An approximation scheme for an Eikonal Equation with discontinuous coefficient

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Outline of the talk

1. Degenerate eikonal equation with discontinuous coefficient
2. Theoretical background
3. A semiLagrangian scheme with good properties
4. Tests and Applications
5. Concluding Remarks

Degenerate Eikonal Equation with discontinuous coefficient

$\Omega \subset \mathbb{R}^n$ open bounded domain with regular boundary

$$\begin{cases} \sum_{i,j=1}^{N} b_{ij}(x) U_{x_i} U_{x_j} = [f(x)]^2 & x \in \Omega \\ U(x) = G(x) & x \in \partial\Omega \end{cases} \qquad \text{(EK)}$$

- $G : \partial\Omega \to [0, +\infty[$ is continuous
- b is symmetric, positive semidefinite
- $(b_{i,j}) = (\sigma_{ik}) \cdot (\sigma^t_{kj})$ with $\sigma(\cdot) : \overline{\Omega} \to \mathbb{R}^{NM}$ is L-Lipschitz continuous but possibly degenerate
- $f : \mathbb{R}^N \to [\rho, +\infty[$, $\rho > 0$ is Borel measureable but possibly discontinuous

Optimal Control InterprGammation

Rewriting the differential operator in the following form

$$\sum_{i,j=1}^{N} b_{ij}(x)p_i p_j = \sum_{k=1}^{M} (p \cdot \sigma_k(x))^2 = |p \cdot \sigma(x)|^2,$$

where $(\sigma_{ik})_k := \sigma_k : \Omega \to \mathbb{R}^N$, $k = 1, ...M$.

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where $(\sigma_{ik})_k := \sigma_k : \Omega \to \mathbb{R}^N$, $k = 1, ...M$.

We get the equivalent Bellman equation

$$\max_{|a|\leq 1} \left\{ -DU(x) \cdot \sum_{k=1} a^k \sigma_k(x) \right\} = f(x) \qquad \text{(BL)}$$

associated to the symmetric optimal control system

$$\dot{y} = \sum_{k=1}^{M} a^k \sigma_k(y), \quad y(0) = x,$$

where $a : [0, +\infty[\to \{a \in \mathbb{R}^M : |a| \leq 1\}$ and $y(\cdot) \equiv y_x(\cdot, a)$ is a solution.

Viscosity solution via semicontinuous envelopes

$$f_*(x) = \lim_{r\to 0^+} \inf\{f(y) : |y-x| \le r\},$$
$$f^*(x) = \lim_{r\to 0^+} \sup\{f(y) : |y-x| \le r\}$$

Definition (Discontinuous Viscosity Solution [Ishii 1985])

A lower semicontinuous (resp. upper) function U is a viscosity super-solution (resp. sub-) of the equation (EK) if for all $\phi \in C^1(\Omega)$, and $x_0 \in argmin_{x\in\Omega}(U-\phi)$, (resp. $x_0 \in argmax_{x\in\Omega}(U-\phi)$), we have

$$\sum_{i,j=1}^{N} b_{ij}(x_0)\phi_{x_i}(x_0)\phi_{x_j}(x_0) \ge [f_*(x_0)]^2 ,$$

$$(\text{resp. } \sum_{i,j=1}^{N} b_{ij}(x_0)\phi_{x_i}(x_0)\phi_{x_j}(x_0) \le [f^*(x_0)]^2).$$

A function U is a discontinuous viscosity solution of the equation (EK) if **U^* is a subsolution and U_* is a supersolution**.

Dirichelet Conditions

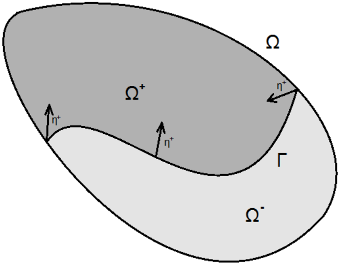
Definition (DC in weaker sense)

We say that an upper semicontinuous function U, subsolution of the equation in (EK), satisfies weakly the DC if for all $\phi \in C^1$ and $\hat{x} \in \partial\Omega$, $\hat{x} \in argmax_{x \in \overline{\Omega}}(U - \phi)$ such that $U(\hat{x}) > G(\hat{x})$, then we have

$$\sum_{i,j=1}^{N} b_{ij}(\hat{x})\phi_{x_i}\phi_{x_j} \leq [f_*(\hat{x})]^2 .$$

Lower semicontinuous functions that satisfy weakly the DC are defined accordingly.

Hypotheses on the discontinuity interface



Assumptions on discontinuities

- $\Gamma = \{x \in \mathbb{R}^N : f \text{ is discontinuous at } x\}$ is a disjoint union of finite Lipschitz hypersurfaces ($\exists \eta^+$ transversal vector)
- f is continuous in each component $\Omega^\pm$
- if $x \in \Gamma$, $\quad f(x) \in \left[\lim_{\Omega^- \ni y \to x} f(y), \lim_{\Omega^+ \ni y \to x} f(y) \right].$
- if $\hat{x} \in \Gamma \cap \partial\Omega$ we can choose η^+, η^- both inward for Ω.

Comparison Results [Soravia 2006]

Theorem

Let Ω be an open domain with Lipschitz boundary.

- *Let $U, V : \overline{\Omega} \to \mathbb{R}$ be upper and a lower- semicontinuous function, a subsolution and a supersolution of* (EK) *with weak Dirichlet conditions.*
- *Suppose that V is **nontangentially continuous** on $\partial\Omega \setminus \Gamma$ in the inward direction η_Ω and on $\Gamma \cap \overline{\Omega}$ in the direction of η^+.*

Then $U \leq V$ in $\overline{\Omega}$.

Corollary

*Let $U : \overline{\Omega} \to \mathbb{R}$ be a **continuous, bounded viscosity solution** of the problem* (EK).

Then U is unique.

Example I

$$\begin{cases} x^2\left(u_x(x,y)\right)^2 + \left(u_y(x,y)\right)^2 = [f(x,y)]^2 &]-1,1[\times]-1,1[\\ u(\pm 1, y) = u(x, \pm 1) = 0 & x,y \in [-1,1] \end{cases}$$

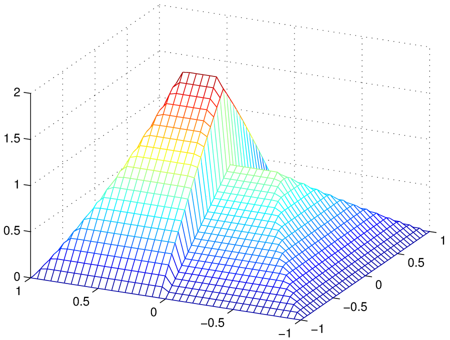
where $f(x,y) = 2$, for $x > 0$, and $f(x,y) = 1$ for $x \leq 0$.

$$b_{i,j} = \begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

therefore the Bellman's equation in this case is

$$\max_{|a|\leq 1}\left\{-Du(x,y)\cdot a_1(x,0)^T - Du(x,y)\cdot a_2(0,1)^T\right\} = f(x,y).$$

Example II



$$u(x,y) = \begin{cases} 2(1-|y|) & x > 0, |y| > 1 + \ln x \\ -2ln(x) & x > 0, |y| \leq 1 + \ln x \\ \frac{u(-x,y)}{2} & x \leq 0. \end{cases} \qquad (1)$$

is a viscosity solution of the problem.

A semiLagrangian Approx.: time discretization

1. (Kruzkov's transform). $V(x) = 1 - e^{-U(x)}$
$$|DV(x)\cdot\sigma(x)| = f(x)(1 - V(x))$$
2. ($1/f$ as velocity)
$$\frac{|DV(x)\cdot\sigma(x)|}{f(x)} = 1 - V(x)$$
3. (Bellman-type equation)
$$\sup_{a\in B(0,1)}\left\{\frac{\sum_k a^k\sigma_k(x)}{f(x)}\cdot DV(x)\right\} = 1 - V(x)$$
4. (discretize as diretional derivative)

$$\begin{cases} V_h(x) = \frac{1}{1+h}\inf_{a\in B(0,1)}\left\{V_h\left(x - h\frac{\sum_k a^k\sigma_k(x)}{f(x)}\right)\right\} + \frac{h}{1+h} & x\in\Omega \\ V_h(x) = 1 - e^{-G(x)} & x\in\partial\Omega \end{cases} \quad \text{(SDE)}$$

The set A(x,h)

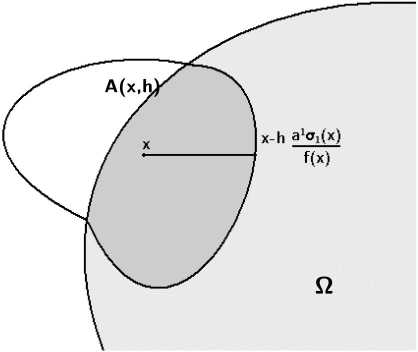


Figure: The set $\overline{A}(x,h) := \left\{ x - h\frac{\sum a^k \sigma_k(x)}{f(x)} ; x \in B(0,1) \right\}$ in dimension 2. In grey $A(x,h) := \Omega \cap \overline{A}(x,h)$

A semiLagrangian Approx.: space discretization

Let us assume that $\Omega = \Pi_{i=1}^{n}(a_i, b_i)$, $\Omega_{\Delta x} := \mathbb{Z}^n_{\Delta x} \cap \Omega$ and that the grind size $\Delta x > 0$.

We look for a solution of

$$\begin{cases} W(x_\alpha) = \frac{1}{1+h} \min\limits_{a \in B(0,1)} I[W](x_\alpha - h\frac{\sum_k a^k \sigma_k(x_\alpha)}{f(x_\alpha)}) + \frac{h}{1+h} & x_\alpha \in \Omega_{\Delta x} \\ W(x_\alpha) = 1 - e^{-\phi(x_\alpha)} & x_\alpha \in \partial\Omega_{\Delta x} \end{cases} \quad \text{(SL)}$$

where $I[W](x)$ is a **linear interpolation**, in the space

$$\mathcal{W}^{\Delta x} := \left\{ W : \overline{\Omega} \to \mathbb{R} | W \in C(\Omega) \text{ and } DW(x) = c_\alpha \right. \\ \left. \text{for any } x \in (x_\alpha, x_{\alpha+1}) \right\}.$$

Properties of the scheme

- (There exists a **fixed point**) Let $A(x_\alpha, h) \in \overline{\Omega}$, for every $x_\alpha \in \Omega_{\Delta x}$, for any $a \in B(0,1)$, so there exists a unique solution W of (SL) in $\mathcal{W}^{\Delta x}$
- (Consistency) Developing with che usual Taylor expansion, like in the DF case, we find consistency
- (**Monotonicity**) The following estimate holds true:

$$||W^n - W||_\infty \leq \left(\frac{1}{1+h}\right)^n ||W_0 - W||_\infty.$$

Error estimates

Theorem

Under the introduced Hypotheses, we have that

$$||V(x) - W(x)||_{L^1(\Omega)} \leq C\sqrt{h} + C'\frac{\Delta x}{h} \quad \text{for all } h > 0$$

for some constant $C, C' > 0$ *independent from h. Moreover, if* $v(x) \in C(\Omega)$ *we have*

$$||V(x) - W(x)||_{L^\infty(\Omega)} \leq C\sqrt{h} + \frac{1+h}{h}(C'\Delta x) \quad \text{for all } h > 0$$

for some constant $C, C' > 0$ *independent from h.*

Sketch of the proof I

We start introducing, for a $\epsilon > 0$, the set $\Lambda_{2\epsilon} := \{x \in \Omega | B(x, 2\epsilon) \cap \Lambda \neq \emptyset\}$. We observe that

$$||V(x) - W(x)||_{L^1(\Omega)} \leq \int_{\Omega \setminus \Lambda_{2\epsilon}} |V(x) - W(x)| dx$$

$$+ \int_{\Lambda_{2\epsilon}} |V(x) - W(x)| dx \leq \int_{\Omega \setminus \Lambda_{2\epsilon}} |V(x) - W(x)| dx + m(\Lambda_{2\epsilon})$$

from the fact that $|V(x) - W(x)| \leq 1$ for all $x \in \Omega$.
If $x \in \partial\Omega$ the assumption is trivially verified because of Dirichlet boundary conditions.

Sketch of the proof II

Let $\hat{x} \in \Omega \setminus \Lambda_{2\epsilon}$. Consider for $\epsilon > 0$ the auxiliary function

$$\psi(x,y) := V(x) - W(y) - \frac{|x-y|^2}{2\epsilon} - \frac{|x-\hat{x}|^2}{2}$$

It is not hard to check that the boundness of v, v_* and the upper semicontinuity of ψ, implies the existence of some $(\overline{x}, \overline{y})$ in $\Omega^{\pm}$ (depending on ϵ) such that

$$\psi(\overline{x}, \overline{y}) \geq \psi(x,y) \quad \text{for all } x, y \in \Omega^{\pm}.$$

After some standard calculations, choosing $\epsilon = \sqrt{h}$ and the boundness of f and σ, we obtain

$$V(\overline{x}) - W(\overline{y}) \leq C\sqrt{h}$$

For C suitable positive constants.

Test 1

Let $\Omega := (-1,1)\times(0,2)$ and $f:\Omega\to\mathbb{R}$ be defined by

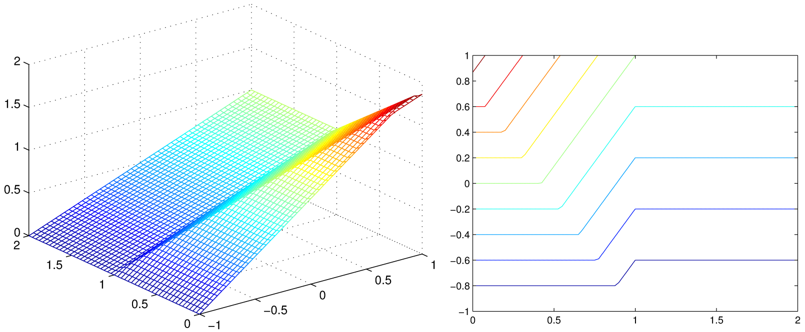
$$f(x_1,x_2) := \begin{cases} 1 & x_1 < 0, \\ 3/4 & x_1 = 0 \\ 1/2 & x_1 > 0 \end{cases}$$

It is not difficult to see that f satisfies our Hypotheses. We can verify that the function

$$u(x_1,x_2) := \begin{cases} \frac{1}{2}x_2, & x_1 \geq 0, \\ -\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, & -\frac{1}{\sqrt{3}}x_2 \leq x_1 \leq 0, \\ x_2, & x_1 < -\frac{1}{\sqrt{3}}x_2. \end{cases}$$

is a viscosity solution of $|Du| = f(x)$ in the sense of our definition.

Test 1 - Results



$\Delta x = h$	$\|\| \cdot \|\|_\infty$	$Ord(L_\infty)$	$\|\| \cdot \|\|_1$	$Ord(L_1)$
0.2	3.812 e-1		1.821 e-1	
0.1	1.734e-1	1.1364	8.112e-2	1.1666
0.05	8.039e-2	1.1095	3.261e-2	1.3148
0.025	4.359e-2	0.8830	1.616e-2	1.0178
0.0125	2.255e-2	0.9509	7.985e-3	1.0271

Test 2

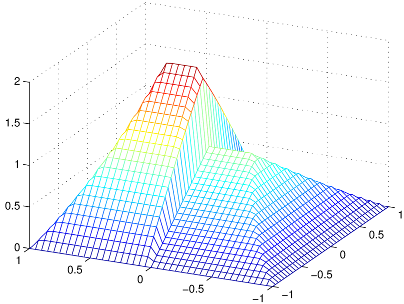
We consider the problem shown in a previous example . As already said, let $\Omega := [-1,1]^2$ we want to solve

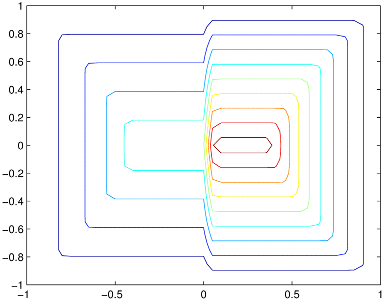
$$\begin{cases} x^2\left(u_x(x,y)\right)^2 + \left(u_y(x,y)\right)^2 = [f(x,y)]^2 &]-1,1[\times]-1,1[\\ u(\pm 1,y) = u(x,\pm 1) = 0 & x,y \in [-1,1] \end{cases}$$

with $f(x,y) = 2$, for $x > 0$, and $f(x,y) = 1$ for $x \leq 0$. The correct viscosity solution is function,

$$u(x,y) = \begin{cases} 2(1-|y|) & x > 0, |y| > 1 + \ln x \\ -2ln(x) & x > 0, |y| \leq 1 + \ln x \\ \frac{u(-x,y)}{2} & x \leq 0. \end{cases}$$

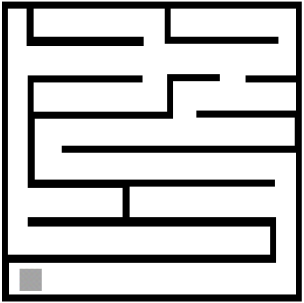
Test 2 - Results





$\Delta x = h$	$\|\| \cdot \|\|_\infty$	$Ord(L_\infty)$	$\|\| \cdot \|\|_1$	$Ord(L_1)$
0.2	1.0884		0.4498	
0.1	1.0469	-	0.2444	0.88
0.05	1.0242	-	0.1270	0.9444
0.025	1.0123	-	0.0628	0.9708
0.0125	1.0062	-	0.0327	0.9867

Applications I - Labyrinths



We consider the labyrinth $I(x)$ as a digital image with $I(x) = 0$ if x is on a wall, $I(x) = 0.5$ if x is on the target, $I(x) = 1$ otherwise.
We solve the eikonal equation

$$|Du(x)| = f(x) \quad x \in \Omega$$

with the discontinuous running cost

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } I(x) = 1 \\ M & \text{if } I(x) = 0. \end{cases}$$

Applications I - Labyrinths

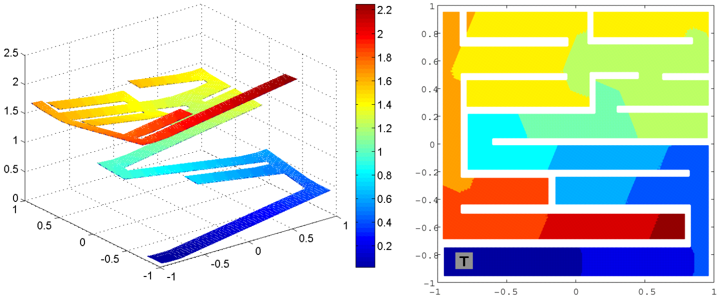
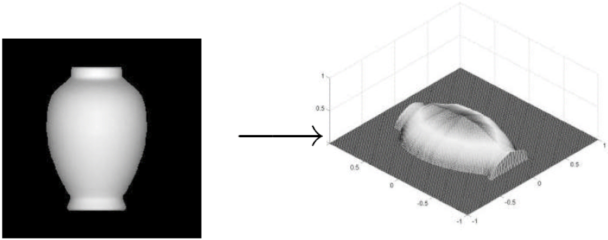


Figure: Mesh and level sets of the value function for the labyrinth problem ($dx = dt = 0.0078$, $M = 10^{10}$).

Applications II - Shape-From-Shading

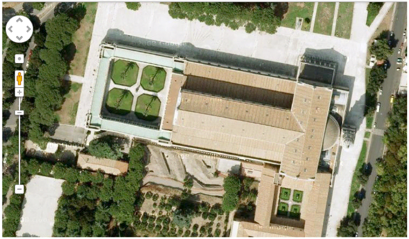


SFS equation

The SFS equation in the case of vertical light is

$$|Du(x,y)| = \left(\sqrt{\frac{1}{I(x,y)^2} - 1}\right), \quad (x,y) \in \Omega.$$

where I is the **brightness function** measured at all points (x,y) in the image.



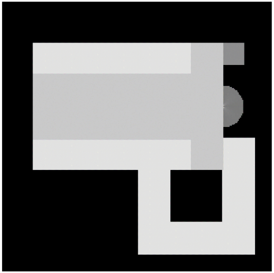


Figure: Basilica of Saint Paul Outside the Walls: satellite image and simplified sfs-datum.

Applications II - Shape-From-Shading

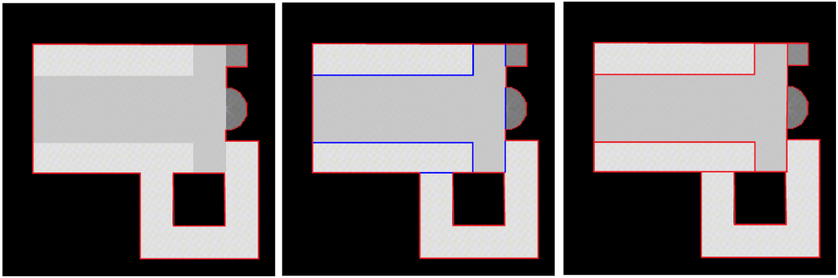


Figure: Choosing boundary conditions.

Applications II - Shape-From-Shading

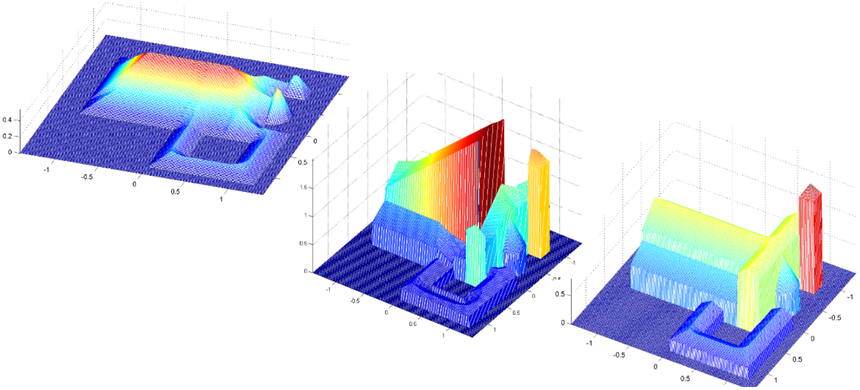


Figure: Solutions with various BC.

Concluding Remarks

- Two different problems overcomed: discontinuity of the running coast and degeneracy of the dynamics
- Error estimations provided
- Large presence in applications
- Generalization to discontinuous equation of a more general kind (Representation formulae [Soravia 2002])

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Thank you.

Definition

The set $\Gamma \subset \mathbb{R}^N$ is said to be a Lipschitz hypersurface if

- for all $\hat{x} \in \Gamma$ one of its neighborhoods is partitioned by Γ into two connected open sets Ω^+, Ω^- and Γ
- there exists a *transversal* unit vector $\eta^+ \in \mathbb{R}^N$, $|\eta^+| = 1$, s.t.: there are $c, r > 0$ s.t. if $x \in B(\hat{x}, r) \cap \overline{\Omega^\pm}$ then $B(x \pm t\eta^+, ct) \subset \Omega^\pm$ for all $0 < t \leq c$, respectively.

We will say that an open set Ω is a Lipschitz domain if $\partial\Omega$ is a Lipschitz hypersurface. In this case if for $\hat{x} \in \partial\Omega$ and a transversal unit vector Λ we have $\Omega^+ \subset \Omega$, then we call $\Lambda = \Lambda_\Omega$ an inward unit vector.

Definition

Given a Lipschitz surface $\Gamma \subset \mathbb{R}^N$ with transversal unit vector Λ, we say that a function $u : \Omega \to \mathbb{R}$ is nontangentially continuous at $\hat{x}$ in the direction of η if there are sequences $t_n \to 0^+$, and $p_n \to 0$, $p_n \in \mathbb{R}^N$, such that

$$\lim_{n\to+\infty} u(\hat{x} + t_n\eta + t_n p_n) = u(\hat{x}).$$