

# Relative entropy methods in the mathematical theory of complete fluid systems

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# Mathematical model

## STATE VARIABLES

**Mass density**

$$\rho = \rho(t, \mathbf{x})$$

**Absolute temperature**

$$\vartheta = \vartheta(t, \mathbf{x})$$

**Velocity field**

$$\mathbf{u} = \mathbf{u}(t, \mathbf{x})$$

## THERMODYNAMIC FUNCTIONS

**Pressure**

$$p = p(\rho, \vartheta)$$

**Internal energy**

$$e = e(\rho, \vartheta)$$

**Entropy**

$$s = s(\rho, \vartheta)$$

## TRANSPORT

**Viscous stress**

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_{\mathbf{x}} \mathbf{u})$$

**Heat flux**

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_{\mathbf{x}} \vartheta)$$



# Field equations



Claude Louis  
Marie Henri  
Navier  
[1785-1836]

## Equation of continuity

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

## Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S} + \varrho \nabla_x F$$



George  
Gabriel  
Stokes  
[1819-1903]

## Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \left( \frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = (\geq) \frac{1}{\vartheta} \left( \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

# Constitutive relations



François Marie Charles Fourier  
[1772-1837]

## Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta$$

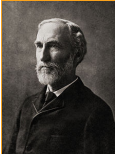


Isaac Newton  
[1643-1727]

## Newton's rheological law

$$\mathbb{S} = \mu(\vartheta) \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}$$

# Gibbs' relation



Willard Gibbs  
[1839-1903]

Gibbs' relation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta)D\left(\frac{1}{\varrho}\right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

# Boundary conditions

## Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

## No-slip

$$\mathbf{u}_{\text{tan}}|_{\partial\Omega} = 0$$

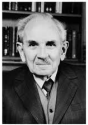
## No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

## Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

# A bit of history of global existence for large data



Jean Leray - Royal society (1998)

**Jean Leray** [1906-1998]  
Global existence of weak  
solutions for the  
incompressible  
Navier-Stokes system (3D)



**Olga Aleksandrovna  
Ladyzhenskaya**  
[1922-2004] Global  
existence of classical  
solutions for the  
incompressible 2D  
Navier-Stokes system



**Pierre-Louis Lions** [\*1956] Global existence of weak  
solutions for the compressible barotropic Navier-Stokes  
system (2,3D)

and many, many others...



# Relative entropy (energy)

## Dynamical system

$$\frac{d}{dt}u(t) = A(t, u(t)), \quad u(t) \in X, \quad u(0) = u_0$$

## Relative entropy

$U : t \mapsto U(t) \in Y$  a “trajectory” in the phase space  $Y \subset X$

$$\mathcal{E} \left\{ u(t) \middle| U(t) \right\}, \quad \mathcal{E} : X \times Y \rightarrow R$$



# Basic properties

## Positivity(distance)

$\mathcal{E} \{u | U\}$  is a “distance” between  $u$ , and  $U$ , meaning  $\mathcal{E}(u|U) \geq 0$  and  $\mathcal{E} \{u|U\} = 0$  only if  $u = U$

## Lyapunov function

$\mathcal{E} \{u(t)|\tilde{U}\}$  is a Lyapunov function provided  $\tilde{U}$  is an equilibrium

$t \mapsto \mathcal{E} \{u(t)|\tilde{U}\}$  is non-increasing

## Gronwall inequality

$$\mathcal{E} \{u(\tau)|U(\tau)\} \leq \mathcal{E} \{u(s)|U(s)\} + c(\mathcal{T}) \int_s^{\mathcal{T}} \mathcal{E} \{u(t)|U(t)\} dt$$

if  $U$  is a solution of the same system (in a “better” space)  $Y$

# Applications

## Stability of equilibria

Any solution ranging in  $X$  stabilizes to an equilibrium belonging to  $Y$  (to be proved!)

## Weak-strong uniqueness

Solutions ranging in the “better” space  $Y$  are unique among solutions in  $X$ .

## Singular limits

Stability and convergence of a family of solutions  $u_\varepsilon$  corresponding to  $A_\varepsilon$  to a solution  $U = u$  of the limit problem with generator  $A$ .

# Navier-Stokes-Fourier system revisited

## Total energy balance (conservation)

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) dx = 0$$

## Total entropy production

$$\frac{d}{dt} \int_{\Omega} \varrho s(\varrho, \vartheta) dx = \int_{\Omega} \sigma dx \geq 0$$

## Total dissipation balance

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta) - \varrho F \right) dx + \int_{\Omega} \sigma dx = 0$$

# Equilibrium (static) solutions

Equilibrium solutions minimize the entropy production

$$\mathbf{u} \equiv 0, \vartheta \equiv \Theta > 0 \text{ a positive constant}$$

Static problem

$$\nabla_x p(\tilde{\varrho}, \Theta) = \tilde{\varrho} \nabla_x F$$

Total mass and energy are constants of motion

$$\int_{\Omega} \tilde{\varrho} \, dx = M_0, \quad \int_{\Omega} \tilde{\varrho} e(\tilde{\varrho}, \Theta) - \tilde{\varrho} F \, dx = E_0$$

# Total dissipation balance revisited

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(\tilde{\varrho}, \Theta)}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\Theta}(\tilde{\varrho}, \Theta) \right) dx + \int_{\Omega} \sigma dx = 0$$

## Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho \left( e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \right)$$

# Coercivity of the ballistic free energy

$$\partial_{\varrho, \varrho}^2 H_{\Theta}(\varrho, \Theta) = \frac{1}{\varrho} \partial_{\varrho} p(\varrho, \Theta)$$

$$\partial_{\vartheta} H_{\Theta}(\varrho, \vartheta) = \varrho(\vartheta - \Theta) \partial_{\vartheta} s(\varrho, \vartheta)$$

## Coercivity

$\varrho \mapsto H_{\Theta}(\varrho, \Theta)$  is convex

$\vartheta \mapsto H_{\Theta}(\varrho, \vartheta)$  attains its global minimum (zero) at  $\vartheta = \Theta$

# Relative entropy

$$\begin{aligned} & \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \\ &= \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right) dx \end{aligned}$$

# Dissipative solutions

## Relative entropy inequality

$$\begin{aligned} & \left[ \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \right]_{t=0}^{\tau} \\ & + \int_0^{\tau} \int_{\Omega} \frac{\Theta}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) dx dt \\ & \leq \int_0^{\tau} \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) dt \end{aligned}$$

for any  $r > 0$ ,  $\Theta > 0$ ,  $\mathbf{U}$  satisfying relevant boundary conditions



# Remainder

$$\begin{aligned} & \boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})} \\ &= \int_{\Omega} \left( \varrho \left( \partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) dx \\ &+ \int_{\Omega} \left[ \left( p(r, \Theta) - p(\varrho, \vartheta) \right) \operatorname{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] dx \\ &- \int_{\Omega} \left( \varrho \left( s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta + \varrho \left( s(\varrho, \vartheta) - s(r, \Theta) \right) \mathbf{u} \cdot \nabla_x \Theta \right. \\ &\quad \left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) dx \\ &+ \int_{\Omega} \frac{r - \varrho}{r} \left( \partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \right) dx \end{aligned}$$

# Applications

## Existence

Dissipative (weak) solutions exist (under certain constitutive restrictions) globally in time for any choice of the initial data.

## Unconditional stability of the equilibrium solutions

Any (weak) solution of the Navier-Stokes-Fourier system stabilizes to an equilibrium (static) solution for  $t \rightarrow \infty$ .

## Weak-strong uniqueness

Weak and strong solutions emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions.

## Singular limit in the incompressible, inviscid regime

Solutions of the Navier-Stokes-Fourier system converge in the limit of low Mach and high Reynolds and Péclet number to the Euler-Boussinesq system.



Ernst Mach  
[1838-1916]



Osborne  
Reynolds  
[1842-1912]



Jean Claude  
Eugène  
Péclet  
[1793-1857]

# Scaled Navier-Stokes-Fourier system

## Equation of continuity

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

## Balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho, \vartheta) = \boxed{\varepsilon^a} \operatorname{div}_x \mathbb{S}$$

## Entropy production

$$\begin{aligned} \partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \boxed{\varepsilon^b} \operatorname{div}_x \left( \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) \\ = \frac{1}{\vartheta} \left( \boxed{\varepsilon^{2+a}} \mathbb{S} : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \end{aligned}$$

# Target system

INCOMPRESSIBILITY

$$\operatorname{div}_x \mathbf{v} = 0$$

EULER SYSTEM

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

TEMPERATURE TRANSPORT

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

## Basic assumption

*The incompressible Euler system possesses a strong solution  $\mathbf{v}$  on a time interval  $(0, T_{\max})$  for the initial data  $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$ .*

# Prepared data

$$\varrho(0, \cdot) = \bar{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \quad \varrho_{0,\varepsilon}^{(1)} \rightarrow \varrho_0^{(1)} \text{ in } L^2(\Omega) \text{ and weakly-}^*(*) \text{ in } L^\infty(\Omega)$$

$$\vartheta(0, \cdot) = \bar{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)}, \quad \vartheta_{0,\varepsilon}^{(1)} \rightarrow \vartheta_0^{(1)} \text{ in } L^2(\Omega) \text{ and weakly-}^*(*) \text{ in } L^\infty(\Omega)$$

$$\mathbf{u}(0, \cdot) = \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{u}_0 \text{ in } L^2(\Omega; \mathbb{R}^3), \quad \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] \in W^{k,2}(\Omega; \mathbb{R}^3), \quad k > \frac{5}{2}$$

# Boundary conditions

## Navier's complete slip condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

# Convergence

$$b > 0, \quad 0 < a < \frac{10}{3}$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \|\varrho_\varepsilon(t, \cdot) - \bar{\varrho}\|_{L^2 + L^{5/3}(\Omega)} \leq \varepsilon C$$

$$\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \rightarrow \sqrt{\bar{\varrho}} \mathbf{v} \text{ in } \boxed{L_{\text{loc}}^\infty((0, T]; L_{\text{loc}}^2(\Omega; \mathbb{R}^3))}$$

and weakly-(\*) in  $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} \rightarrow T \text{ in } \boxed{L_{\text{loc}}^\infty((0, T]; L_{\text{loc}}^q(\Omega; \mathbb{R}^3)), \quad 1 \leq q < 2,}$$

and weakly-(\*) in  $L^\infty(0, T; L^2(\Omega))$



# Uniform bounds

The uniform bounds independent of  $\varepsilon$  are obtained by means of the choice

$$r = \bar{\varrho}, \quad \Theta = \bar{\vartheta}, \quad \mathbf{U} = 0$$

in the relative entropy inequality:

$$\operatorname{ess\,sup}_{t \in (0, T)} \left\| \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \right\|_{L^2 + L^{5/3}(\Omega)} \leq c,$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \left\| \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \left\| \sqrt{\varrho} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega; \mathbb{R}^3)} \leq c$$

# Stability

## MAIN IDEA OF THE PROOF

Take

$$r_\varepsilon = \bar{\rho} + \varepsilon R_\varepsilon, \quad \Theta_\varepsilon = \bar{\vartheta} + \varepsilon T_\varepsilon, \quad \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

### Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta T_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta T_\varepsilon) = 0$$

### Transport equation

$$\partial_t (\delta T_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon) \operatorname{div}_x \mathbf{U}_\varepsilon = 0$$

# Dispersion of acoustic waves

$-\Delta_N$  Neumann Laplacian

$$\partial_{t,t}^2 \Phi - \omega \Delta_N \Phi = 0$$

Hypotheses imposed on  $\Omega$

- Limiting absorption principle. The operator  $\Delta_N$  satisfies the limiting absorption principle in  $\Omega$ :

$$\varphi \circ [-\Delta_N^{-1} - \lambda \pm i\delta]^{-1} \circ \varphi, \varphi \in C_c^\infty(\Omega) \text{ bounded in } L^2(\Omega)$$

for  $\lambda$  belonging to compact subintervals of  $(0, \infty)$ ,  $\delta > 0$ .

- There is a compact set  $B$  such that  $\Delta_N$  satisfies the Strichartz estimates on  $D = \Omega \cup B$ .
- The operator  $\Delta_N$  satisfies the local energy decay.

# Strichartz estimates and local energy decay

$$\|\Phi\|_{L^p(\mathbb{R}; L^q(D))} \leq c \left( \|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)} \right)$$

$$2 \leq q < \infty, \quad \frac{2}{p} \leq \left(1 - \frac{2}{q}\right), \quad \gamma = \frac{3}{2} - \frac{3}{q} - \frac{1}{p}$$

$$\int_{-\infty}^{\infty} \left( \|\chi \Phi(t, \cdot)\|_{H^\gamma(D)}^2 + \|\chi \partial_t \Phi(t, \cdot)\|_{H^{\gamma-1}(D)}^2 \right) dt$$
$$\leq c \left( \|\Phi(0)\|_{H^\gamma(D)} + \|\partial_t \Phi(0)\|_{H^{\gamma-1}(D)} \right),$$

$$\chi \in C_c^\infty(D).$$