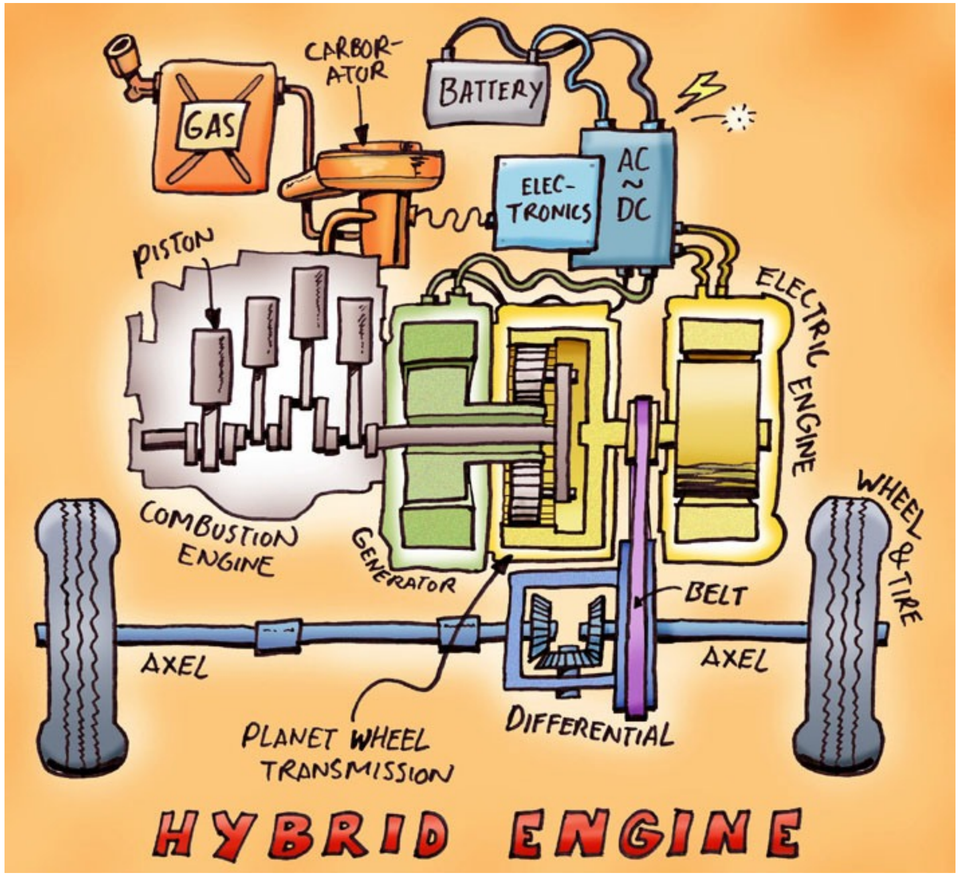
Monotone numerical approximations for optimal control of hybrid systems

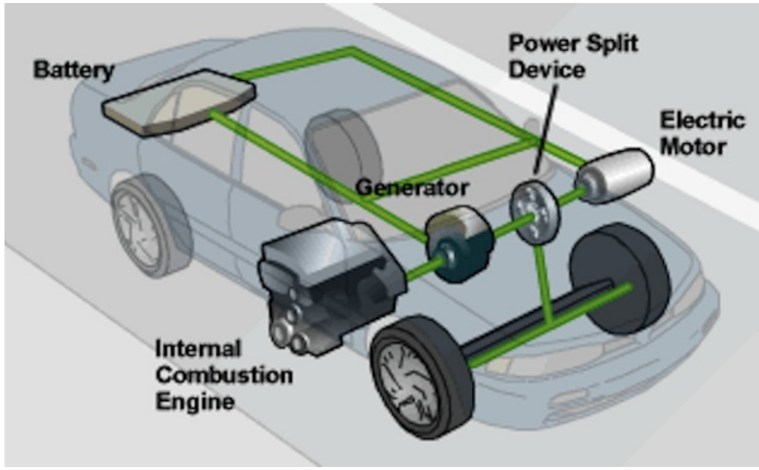
Roberto Ferretti

(joint work with J. Zhao and H. Zidani)

- A basic example in hybrid control
- A general framework for hybrid systems
- The use of monotone numerical schemes
- Numerical examples
- References

A basic example in hybrid control





A hybrid vehicle may use both an internal combustion engine and an electric engine:

- ICE: motion + battery recharging

- EE: motion

- Generator: battery recharging + braking

Simplified structure: discontinuous switching between ICE and EE.

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• When the battery is fully discharged the system must turn to the ICE, while at any intermediate level the switching may or may not be performed on the basis of the optimal strategy

• There exists a switching cost for changing the active engine

- Two different dynamics

$$\begin{cases} \dot{X}(t) = f_t(X(t), u(t)) = au(t) > 0 & \text{(thermic engine)} \\ \dot{X}(t) = f_e(X(t), u(t)) = -(b - cX(t))u(t) < 0 & \text{(electric engine)} \end{cases}$$

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- A running cost related to each of the dynamics, e.g., the fuel consumption

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• An optimization criterion, e.g., fuel consumption with a given minimal speed

• A set on which switching is mandatory + a set on which switching is optional

$$\begin{cases} X(t) = 0 & \text{(electric} \to \text{thermic, mandatory)} \\ X(t) > 0 & \text{(any switching is optional)} \end{cases}$$

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- A cost related to switching

$$\text{switching cost} = \begin{cases} c_{te} = 0 & \text{(thermic} \to \text{electric)} \\ c_{et} > 0 & \text{(electric} \to \text{thermic)} \end{cases}$$

- An infinite number of commutations in finite time should be avoided

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A general framework for hybrid systems

• State of the system: $(X(t), Q(t)) \in \Omega \times \mathbb{I}$, with $\Omega \subseteq \mathbb{R}^d$, $\mathbb{I} = \{1, \ldots, Q_m\}$. The discrete variable $Q(t)$ (with initial value $q = Q(0)$) tells which dynamics is active at time t

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- A measurable control $u(t)$ mapping $(0, +\infty)$ into a compact set U

- A state equation describing the evolution for a given value of Q:

$$\begin{cases} \dot{X}(t) = f(X(t), Q(t), u(t)), \\ X(0) = x \end{cases}$$

Commutations (i.e., changes in $Q(t)$) may occur in two prescribed sets A and C:

• On hitting the set A at a point (x, q), the state *must jump* to a destination set D. The new state is given by a transition map

$$(x', q') = g(x, q, w)$$

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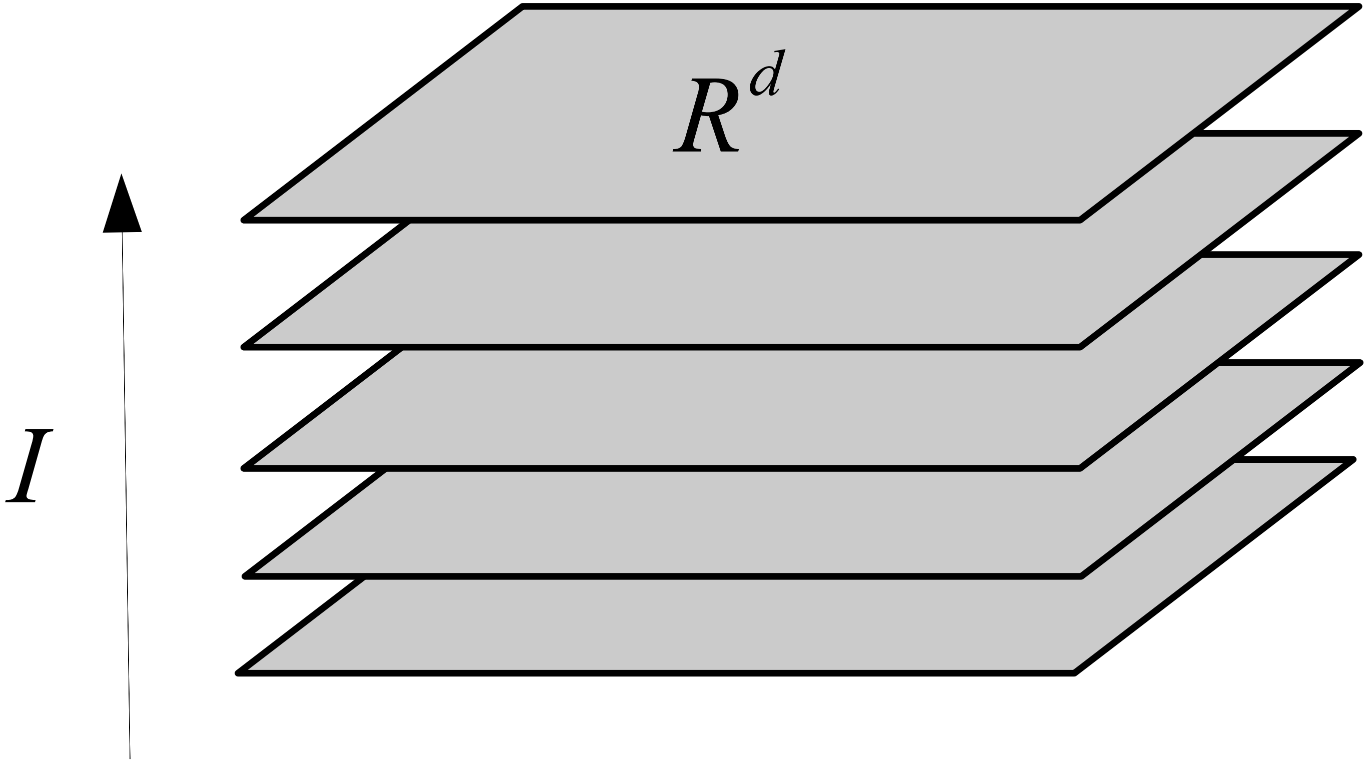
• Inside the set C, the state *may jump* from a state (x, q) to a different state $(x', q') \in D \times \mathbb{I}$



The state space is endowed with the product topology (metric in x, discrete in q)

A control strategy for this hybrid system is a quadruple:

$$\theta = \left(u, w, \{\xi_k\}, (X, Q)\left(\xi_k^+\right)\right)$$

- u and w are the controls for respectively the continuous system f and the transition map g

- ξ_k is a sequence of switching times for the optional jumps and $(X, Q)(\xi_k^+)$ is the corresponding state after the jump

In the discounted infinite horizon case, the cost functional is defined by

$$J(x,q,\theta) = \int_0^{+\infty} \ell(X(t),Q(t),u(t))e^{-\lambda t}\,dt + \quad (1)$$

$$+ \sum_{k=0}^{\infty} C_a(X(\tau_k^-),Q(\tau_k^-),w(\tau_k))e^{-\lambda\tau_k} + \quad (2)$$

$$+ \sum_{i=0}^{\infty} C_c(X(\xi_i^-),Q(\xi_i^-),X(\xi_i^+),Q(\xi_i^+))e^{-\lambda\xi_i} \quad (3)$$

where:

- (1) is the cost related to continuous control
- (2) is the cost related to mandatory (*autonomous*) commutations
- (3) is the cost related to optional (*controlled*) commutations

In order to use the Dynamic Programming approach, the value function of the problem is defined as

$$V(x,q) = \inf_{\theta} J(x,q,\theta)$$

and it can be proved that V satisfies (in a suitably adapted viscosity sense) the Quasi-Variational Inequality

$$\begin{cases} V(x,q) - \mathcal{M}V(x,q) = 0 & x \in A_q, \\ \max(V(x,q) - \mathcal{N}V(x,q), \lambda V(x,q) + H(x,q,D_x V(x,q))) = 0 & x \in C_q, \\ \lambda V(x,q) + H(x, D_x V(x,q)) = 0 & \text{else} \end{cases}$$

• $H(x,q,p) = \sup_{u \in U}\{-\ell(x,q,u) - f(x,q,u) \cdot p\}$
is the term related to continuous control (it selects the best continuous control strategy)

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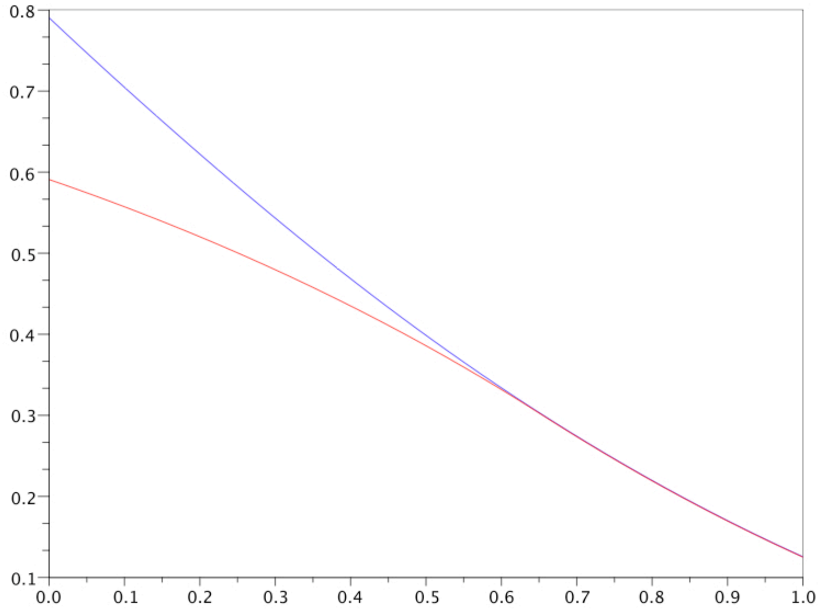
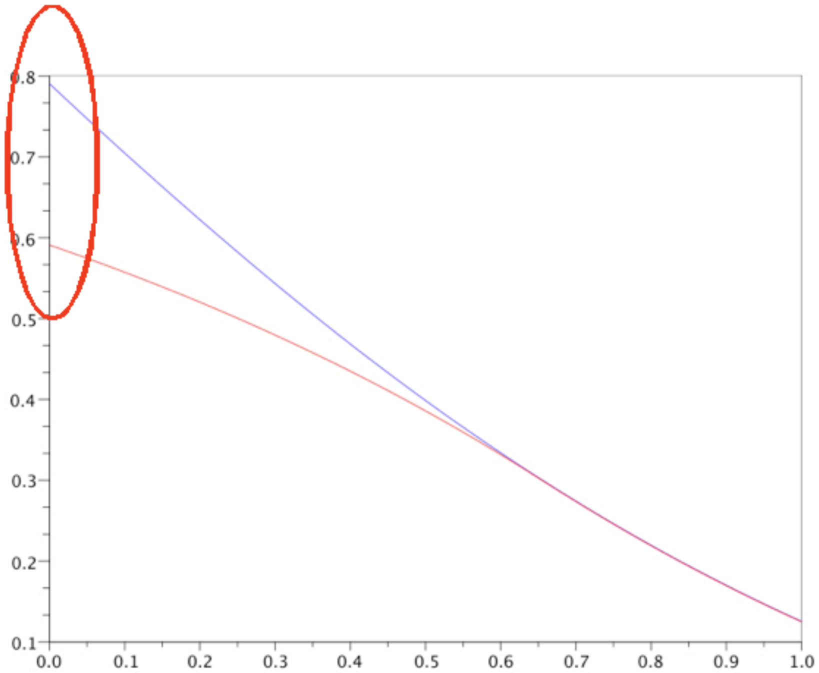
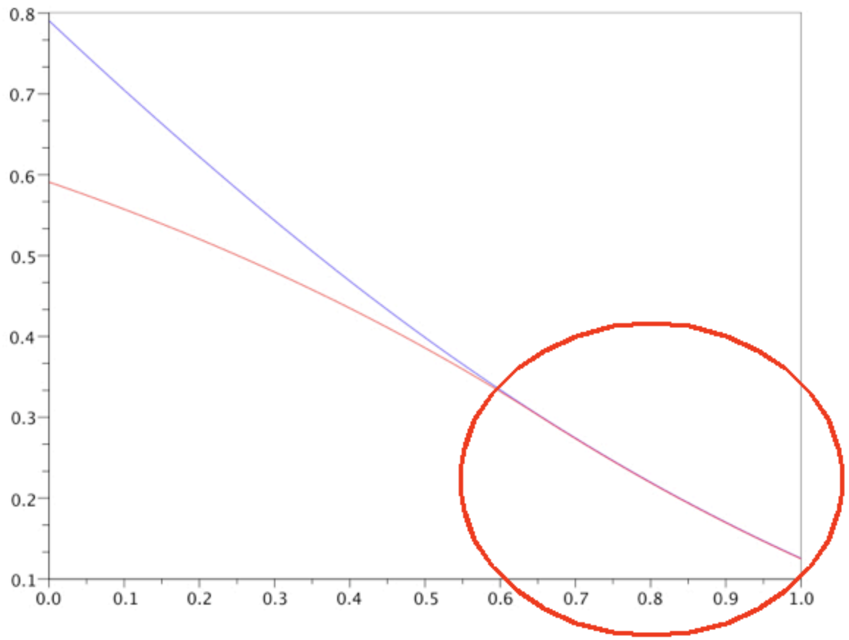
- $\mathcal{N}\phi(x,q) = \inf_{(x',q') \in D \times \mathbb{I}}\{\phi(x',q') + C_c(x,q,x',q')\}$
is the term related to controlled jumps (it selects the best jump strategy on C, and the complementarity condition further chooses the optimal strategy between jumps and continuous control)



Back to the example of the hybrid vehicle: value functions for both the thermic (red) and the electric (blue) dynamics



At $x = 0$, the difference between the two value functions is given by the electric to thermic switching cost $c_{et} = 0.2$



For $x \gtrsim 0.6$, the contact set between the two value functions swows that the optimal strategy is to commute to electric power ($c_{te} = 0$)

A certain number of technical assumptions are to be satisfied to develop the theory. We skip here a complete overview, but the numerical examples show that the framework is robust enough to drop some of them (at least in some special case).

Main Theoretical results:

- The value function V is Hölder continuous (Lipschitz continuous under a slightly less general formulation of the problem)

- A strong comparison principle holds

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The use of monotone numerical schemes

A natural approximation approach is to discretize (in a consistent form) each of the operators appearing in the QVI:

$$\begin{aligned}
\frac{V^h(x,q) - S^h(x,q,V^h)}{h} &\leftrightarrow \lambda V(x,q) + H\left(x,q,D_x V(x,q)\right) \\
V^h(x,q) - N^h V^h(x,q) &\leftrightarrow V(x,q) - \mathcal{N} V(x,q) \\
V^h(x,q) - M^h V^h(x,q) &\leftrightarrow V(x,q) - \mathcal{M} V(x,q)
\end{aligned}$$

• The approximation V^h of V is indexed by a discretization step h, i.e., space and time steps are related (in this situation, typically h is to be intended as a time step)

- S^h, M^h and N^h are monotone, i.e., for any $\Phi^h(x,q) \geq \Psi^h(x,q)$:

$$\begin{aligned} M^h \Phi^h(x,q) &\geq M^h \Psi^h(x,q) \\ N^h \Phi^h(x,q) &\geq N^h \Psi^h(x,q) \\ S^h(x,q,\Phi^h) &\geq S^h(x,q,\Psi^h) \end{aligned}$$

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- M^h and N^h are invariant w.r.t. the addition of constants:

$$\begin{aligned} M^h(\Phi^h + c)(x,q) &= M^h\Phi^h(x,q) + c \\ N^h(\Phi^h + c)(x,q) &= N^h\Phi^h(x,q) + c \end{aligned}$$

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- The operator S^h is a contraction for $h < h_0$, that is,

$$\left\| S^h(\Phi^h) - S^h(\Psi^h) \right\|_\infty \leq (1 - Ch) \left\| \Phi^h - \Psi^h \right\|_\infty$$

The resulting scheme reads $\mathcal{S}^h\left(x, q, V^h\right) = 0$, with

$$\mathcal{S}^h(x,q,V^h) = \begin{cases} V^h(x,q) - M^h V^h(x,q) & \text{if } x \in A_q \\ \max\left\{V^h(x,q) - N^h V^h(x,q), \dfrac{V^h(x,q) - S^h(x,q,V^h)}{h}\right\} & \text{if } x \in C_q \\ \dfrac{V^h(x,q) - S^h(x,q,V^h)}{h} & \text{else.} \end{cases}$$

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- Due to the monotonicity of each building block, the scheme is monotone and L^∞ stable

- Since a strong comparison principle holds for the continuous problem, convergence follows via the Barles–Souganidis theorem

The time-marching origin of the scheme allows to put it in a fixed-point form:

$$V^h(x,q) = T^h(x,q,V^h) = \begin{cases} M^h V^h(x,q) & \text{if } x \in A_q \\ \min\left\{N^h V^h(x,q), S^h(x,q,V^h)\right\} & \text{if } x \in C_q \\ S^h(x,q,V^h) & \text{else.} \end{cases}$$

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• This formulation has a fixed point, attainable by the iteration $V^h_{k+1} = T^h(V^h_k)$

Situations covered by this theory:

• Treatment of the discounted continuous problem: Upwind, Lax–Friedrichs, low-order Semi-Lagrangian,...

• Treatment of the jump operators: in the form

$$M^h\Phi^h(x,q) := \min_{w\in\mathcal{V}}\{I[\Phi](g(x,q,w)) + C_a(x,q,w)\},$$
$$N^h\Phi^h(x,q) := \min_{(x',q')\in D\times\mathbb{I}}\{I[\Phi](x',q') + C_c(x,q,x',q')\},$$

for an interpolation operator $I[\Phi](x,q)$, monotone w.r.t. Φ

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Numerical examples

Example 1 – an unstable system in $\Omega = [-1, 1]$ (with controls in $U = [-1, 1]$) can be stabilized by means of two different dynamics:

- A weaker, less expensive controller
- A stronger, more expensive controller

The first controller cannot stabilize the system for any $x \in [-1, 1]$, so it must switch to the second dynamics on hitting the boundary of Ω, whereas at internal points switching between the two dynamics is optional.

- The two controllers are described by the dynamics

$$f(X(t), 1, u(t)) = X(t) + c_1 u(t)$$
$$f(X(t), 2, u(t)) = X(t) + c_2 u(t),$$

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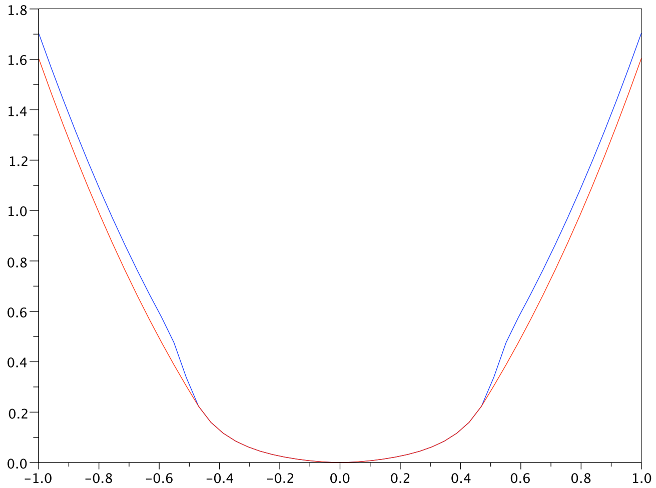
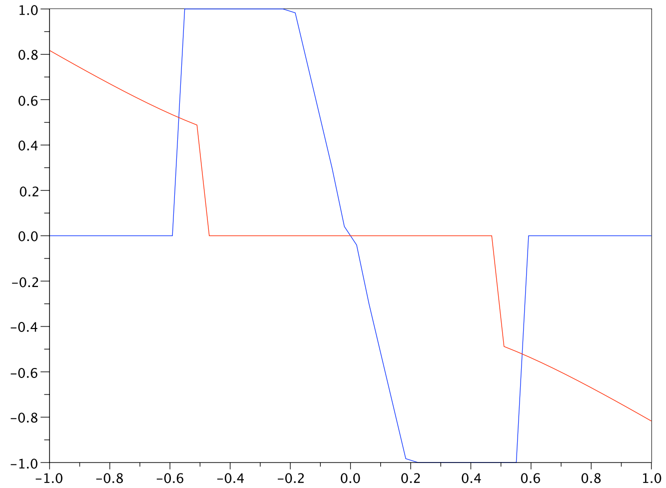
• Accordingly, the running costs associated to the two dynamics are

$$\ell(X(t), 1, u(t)) = X(t)^2 + c_3 u(t)^2,$$
$$\ell(X(t), 2, u(t)) = X(t)^2 + c_4 u(t)^2.$$

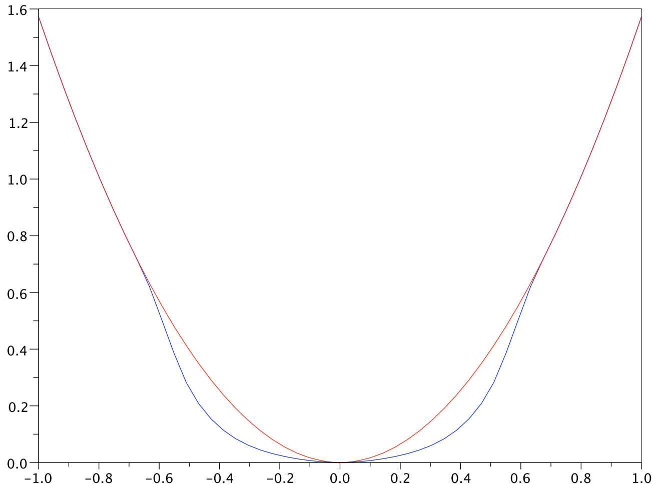
with $c_4 \gg c_3$, this meaning that the second controller is "more expensive";

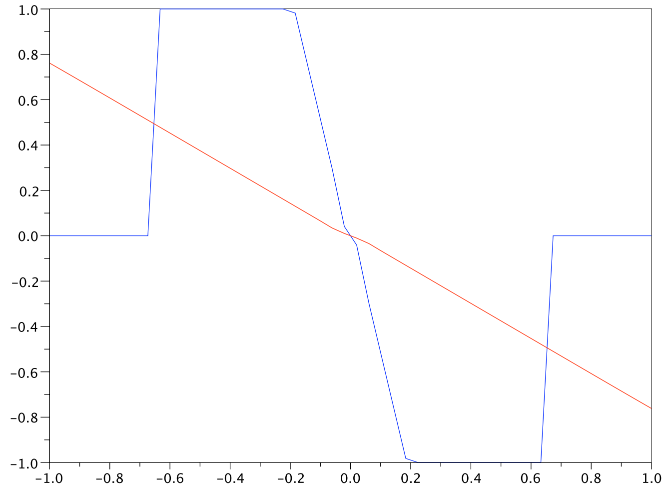




First case: value functions and feedback.

- transition cost weak $\rightarrow$ strong > 0
- transition cost strong $\rightarrow$ weak $= 0$.

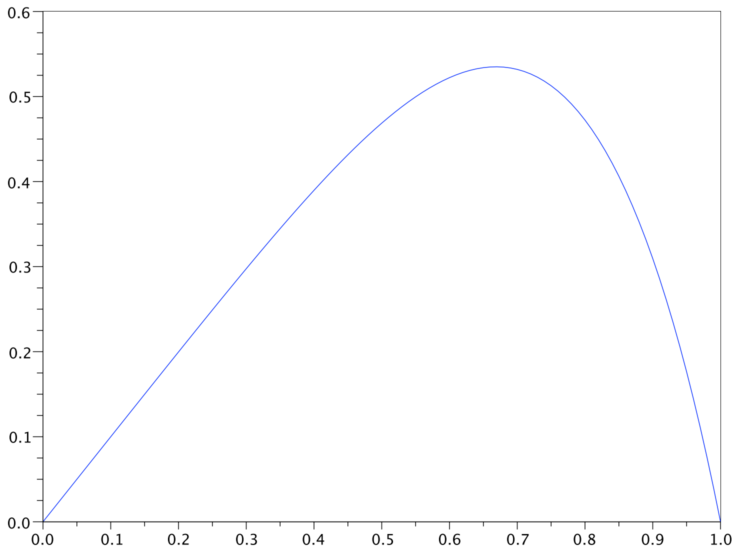




Second case: value functions and feedback.

- transition cost weak $\rightarrow$ strong $= 0$
- transition cost strong $\rightarrow$ weak > 0.

Example 2 – a vehicle with speed in $\Omega = [0,1]$ (and controls in $U = [0,1]$) and two gears. The engine has a given relationship $T(\omega)$ for torque vs angular speed:



- The dynamics associated to the two gears are described by

$$f(X(t), q(t), u(t)) = \frac{T\left(\frac{X(t)}{\rho(q(t))}\right)}{m\,\rho(q(t))} u(t) - \alpha X(t)$$

with $\rho(1)$, $\rho(2)$ the ratios of the gears

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• The associated running cost is given by a combination of travel time and fuel consumption:

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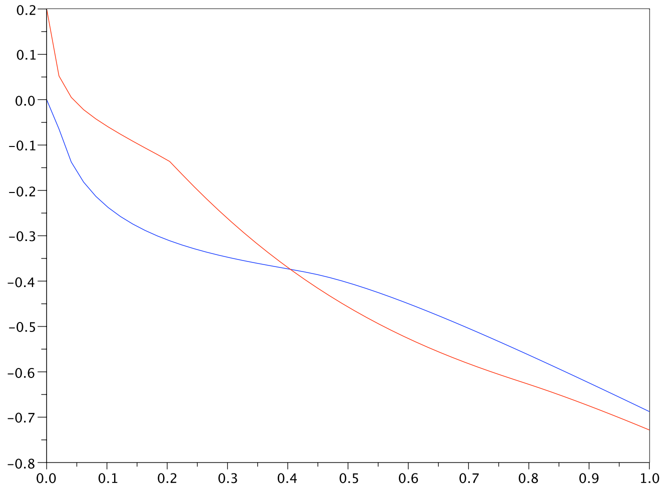
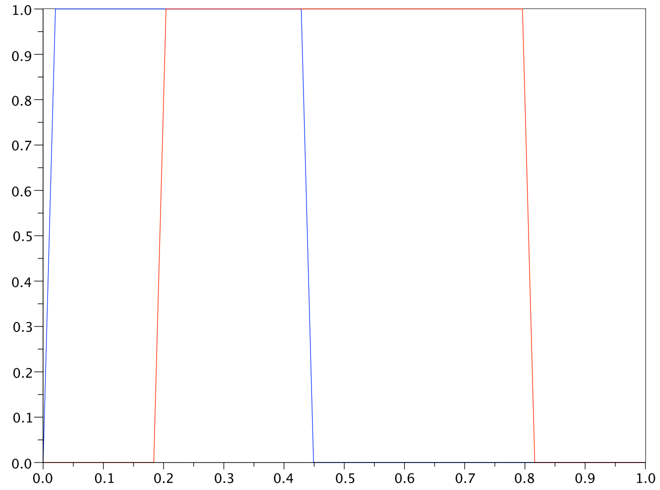
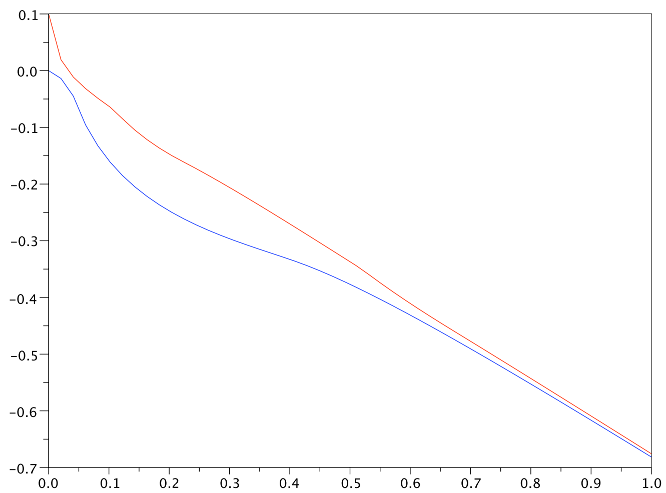
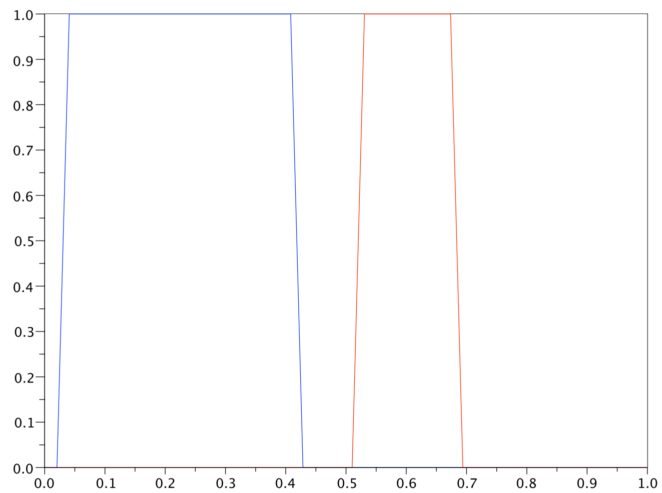
• Constant switching costs c_3, c_4 for changing gear





First case: value functions and feedback (low fuel cost)





Second case: value functions and feedback (high fuel cost)

Forthcoming work:

- More in-depth numerical tests, higher dimension
- Error estimates for the approximate value function
- Fast solvers for the scheme

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