - \beta_{j,i}$ depends on the sign of $f^h_{\Delta x}(W^n_{j,i-1}, W^n_{j,i})$ and $f^h_{\Delta x}(W^n_{j,i}, W^n_{j,i+1})$. We find $\gamma_{j,i} > 0$ in all cases, under the additional CFL like condition :

$$\frac{\Delta t}{\Delta x}\left(\max(0, f^h_{\Delta x}(W^n_{j,i}, W^n_{j,i+1})) - \min(0, f^h_{\Delta x}(W^n_{j,i-1}, W^n_{j,i}))\right) \leq H^n_{j,i}$$

- Finally, the quotient $\frac{h^{n+1}_{j,i}}{\tilde{H}^{n+1}_{j,i}}$ is a convex combination of $X^n_{j,i-1}$, $X^n_{j,i}$ and $X^n_{j,i+1}$ which are in $[0, 1]$ by definition.
- We deduce $0 \leq h^{n+1}_{j,i} \leq \tilde{H}^{n+1}_{j,i}$.

Second-order MUSCL extension

- We reconstruct new values on the two sub-cells

$$\begin{cases} [x_{i-\frac{1}{2}}, x_i] \text{ (values with exposant -)} \\ [x_i, x_{i+\frac{1}{2}}] \text{ (values with exposant +)} \end{cases} :$$

$$h_{j,i}^{n,\pm} = h_{j,i}^{n} \pm (\Delta h)_{j,i} \geq 0,\ (hu)_{j,i}^{n,\pm} = (hu)_{j,i}^{n} \pm (\Delta hu)_{j,i},$$

$$H_{j,i}^{n,\pm} = H_{j,i}^{n} \pm (\Delta H)_{j,i} > 0,$$

$$X_{j,i}^{n,\pm} = \frac{h_{j,i}^{n,\pm}}{H_{j,i}^{n,\pm}},\ (Hu)_{j,i}^{n,\pm} = \frac{H_{j,i}^{n,\pm}(hu)_{j,i}^{n,\pm}}{h_{j,i}^{n,\pm}}$$

$W_{j,i-1}^{n,+}$	$W_{j,i}^{n,-}$	$W_{j,i}^{n,+}$	$W_{j,i+1}^{n,-}$

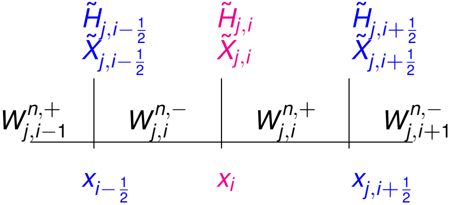
$x_{i-\frac{1}{2}}$ $\quad x_i \quad$ $x_{i+\frac{1}{2}}$

- We define upwind values on the interfaces x_i and $x_{i+\frac{1}{2}}$:

$$\tilde{H}_{j,i},\ \tilde{X}_{j,i} = \begin{cases} H_{j,i}^{n,-},\ X_{j,i}^{n,-} \text{ if } f_{\Delta x}^{h}(W_{j,i}^{n,-}, W_{j,i}^{n,+}) > 0 \\ H_{j,i}^{n,+},\ X_{j,i}^{n,+} \text{ else} \end{cases}$$

and

$$\tilde{H}_{j,i+\frac{1}{2}},\ \tilde{X}_{j,i+\frac{1}{2}} = \begin{cases} H_{j,i}^{n,+},\ X_{j,i}^{n,+} \text{ if } f_{\Delta x}^{h}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}) > 0 \\ H_{j,i+1}^{n,-},\ X_{j,i+1}^{n,-} \text{ else} \end{cases}$$



Second-order scheme

We deduce a conservative reconstruction of w_i^{n+1} on the initial cell $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ with values $w_i^{n+1,\pm}$ calculated with the first-order scheme on each sub-cell :

$$w_{j,i}^{n+1} = \frac{w_{j,i}^{n+1,-} + w_{j,i}^{n+1,+}}{2}$$

We obtain the new scheme :

$$\begin{aligned} w_{j,i}^{n+1} = w_{j,i}^{n} - \frac{\Delta t}{\Delta x}\Big(\tilde{X}_{j,i+\frac{1}{2}} f_{\Delta x}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}) - \tilde{X}_{j,i-\frac{1}{2}} f_{\Delta x}(W_{j,i-1}^{n,+}, W_{j,i}^{n,-})\Big) \\ + \frac{g}{2}\frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ \tilde{H}_{j,i}\tilde{H}_{j,i+\frac{1}{2}}(\tilde{X}_{j,i+\frac{1}{2}} - \tilde{X}_{j,i}) + \tilde{H}_{j,i}\tilde{H}_{j,i-\frac{1}{2}}(\tilde{X}_{j,i} - \tilde{X}_{j,i-\frac{1}{2}}) \end{pmatrix} \end{aligned}$$

Test 1 on flat bottom with $r = 0.7$

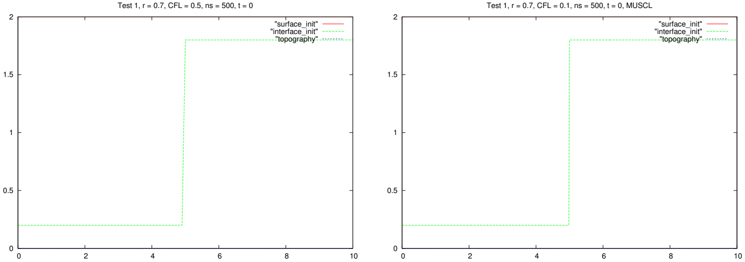


FIGURE: left : first-order, ns = 500, CFL = 0.5 ; right : second-order, ns = 500, CFL = 0.1

We put boundary conditions to simulate infinite reservoirs at both ends :

- Neumann conditions for h_1 and h_2,
- $h_1 u_1 = -h_2 u_2$

We let fluids evolve until a stationary state is reached.

Test 1 on flat bottom with $r = 0.7$

Let us stop earlier to better compare the two schemes :

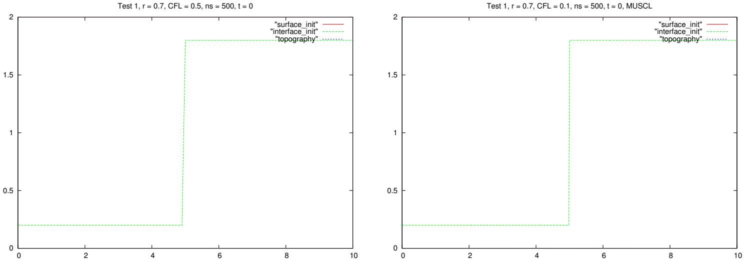
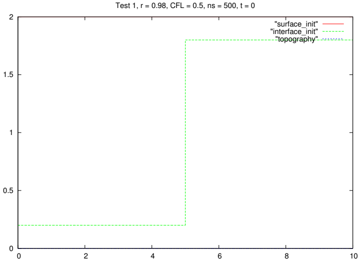
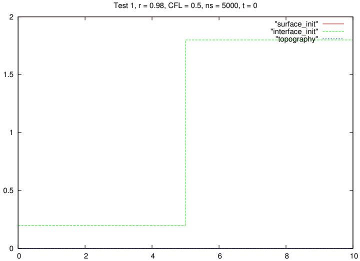


FIGURE: left : first-order, ns = 500, CFL = 0.5 ; right : second-order, ns = 500, CFL = 0.1

Test 1 on flat bottom with $r = 0.98$





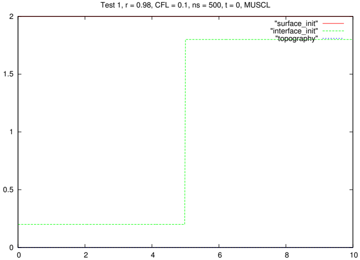


FIGURE: left : first-order, ns = 500 ; right : first order, ns = 5000 ; lower : second-order, ns = 500

Test 1 on flat bottom with $r = 0.98$

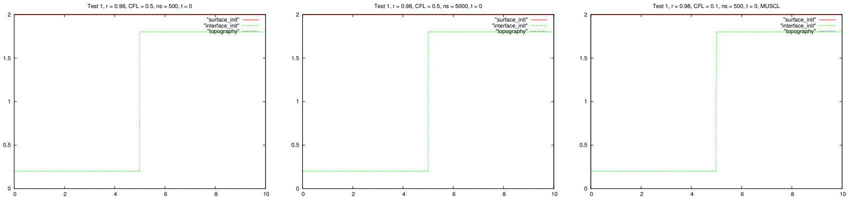


FIGURE: left : first-order, ns = 500 ; right : first order ns = 5000 ; lower : second-order, ns = 500

- first-order with 5000 cells $\simeq$ second-order with 500 cells,
- improvement when comparing with result obtained by Bouchut and Morales.

Test 2 : lock-exchanged on flat bottom with $r = 0.85$

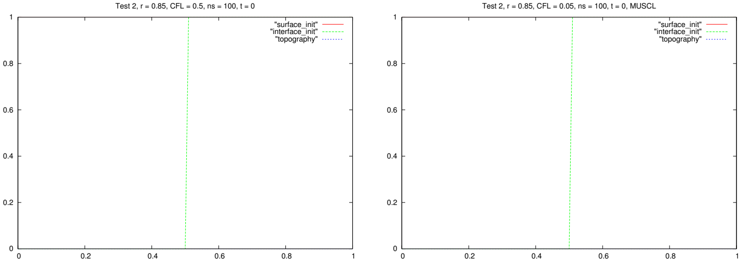


FIGURE: left : first-order, ns = 100, CFL = 0.5 ; right : second-order, ns = 100, CFL = 0.05

Test 2 : lock-exchanged on flat bottom with $r = 0.95$

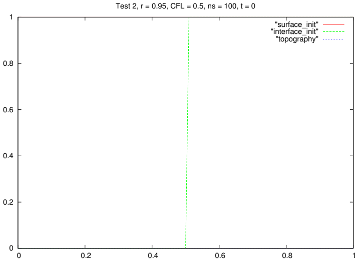


FIGURE: left : first-order, ns = 100, CFL = 0.5, right : second-order, ns = 100, CFL = 0.05

Lock-exchanged over a bump with $r = 0.98$

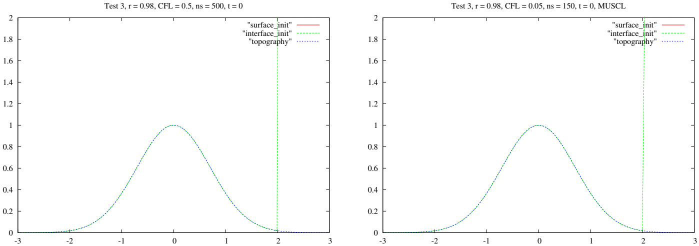


FIGURE: left : first-order, ns = 500, CFL = 0.5 ; right : second-order, ns = 150, CFL = 0.05

- we compute the approximate solution until a steady state is reached,
- we get values $q_1 = -q_2$, $h_1(x = -3)$, $h_2(x = -3)$,
- we calculate the corresponding $(h_1(x), h_2(x))$ by solving steady state equations,
- we obtain good agreement between approximation and reference solutions.

Conclusion

- We have find a way to split the two-layer system into two one-layer systems.
- The first numerical results are satisfying. We will next look for further results.
- It seems that improvement can be obtained for the CFL value.