High-order accuracy, entropy stability and convergence for finite difference methods for hyperbolic conservation laws

Ulrik Skre Fjordholm
with Siddharth Mishra and Eitan Tadmor

ETH Zürich

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Entropy conditions

- Consider the scalar conservation law

$$u_t + f(u)_x = 0. \tag{1}$$

Definition (Entropy pair)

A pair of functions $(\eta(u), q(u))$ such that $q'(u) = \eta'(u)f'(u)$.

- Multiply (1) by **entropy variables** $v := \eta'(u)$:

$$\eta(u)_t + q(u)_x = 0$$

(entropy is conserved). However, entropy should be *dissipated* at shocks.

Entropy condition: For all convex entropy pairs (η, q),

$$\eta(u)_t + q(u)_x \leqslant 0.$$

Implies existence and uniqueness for scalar conservation laws and linear systems.

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- Sequences u^ε approximating (1) might satisfy

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x = \text{SOMETHING}.$$

We will study SOMETHING to derive stability bounds for high-order accurate finite difference schemes.

Convergence of viscous approximations

Consider the viscous approximation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon(D(u^\varepsilon)u_x^\varepsilon)_x \tag{2}$$

for some $D(u^\varepsilon) \geqslant c > 0$. Multiply (2) by $v = \eta'(u)$:

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x = \varepsilon\left(D(u^\varepsilon)\eta'(u^\varepsilon)u_x^\varepsilon\right)_x - \varepsilon\eta''(u^\varepsilon)D(u^\varepsilon)(u_x^\varepsilon)^2 \tag{3}$$

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- From (3) we get that for every convex $\eta(u)$ with $\eta''(u) \geqslant d > 0$,

  $$\|\eta(u^\varepsilon(T))\|_{L^1(\mathbb{R})} - \|\eta(u^\varepsilon(0))\|_{L^1(\mathbb{R})} \leqslant -C\varepsilon\|u_x^\varepsilon\|_{L^2(\mathbb{R}\times[0,T])}^2 \qquad \forall\ T > 0.$$

  This gives the oscillation bound

  $$\|u_x^\varepsilon\|_{L^2(\mathbb{R}\times[0,T])} \lesssim \frac{1}{\sqrt{\varepsilon}}$$

  The maximum principle for (2) gives

  $$\|u^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,T])} \leqslant \|u^\varepsilon(0)\|_{L^\infty(\mathbb{R})}.$$

- Using compensated compactness, these stability estimates are enough to show that u^ε converges strongly to the entropy solution of (1).

First-order finite difference methods

- We consider the finite difference scheme

$$\frac{d}{dt}u_i + \frac{1}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0.$$

- First-order schemes are generally of the form

$$\underbrace{\frac{d}{dt}u_i}_{\approx u_t} + \underbrace{\frac{\overline{f}_{i+1/2} - \overline{f}_{i+1/2}}{\Delta x}}_{\approx f(u)_x} = \underbrace{\frac{D_{i+1/2}(u_{i+1} - u_i) - D_{i-1/2}(u_i - u_{i-1})}{\Delta x}}_{\approx \Delta x (D(u) u_x)_x}$$

$\left(\overline{f}_{i+1/2} = \left(f(u_i) + f(u_{i+1})\right)/2\right)$ for some $D_{i+1/2} \geqslant c > 0$.

- [Lax 1971] showed that the (fully discrete) Lax-Friedrichs scheme satisfies

$$\sum_i \eta(u_i^n)\Delta x - \sum_i \eta(u_i^0)\Delta x \leqslant -C \sum_{m\leqslant n} \sum_i |u_{i+1}^m - u_i^m|^2 \Delta t \qquad \forall\ n \in \mathbb{N}$$

(generalized to E-schemes by Coquel and LeFloch, 1991). This gives the oscillation bound

$$\sum_{m\leqslant n} \sum_i |u_{i+1}^m - u_i^m|^2 \Delta t \leqslant C,$$

analogous to the bound $\|u_x^\varepsilon\|_{L^2} \lesssim \frac{1}{\sqrt{\varepsilon}}$. The scheme is uniformly bounded,

$$\|u^n\|_{L^\infty(\mathbb{R})} \leqslant \|u^0\|_{L^\infty(\mathbb{R})}.$$

- Using these estimates, [DiPerna 1983] proved convergence to the entropy solution of (1) for all $u^0 \in L^\infty(\mathbb{R})$, using compensated compactness.

Higher-order finite difference methods

- Higher-order schemes are of the form

$$\frac{d}{dt}u_i + \frac{\widehat{f}_{i+1/2} - \widehat{f}_{i+1/2}}{\Delta x} = \frac{D_{i+1/2}(u_{i+1} - u_i) - D_{i-1/2}(u_i - u_{i-1})}{\Delta x}$$

$(\widehat{f}_{i+1/2} = (f(u_i^+) + f(u_{i+1}^-))/2)$.

- $D_{i+1/2}$ is in general $O(\Delta x^k)$ for a $k \geqslant 1$

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- $D_{i+1/2}$ is in general $O(\Delta x^k)$ for a $k \geqslant 1$
- Unlike in the first-order setting, $D_{i+1/2}$ can vanish, even when $|u_{i+1} - u_i| \neq 0$. This degeneracy makes it hard to obtain bounds on $\|u_x\|_{L^2}$, or even on

$$\sum_{m\leqslant n}\sum_i |u_{i+1}^m - u_i^m|^p \Delta t$$

$(p > 2)$.

- Common "dirty fix": Replace $D_{i+1/2}$ by $\widetilde{D}_{i+1/2} = \max(D_{i+1/2}, C\Delta x^\beta)$.

- We consider the finite difference scheme

$$\frac{d}{dt}u_i + \frac{1}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0.$$

- An **entropy conservative** flux F produces solutions satisfying

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) = 0$$

  for a given entropy $\eta(u)$ and numerical entropy flux function $Q_{i+1/2} = Q(u_i, u_{i+1})$ consistent with $q(u)$.

- We add numerical diffusion to obtain **entropy stability**:

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) \leqslant 0. \tag{4}$$

- We wish to design high-order accurate entropy stable schemes.

Entropy conservative schemes

Denote $\overline{v}_{i+1/2} = \frac{v_i + v_{i+1}}{2}$, $[\![v]\!]_{i+1/2} = v_{i+1} - v_i$. Recall that $v(u) = \eta'(u)$.

Theorem (Tadmor 1987)

If $\tilde{F}$ is given by

$$\tilde{F}_{i+1/2} = \frac{[\![\psi]\!]_{i+1/2}}{[\![v]\!]_{i+1/2}}$$

($\psi(u) := v(u)f(u) - q(u)$) then computed solutions u_i satisfy

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) = 0$$

with $Q_{i+1/2} = \overline{v}_{i+1/2}\tilde{F}_{i+1/2} - \overline{\psi}_{i+1/2}$.

(Formally) p-th order accurate ($p \geqslant 2$) entropy conservative fluxes $\tilde{F}^p$ are obtained through linear combinations of $\tilde{F}$ using Richardson extrapolation [LeFloch, Mercier, Rohde, 2002].

Entropy stable schemes

- Need dissipation of entropy in the vicinity of shocks:

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) \leqslant 0. \tag{5}$$

Theorem (Tadmor 1987)

If

$$F_{i+1/2} = \tilde{F}_{i+1/2} - P_{i+1/2}[\![v]\!]_{i+1/2} \tag{6}$$

for an entropy conservative flux $\tilde{F}$ *and a* $P_{i+1/2} \geqslant 0$, *then*

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) = -\frac{1}{2\Delta x}\left(P_{i+1/2}[\![v]\!]^2_{i+1/2} + P_{i-1/2}[\![v]\!]^2_{i-1/2}\right) \leqslant 0.$$

- The scheme is then

$$\begin{aligned}\frac{d}{dt}u_i + \frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta x} &= \frac{P_{i+1/2}[\![v]\!]_{i+1/2} - P_{i-1/2}[\![v]\!]_{i-1/2}}{\Delta x} \\ &= \frac{D_{i+1/2}[\![u]\!]_{i+1/2} - D_{i-1/2}[\![u]\!]_{i-1/2}}{\Delta x}\end{aligned}$$

with $D_{i+1/2} = P_{i+1/2}\eta''(\xi) \geqslant 0$, for some ξ between u_i and u_{i+1}.

High-order entropy stable schemes

Recall: entropy stable fluxes have the form $F_{i+1/2} = \tilde{F}_{i+1/2} - P_{i+1/2}[\![v]\!]_{i+1/2}$.

- Use a p-th order accurate, entropy conservative flux $\tilde{F}^p$ in place of $\tilde{F}$.
- Choose P such that $P_{i+1/2} = O(|u_{i+1} - u_i|^{p-1})$, for instance

$$P_{i+1/2} = c_{i+1/2}|u_{i+1} - u_i|^{p-1}$$

for a $c_{i+1/2} \geqslant c > 0$. With this choice,

$$\sum_i \eta(u_i(T))\Delta x - \sum_i \eta(u_i(0))\Delta x \leqslant -C \int_0^T \sum_i |u_{i+1} - u_i|^{p+1} \Delta t \qquad \forall\ T > 0$$

for a $C > 0$, provided the entropy η is strongly convex ($\eta''(u) \geqslant d > 0$ for all u).

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Proposition

Assume that $\|u\|_{L^\infty}$ is uniformly bounded. Then the semi-discrete scheme with flux

$$F_{i+1/2} = \tilde{F}^p_{i+1/2} - c_{i+1/2}|u_{i+1} - u_i|^{p-1}(u_{i+1} - u_i)$$

is formally p-th order accurate and converges to a weak solution of (1) in L^q for all $q \in [1, \infty)$.

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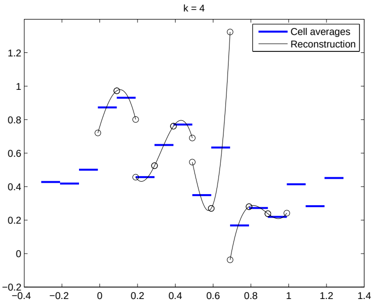
- This Lax-Wendroff-type scheme will be very oscillatory around shocks.

High-order reconstruction

- Instead, we perform an ENO (Essentially Non-Oscillatory) reconstruction of u_i.



- In each cell $[x_{i-1/2}, x_{i+1/2})$ we obtain *reconstructed values* $u = u_i(x)$. The values of interest are the *cell interface values*

$$u_i^+ = u_i(x_{i+1/2}), \qquad u_{i+1}^- = u_{i+1}(x_{i+1/2}).$$

- Set the diffusion operator to $c_{i+1/2}\langle\langle u\rangle\rangle_{i+1/2}$, where $c_{i+1/2} \geqslant c > 0$ and $\langle\langle u\rangle\rangle_{i+1/2} = u_{i+1}^{-} - u_i^{+}$.
- This gives the flux

$$F_{i+1/2} = \tilde{F}^p_{i+1/2} - c_{i+1/2}\langle\langle u\rangle\rangle_{i+1/2}. \tag{7}$$

  In smooth regions, $\langle\langle u\rangle\rangle_{i+1/2} = O(\Delta x^p)$, while near discontinuities, $\langle\langle u\rangle\rangle_{i+1/2} = O(1)$. We name the scheme the **TeCNO scheme**.
- **Recall:** entropy stable fluxes have the general form

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- Set the diffusion operator to $c_{i+1/2}\langle\langle u\rangle\rangle_{i+1/2}$, where $c_{i+1/2} \geqslant c > 0$ and $\langle\langle u\rangle\rangle_{i+1/2} = u_{i+1}^{-} - u_i^{+}$.
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- **Recall:** entropy stable fluxes have the general form

$$F_{i+1/2} = \tilde{F}_{i+1/2} - P_{i+1/2}[\![v]\!]_{i+1/2}.$$

- We rewrite (7) as

$$F_{i+1/2} = \tilde{F}^p_{i+1/2} - c_{i+1/2}\underbrace{\left(\frac{\langle\langle u\rangle\rangle_{i+1/2}}{[\![u]\!]_{i+1/2}}\right)}_{(*)}\underbrace{\left(\frac{[\![u]\!]_{i+1/2}}{[\![v]\!]_{i+1/2}}\right)}_{(**)}[\![v]\!]_{i+1/2}.$$

  $c_{i+1/2}$ and $(**)$ are positive by definition.
- If $\operatorname{sgn}\langle\langle u\rangle\rangle_{i+1/2} = \operatorname{sgn}[\![u]\!]_{i+1/2}$, then $(*)$ is positive, and so the scheme is entropy stable.

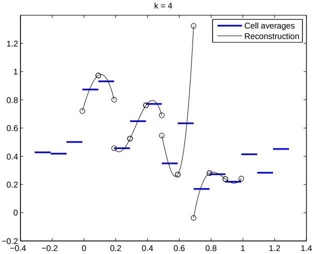
Sign property of ENO reconstruction

Theorem ([USF, Mishra, Tadmor 2012a])

There is a constant $C_p > 0$ such that the p-th order accurate ENO reconstruction satisfies

$$0 \leqslant \frac{\langle\langle u \rangle\rangle_{i+1/2}}{[\![u]\!]_{i+1/2}} \leqslant C_p.$$

C_p only depends on p.



Entropy dissipation rates

- Our scheme has the form

$$\frac{d}{dt}u_i + \frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta x} = \frac{c_{i+1/2}\langle\langle u\rangle\rangle_{i+1/2} - c_{i-1/2}\langle\langle u\rangle\rangle_{i-1/2}}{\Delta x},$$

and satisfies the entropy dissipation estimate

$$\frac{d}{dt}\eta(u_i) + \frac{Q_{i+1/2} - Q_{i-1/2}}{\Delta x} = -\frac{(D[\![u]\!]\langle\langle u\rangle\rangle)_{i+1/2} + (D[\![u]\!]\langle\langle u\rangle\rangle)_{i-1/2}}{2\Delta x} \leqslant 0,$$

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- This implies the estimates

$$\sum_i \eta(u_i(T))\Delta x - \sum_i \eta(u_i(0))\Delta x \leqslant -C\int_0^T \sum_i [\![u]\!]\langle\langle u \rangle\rangle_{i+1/2}\,dt \qquad \forall\ T > 0.$$

Hence,

$$\int_0^T \sum_i [\![u]\!]\langle\langle u \rangle\rangle_{i+1/2}\,dt = \int_0^T \sum_i (u_{i+1} - u_i)\left(u_{i+1}^- - u_i^+\right)dt \leqslant C. \tag{8}$$

- **Question:** Does (8) imply that

$$\int_0^T \sum_i |u_{i+1} - u_i|^2\,dt \leqslant C,$$

that is, "$\|u_x^\varepsilon\|_{L^2} \lesssim \frac{1}{\sqrt{\varepsilon}}$"?

Stability estimates (?)

Let $\{u_i\}_{i\in\mathbb{Z}}$ be any given sequence, and define $\Delta u_i = u_{i+1} - u_i$ and

$$\nu_i^1 = i, \qquad \nu_i^{k+1} = \begin{cases} \nu_i^k & \text{if } |\Delta^k u_i| > |\Delta^k u_{i+1}| \\ \nu_{i+1}^k & \text{if } |\Delta^k u_i| \leqslant |\Delta^k u_{i+1}| \end{cases} \qquad (k \in \mathbb{N}).$$

Lemma

For every $p \in \mathbb{N}$ there is a $C = C(p)$ such that

$$\sum_i |\Delta u_{\nu_i^p}||\Delta^p u_i| \leqslant C \sum_i (u_{i+1} - u_i)(u_{i+1}^- - u_i^+)$$

where u_i^+, u_{i+1}^- are p-th order ENO-reconstructed values at $x = x_{i+1/2}$.

Conjecture

For every $k, n \in \mathbb{N}$ there is a $C = C(k, n)$ such that

$$\sum_i |\Delta u_{\nu_i^k}|^n |\Delta^k u_i| \leqslant C\|u\|_\infty \sum_i |\Delta u_{\nu_i^{k+1}}|^{n-1} |\Delta^{k+1} u_i|. \tag{9}$$

Applying (9) $p-2$ times, we get

$$\sum_i |\Delta u_i|^p \leqslant C\|u\|_\infty^{p-2} \sum_i |\Delta u_{\nu_i^p}||\Delta^p u_i|.$$

Convergence of the TeCNO method

Corollary (1)

There is a $C = C(p)$ such that

$$\sum_i |u_{i+1} - u_i|^p \leqslant C\|u\|_\infty^{p-2} \sum_i (u_{i+1} - u_i)(u_{i+1}^- - u_i^+).$$

Since $\|u\|_\infty$ and $\int \sum_i (u_{i+1} - u_i)(u_{i+1}^- - u_i^+) dt$ are bounded, we get

Corollary (2)

There is a $C = C(p)$ such that

$$\int_0^T \sum_i |u_{i+1} - u_i|^p dt \leqslant C.$$

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Since $\|u\|_\infty$ and $\int \sum_i (u_{i+1} - u_i)(u_{i+1}^- - u_i^+)dt$ are bounded, we get

Corollary (2)

There is a $C = C(p)$ such that

$$\int_0^T \sum_i |u_{i+1} - u_i|^p \, dt \leqslant C.$$

Corollary (3)

Assume that $\|u\|_{L^\infty}$ is uniformly bounded. Then the TeCNO method is formally p-th order accurate and converges to a weak solution of (1) in L^q for all $q \in [1, \infty)$.

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